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TRANSCENDENCE OF GENERALISED EULER–KRONECKER CONSTANTS

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Abstract

We introduce some generalisations of the Euler–Kronecker constant of a number field and study their arithmetic nature.

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1. Introduction and preliminaries

In 1740, Euler [2] introduced the Euler–Mascheroni constant, which is defined as

$$\gamma = \lim_{x\to\infty}\Big(\sum_{n\le x}\frac{1}{n} - \log x\Big). \tag{1.1}$$

This constant has been extensively studied (see [4]), but many questions about its behaviour are unanswered. For example, it is not known if γ is rational or irrational. Diamond and Ford [1] introduced a generalisation of Euler's constant as follows. For a nonempty finite set of distinct primes Ω, let P_Ω denote the product of the elements of Ω and $\delta_\Omega = \prod_{p\in\Omega}(1-1/p)$. Then the generalised Euler constant is defined as

$$\gamma(\Omega) = \lim_{x\to\infty}\Big(\sum_{\substack{n\le x\\ (n,P_\Omega)=1}}\frac{1}{n} - \delta_\Omega\log x\Big).$$

Note that when $\Omega = \emptyset$, we have $P_\Omega = 1 = \delta_\Omega$ and $\gamma(\Omega) = \gamma$. In this context, Murty and Zaytseva proved the following theorem.

THEOREM 1.1 (Murty and Zaytseva, [8]). *At most one number in the infinite list $\{\gamma(\Omega)\}$, as Ω varies over all finite subsets of distinct primes, is algebraic.*

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We note that γ appears as the constant term in the Laurent series expansion of $\zeta(s)$ around $s = 1$. This observation led Ihara [3] to define the Euler–Kronecker constant associated to a number field as follows.

Let $\mathbf{K}$ be a number field of degree n and let $\mathcal{O}_{\mathbf{K}}$ denote its ring of integers. The Dedekind zeta function of $\mathbf{K}$ is given by

$$\zeta_{\mathbf{K}}(s) = \sum_{(0)\neq \mathfrak{a}\subseteq \mathcal{O}_{\mathbf{K}}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s}, \quad \Re(s) > 1.$$

It has a meromorphic continuation to the entire complex plane with only a simple pole at the point $s = 1$. Its Laurent series expansion around $s = 1$ is given by

$$\zeta_{\mathbf{K}}(s) = \frac{\rho_{\mathbf{K}}}{s-1} + c_{\mathbf{K}} + O(s-1),$$

where $\rho_{\mathbf{K}} \neq 0$ is the residue of $\zeta_{\mathbf{K}}$ at $s = 1$. Ihara defined the ratio

$$\gamma_{\mathbf{K}} := c_{\mathbf{K}}/\rho_{\mathbf{K}}$$

as the Euler–Kronecker constant of $\mathbf{K}$. In the next section, an expression analogous to (1.1) is given for $\gamma_{\mathbf{K}}$.

The aim of this article is to study the arithmetic nature of generalisations of Euler–Kronecker constants. To do so, we introduce some notation. Let $\mathcal{P}_{\mathbf{K}}$ denote the set of nonzero prime ideals $\mathfrak{p}$ of $\mathcal{O}_{\mathbf{K}}$ and let Ω be a nonempty subset of $\mathcal{P}_{\mathbf{K}}$ (possibly infinite) such that

$$\sum_{\mathfrak{p}\in\Omega} \frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p}) - 1} < \infty. \tag{1.2}$$

For $\mathbf{K} = \mathbb{Q}$, the set of Pjateckii–Šapiro primes is an example of such an infinite subset. Let $N_\Omega = \{\mathfrak{p} \cap \mathbb{Z} \mid \mathfrak{p} \in \Omega\}$. We set

$$P(\Omega(x)) = \prod_{\mathfrak{p}\in\Omega(x)} \mathfrak{p} \quad \text{and} \quad \delta_{\mathbf{K}}(\Omega(x)) = \prod_{\mathfrak{p}\in\Omega(x)} \Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\Big),$$

where $\Omega(x) = \{\mathfrak{p} \in \Omega \mid \mathfrak{N}(\mathfrak{p}) \leq x\}$. Then by (1.2), $\lim_{x\to\infty} \delta_{\mathbf{K}}(\Omega(x))$ exists and equals

$$\delta_{\mathbf{K}}(\Omega) = \prod_{\mathfrak{p}\in\Omega} \Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\Big).$$

Note that $\delta_{\mathbf{K}}(\Omega) = 1$ for $\Omega = \emptyset$. The generalised Euler–Kronecker constant associated to Ω is denoted by $\gamma_{\mathbf{K}}(\Omega)$ and is defined as

$$\lim_{x\to\infty} \Bigg(\frac{1}{\rho_{\mathbf{K}}} \sum_{\substack{0\neq\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}} \\ \mathfrak{N}(\mathfrak{a})\leq x \\ (\mathfrak{a}, P(\Omega(x)))=1}} \frac{1}{\mathfrak{N}(\mathfrak{a})} - \delta_{\mathbf{K}}(\Omega(x)) \log x \Bigg).$$

In Section 3, we will show that this limit exists. We note that $\gamma_{\mathbf{K}}(\Omega) = \gamma_{\mathbf{K}}$ when $\Omega = \emptyset$. With this set up, we have the following theorem.

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THEOREM 1.2. *Let $\{\Omega_i\}_{i\in I}$ be a family of subsets of $\mathcal{P}_{\mathbf{K}}$ satisfying (1.2). Further, suppose that $N_{\Omega_i}\backslash N_{\Omega_j}$ is nonempty and finite for all $i,j \in I$ and $i \neq j$. Then at most one number from the infinite list*

$$\left\{ \frac{\gamma_{\mathbf{K}}(\Omega_i)}{\delta_{\mathbf{K}}(\Omega_i)} \,\middle|\, i \in I \right\}$$

is algebraic.

We digress here a little to make an interesting observation. For $K = \mathbb{Q}$, it is known by Merten's theorem that as $x \to \infty$,

$$\delta_{\mathbb{Q}}(\Omega(x)) \sim \frac{e^{-\gamma}}{\log x}.$$

This makes one wonder if $\gamma_{\mathbf{K}}$ appears as an exponent in the expression for $\mathbf{K} \neq \mathbb{Q}$. A result of Rosen [9] shows that this is not true in general. More precisely, he showed that as $x \to \infty$,

$$\delta_{\mathbf{K}}(\Omega(x)) \sim \frac{e^{-\gamma}}{\rho_{\mathbf{K}} \log x}.$$

2. Preliminaries and lemmas

Let $\mathbf{K}$ be a number field of degree n. Throughout this section, $\mathfrak{p}$ denotes a nonzero prime ideal of $\mathcal{O}_{\mathbf{K}}$. We recall the following result on counting the number of integral ideals of $\mathcal{O}_{\mathbf{K}}$.

LEMMA 2.1 [7, Ch. 11]. *Let a_m be the number of integral ideals of $\mathcal{O}_{\mathbf{K}}$ with norm m. Then, as x tends to infinity,*

$$\sum_{m=1}^{x} a_m = \rho_{\mathbf{K}} x + O(x^{1-1/n}).$$

Using this result, we find the following expression for $\gamma_{\mathbf{K}}$, analogous to (1.1).

LEMMA 2.2. *For any number field $\mathbf{K}$, the limit*

$$\lim_{x\to\infty} \left(\frac{1}{\rho_{\mathbf{K}}} \sum_{\substack{0 \neq \mathfrak{a} \subset \mathcal{O}_{\mathbf{K}} \\ \mathfrak{N}(\mathfrak{a}) \leq x}} \frac{1}{\mathfrak{N}(\mathfrak{a})} - \log x \right)$$

exists and equals $\gamma_{\mathbf{K}}$.

PROOF. Applying partial summation and Lemma 2.1, the result follows. □

The Möbius function $\mu_{\mathbf{K}}$ and the von Mangoldt function $\Lambda_{\mathbf{K}}$ are defined on $\mathcal{O}_{\mathbf{K}}$ as follows:

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$$\mu_{\mathbf{K}}(\mathfrak{a}) = \begin{cases} 1 & \text{if } \mathfrak{a} = \mathcal{O}_{\mathbf{K}}, \\ (-1)^r & \text{if } \mathfrak{a} \text{ is a product of } r \text{ distinct prime ideals,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\Lambda_{\mathbf{K}}(\mathfrak{a}) = \begin{cases} \log \mathfrak{N}(\mathfrak{p}) & \text{if } \mathfrak{a} = \mathfrak{p}^m \text{ for some } \mathfrak{p} \text{ and some integer } m \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

We record the following identities satisfied by these functions which can be derived using techniques similar to [6, Exercises 1.1.2, 1.1.4, 1.1.6].

$$\sum_{J|I} \frac{\mu_{\mathbf{K}}(J)}{\mathfrak{N}(J)} = \prod_{\mathfrak{p}|I} \Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\Big),$$

$$\mu_{\mathbf{K}}(I) \log \mathfrak{N}(I) = -\sum_{J|I} \Lambda_{\mathbf{K}}(J) \mu_{\mathbf{K}}(IJ^{-1}).$$

We end this section by stating the key ingredient in the proof of Theorem 1.2.

LEMMA 2.3 (Lindemann, [5]). *If* $\alpha \neq 0, 1$ *is an algebraic number, then* $\log \alpha$ *is transcendental, where* log *denotes any branch of the logarithm.*

3. Generalised Euler–Kronecker constants

Let $\mathcal{P}_{\mathbf{K}}$ denote the set of nonzero prime ideals of $\mathcal{O}_{\mathbf{K}}$. For any nonempty finite set $\Omega_f \subset \mathcal{P}_{\mathbf{K}}$, we set

$$P(\Omega_f) = \prod_{\mathfrak{p} \in \Omega_f} \mathfrak{p} \quad \text{and} \quad \delta_{\mathbf{K}}(\Omega_f) = \prod_{\mathfrak{p} \in \Omega_f} \Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\Big),$$

with the convention that $P(\Omega_f) = 1 = \delta_{\mathbf{K}}(\Omega_f)$, when $\Omega_f = \emptyset$. Since $\mathcal{O}_{\mathbf{K}}$ is a Dedekind domain, every integral ideal can be uniquely expressed as a product of prime ideals. For ideals

$$\mathfrak{a} = \prod_{\mathfrak{p} \in \mathcal{P}_{\mathbf{K}}} \mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{a})}, \quad \mathfrak{b} = \prod_{\mathfrak{p} \in \mathcal{P}_{\mathbf{K}}} \mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{b})},$$

where all but finitely many $\nu_{\mathfrak{p}}(\mathfrak{a}), \nu_{\mathfrak{p}}(\mathfrak{b})$ are zero, we define the greatest common divisor (gcd) of $\mathfrak{a}$ and $\mathfrak{b}$ by

$$(\mathfrak{a}, \mathfrak{b}) = \gcd(\mathfrak{a}, \mathfrak{b}) = \prod_{\mathfrak{p} \in \mathcal{P}_{\mathbf{K}}} \mathfrak{p}^{\min(\nu_{\mathfrak{p}}(\mathfrak{a}), \nu_{\mathfrak{p}}(\mathfrak{b}))},$$

where we have denoted $\mathfrak{p}^0$ by $\mathcal{O}_{\mathbf{K}}$. Hence, if the prime factors of $\mathfrak{a}$ and $\mathfrak{b}$ are all distinct, $(\mathfrak{a}, \mathfrak{b}) = \mathcal{O}_{\mathbf{K}}$. We notice that $(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} + \mathfrak{b}$ as $\nu_{\mathfrak{p}}(\mathfrak{a} + \mathfrak{b}) = \min(\nu_{\mathfrak{p}}(\mathfrak{a}), \nu_{\mathfrak{p}}(\mathfrak{b}))$. From now on, $\mathcal{O}_{\mathbf{K}}$ will be denoted by 1.

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LEMMA 3.1. *For a number field $\mathbf{K}$ and a finite set Ω_f, the limit*

$$\lim_{x\to\infty}\bigg(\frac{1}{\rho_{\mathbf{K}}}\sum_{\substack{0\neq I\subset O_{\mathbf{K}}\\ \mathfrak{N}(I)\leq x\\ (I,P(\Omega_f))=1}}\frac{1}{\mathfrak{N}(I)}-\delta_{\mathbf{K}}(\Omega_f)\log x\bigg)$$

exists and is denoted by $\gamma_{\mathbf{K}}(\Omega_f)$.

PROOF. Let $\Omega_f\subset\mathcal{P}_{\mathbf{K}}$ and $\mathfrak{p}\in\mathcal{P}_{\mathbf{K}}$ be a prime ideal not in Ω_f. Using

$$\sum_{\substack{0\neq I\subseteq O_{\mathbf{K}}\\ \mathfrak{N}(I)\leq x\\ (I,\mathfrak{p}P(\Omega_f))=1}}\frac{1}{\mathfrak{N}(I)}=\sum_{\substack{0\neq I\subseteq O_{\mathbf{K}}\\ \mathfrak{N}(I)\leq x\\ (I,P(\Omega_f))=1}}\frac{1}{\mathfrak{N}(I)}-\frac{1}{\mathfrak{N}(\mathfrak{p})}\sum_{\substack{0\neq I\subseteq O_{\mathbf{K}}\\ \mathfrak{N}(I)\leq x/\mathfrak{N}(\mathfrak{p})\\ (I,P(\Omega_f))=1}}\frac{1}{\mathfrak{N}(I)},$$

the result follows by induction on the cardinality of Ω_f. □

LEMMA 3.2. *Let Ω_f be a finite set of nonzero prime ideals. Then,*

$$\gamma_{\mathbf{K}}(\Omega_f)=\delta_{\mathbf{K}}(\Omega_f)\bigg(\gamma_{\mathbf{K}}+\sum_{\mathfrak{p}\in\Omega_f}\frac{\log\mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})-1}\bigg).$$

PROOF. We have

$$\begin{aligned}\sum_{\substack{0\neq I\subset O_{\mathbf{K}}\\ \mathfrak{N}(I)\leq x\\ (I,P(\Omega_f))=1}}\frac{1}{\mathfrak{N}(I)}&=\sum_{\substack{0\neq I\subset O_{\mathbf{K}}\\ \mathfrak{N}(I)\leq x}}\frac{1}{\mathfrak{N}(I)}\sum_{J\mid(I,P(\Omega_f))}\mu(J)\\ &=\sum_{J\mid P(\Omega_f)}\frac{\mu(J)}{\mathfrak{N}(J)}\sum_{\substack{0\neq J_0\subset O_{\mathbf{K}}\\ \mathfrak{N}(J_0)\leq x/\mathfrak{N}(J)}}\frac{1}{\mathfrak{N}(J_0)}\\ &=\sum_{J\mid P(\Omega_f)}\frac{\mu(J)}{\mathfrak{N}(J)}\Big\{\rho_{\mathbf{K}}\log\frac{x}{\mathfrak{N}(J)}+\rho_{\mathbf{K}}\gamma_{\mathbf{K}}+o(1)\Big\}\\ &=\delta_{\mathbf{K}}(\Omega_f)(\rho_{\mathbf{K}}\log x+\rho_{\mathbf{K}}\gamma_{\mathbf{K}}+o(1))-\rho_{\mathbf{K}}\sum_{J\mid P(\Omega_f)}\frac{\mu(J)}{\mathfrak{N}(J)}\log\mathfrak{N}(J).\end{aligned}$$

We now consider the last term:

$$\begin{aligned}-\sum_{J\mid P(\Omega_f)}\frac{\mu(J)}{\mathfrak{N}(J)}\log\mathfrak{N}(J)&=\sum_{J\mid P(\Omega_f)}\frac{1}{\mathfrak{N}(J)}\sum_{J_0\mid J}\Lambda(J_0)\mu(JJ_0^{-1})\\ &=\sum_{J_0\mid P(\Omega_f)}\frac{\Lambda(J_0)}{\mathfrak{N}(J_0)}\sum_{J_1\mid P(\Omega_f)J_0^{-1}}\frac{\mu(J_1)}{\mathfrak{N}(J_1)}\\ &=\sum_{\mathfrak{p}'\in\Omega_f}\frac{\Lambda(\mathfrak{p}')}{\mathfrak{N}(\mathfrak{p}')}\sum_{J_1\mid P(\Omega_f)\mathfrak{p}'^{-1}}\frac{\mu(J_1)}{\mathfrak{N}(J_1)}\end{aligned}$$

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$$= \sum_{\mathfrak{p}' \in \Omega_f} \frac{\log \mathfrak{N}(\mathfrak{p}')}{\mathfrak{N}(\mathfrak{p}')} \Big(\frac{\delta_{\mathbf{K}}(\Omega_f)}{1 - 1/\mathfrak{N}(\mathfrak{p}')} \Big)$$

$$= \delta_{\mathbf{K}}(\Omega_f) \sum_{\mathfrak{p} \in \Omega_f} \frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p}) - 1}.$$

Thus,

$$\lim_{x \to \infty} \Big(\frac{1}{\rho_{\mathbf{K}}} \sum_{\substack{0 \neq I \subset \mathcal{O}_{\mathbf{K}} \\ \mathfrak{N}(I) \leq x \\ (I, P(\Omega_f)) = 1}} \frac{1}{\mathfrak{N}(I)} - \delta_{\mathbf{K}}(\Omega_f) \log x \Big) = \delta_{\mathbf{K}}(\Omega_f) \Big(\gamma_{\mathbf{K}} + \sum_{\mathfrak{p} \in \Omega_f} \frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p}) - 1} \Big). \qquad \square$$

COROLLARY 3.3. *For a number field* $\mathbf{K}$ *and any set* $\Omega \subset \mathcal{P}_{\mathbf{K}}$ *satisfying (1.2), the limit*

$$\lim_{x \to \infty} \Big(\frac{1}{\rho_{\mathbf{K}}} \sum_{\substack{0 \neq I \subset \mathcal{O}_{\mathbf{K}} \\ \mathfrak{N}(I) \leq x \\ (I, P(\Omega(x))) = 1}} \frac{1}{\mathfrak{N}(I)} - \delta_{\mathbf{K}}(\Omega(x)) \log x \Big)$$

exists and equals

$$\delta_{\mathbf{K}}(\Omega) \Big(\gamma_{\mathbf{K}} + \sum_{\mathfrak{p} \in \Omega} \frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p}) - 1} \Big).$$

We denote this limit by $\gamma_{\mathbf{K}}(\Omega)$.

PROOF. Follows from Lemma 3.2 since $\Omega(x)$ is a finite set. $\square$

4. Proof of Theorem 1.2

Suppose there exist $i, j \in I$ such that

$$\frac{\gamma_{\mathbf{K}}(\Omega_i)}{\delta_{\mathbf{K}}(\Omega_i)} \quad \text{and} \quad \frac{\gamma_{\mathbf{K}}(\Omega_j)}{\delta_{\mathbf{K}}(\Omega_j)}$$

are algebraic. Using Corollary 3.3,

$$\frac{\gamma_{\mathbf{K}}(\Omega_i)}{\delta_{\mathbf{K}}(\Omega_i)} - \frac{\gamma_{\mathbf{K}}(\Omega_j)}{\delta_{\mathbf{K}}(\Omega_j)} = \sum_{\mathfrak{p} \in \Omega_i} \frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p}) - 1} - \sum_{\mathfrak{p} \in \Omega_j} \frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p}) - 1}, \tag{4.1}$$

which is also an algebraic number. Since the sets $N_{\Omega_i} \backslash N_{\Omega_j}$ and $N_{\Omega_i} \backslash N_{\Omega_j}$ are nonempty and finite, the sets $\Omega_i \backslash \Omega_j$ and $\Omega_j \backslash \Omega_i$ are also finite. Let

$$\Omega_i \backslash \Omega_j = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}, \quad \Omega_j \backslash \Omega_i = \{\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_m\}.$$

Then (4.1) implies

$$\sum_{\mathfrak{p} \in \Omega_i} \frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p}) - 1} - \sum_{\mathfrak{p} \in \Omega_j} \frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p}) - 1} = \sum_{s=1}^{n} \frac{\log p_s^{f_s}}{p_s^{f_s} - 1} - \sum_{t=1}^{m} \frac{\log q_t^{g_t}}{q_t^{g_t} - 1} = \log \Big(\frac{\prod_{s=1}^{n} p_s^{(f_s/p_s^{f_s} - 1)}}{\prod_{t=1}^{m} q_t^{(g_t/q_t^{g_t} - 1)}} \Big), \tag{4.2}$$

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where $\mathfrak{N}(\mathfrak{p}_s) = p_s^{f_s}$ and $\mathfrak{N}(\mathfrak{q}_t) = q_t^{g_t}$. Using Lemma 2.3 and unique prime factorisation of natural numbers, the expression in (4.2) becomes a transcendental number, which gives a contradiction.

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