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On homologies in finite combinatorial geometries

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Any subset Π^* of the set of all planes through a line in a finite projective space $\mathrm{PG}(m, q)$ determines a subgeometry $G(\Pi^*)$ of the combinatorial geometry associated with $\mathrm{PG}(m, q)$. In this paper the geometries $G(\Pi^*)$ of rank greater than three in which every line contains at least four points, are characterized in terms of the existence of a certain set of automorphism groups $\Gamma(C, X)$; where X is a copoint and C a point not in X, and each non-trivial element of $\Gamma(C, X)$ fixes X and every copoint through C and fixes C and every point in X, but no other point; and where $\Gamma(C, X)$ acts transitively on the points distinct from C and not in X of some line through C. As a corollary of the main theorem we obtain a characterization of the finite projective spaces $\mathrm{PG}(m, q)$ with $m \geq 3$ and $q \geq 3$.

1. Introduction

The Lenz-Barlotti classification of projective planes (see Lenz [6] and Barlotti [2], or Dembowski [4]) in terms of "(C, a)-transitivity", where C is a point and a is a line, has been extremely useful and much studied. In this paper we illustrate the fact that some results along similar lines can be obtained for finite combinatorial geometries (of which finite projective planes are a special case) or, equivalently, finite geometric lattices. We shall conduct our main discussion in terms of lattices, for reasons of economy.

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One difficulty which arises immediately when we generalize the notion of a central collineation to combinatorial geometries is that an automorphism of a combinatorial geometry may be central and yet not axial, or axial and yet not central. Another difficulty is that a non-trivial automorphism with a centre and an axis may fix a point which is distinct from the centre and does not lie on the axis. This leads us to use, rather than the concept of (C, X)-transitivity, the narrower one of "(C, X)-regularity" (X here denotes a copoint). In Theorem 1 we give a sufficient condition, in terms of the existence of certain transversal lines, for (C, X)-transitivity to imply (C, X)-regularity.

The main Theorem 2, proved in §4, yields a characterization of the geometry determined by a subset of the set of all planes passing through a line, in a finite projective space $\mathrm{PG}(m, q)$. As a corollary, we deduce in §5 a new characterization for finite projective spaces $\mathrm{PG}(m, q)$ with $m \geq 3$, $q \geq 3$.

2. Definitions

A finite lattice is *geometric* if it is semi-modular (x and y cover $x \wedge y$ implies $x \vee y$ covers x and y) and if every non-zero element is a join of points (elements which cover zero). Every geometric lattice has a semi-modular rank function (see Crapo and Rota [3]).

We shall use geometric language quite freely. Thus, if $\underline{\underline{L}}$ is a geometric lattice of rank $n \geq 3$, we refer to the elements of rank 1, 2, 3, $n-3$, $n-2$ or $n-1$ in $\underline{\underline{L}}$ as *points*, *lines*, *planes*, *coplanes*, *colines*, or *copoints*, respectively. Points and copoints we denote by upper case Roman letters, lines and colines by lower case Roman letters, and planes and coplanes by lower case Greek letters.

A point C is a *centre* for the automorphism θ of $\underline{\underline{L}}$ if θ fixes every copoint above (through, containing) C; and a copoint X is an *axis* for θ if θ fixes every point below (in, on, of, and so on) X. An automorphism may have a centre but no axis, or an axis but no centre. If the automorphism θ has centre C and axis X we shall call θ a (C, X)-*automorphism*; or a (C, X)-*homology* if $C \nless X$; or a (C, X)-*elation* if $C < X$.

We say that $\underline{\underline{L}}$ is (C, X)-*transitive* if the group $\Gamma(C, X)$ of all

(C, X)-automorphisms acts transitively on the points distinct from C and not in X of some line, $\nleq X$, through C; and that $\underline{\underline{L}}$ is (C, X)-*regular* if it is (C, X)-transitive and no non-trivial (C, X)-automorphism fixes any point A such that $A \neq C$ and $A \nleq X$.

That a geometric lattice may be (C, X)-transitive and yet not (C, X)-regular may be seen by considering the lattice of rank 3 whose points and copoints are the points and lines of a 3-dimensional projective space $PG(3, F)$. The rank 4 lattice usually associated with $PG(3, F)$ is, as is well known, (C, X)-regular for all point - copoint pairs (C, X).

Since a (C, X)-automorphism θ automatically fixes C and all the points of X, we shall call any other fixed point of θ an *extra fixed point*. If $\underline{\underline{L}}$ is (C, X)-transitive then $\underline{\underline{L}}$ is (C, X)-regular if and only if no non-trivial (C, X)-automorphism has extra fixed points.

Let X and Y be copoints such that $X \wedge Y$ is a coline, and let A be a point with $A \nleq X$, $A \nleq Y$. Then by a *proper transversal* for (A, X, Y) we shall mean a line d through A such that $d \wedge X$ and $d \wedge Y$ are distinct points.

Finally, two lattice elements (flats) x and y are said to *meet* if $x \wedge y \neq 0$, that is if there is at least one point which is contained in both x and y.

We assume implicitly the elementary theories of geometric lattices and combinatorial geometries (see Crapo and Rota [3]), of projective planes, projective spaces, tactical configurations and block designs (see Dembowski [4]), and of finite permutation groups (see Wielandt [7]).

3. Homologies and elations in geometric lattices

The following theorem describes a sufficient condition for the non-existence of extra fixed points of homologies and elations.

THEOREM 1. *Suppose that* θ *is a non-trivial* (C, X)*-automorphism of a finite geometric lattice* $\underline{\underline{L}}$ *of rank* $n \geq 3$, *and that:*

(i) every triple (A, Y, Z) *with* Y *and* Z *copoints of* $\underline{\underline{L}}$, $Y \wedge Z$ *a coline of* $\underline{\underline{L}}$, *and* A *a point of* $\underline{\underline{L}}$ *such that* $A \nleq Y$ *and* $A \nleq Z$, *has a proper transversal in* $\underline{\underline{L}}$;

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(ii) every line of $\underline{\underline{L}}$ *contains at least three points of* $\underline{\underline{L}}$.

Then θ *has no extra fixed points in* $\underline{\underline{L}}$.

Proof. Suppose that θ has an extra fixed point F . Choose a coline y with $C \nleq y < X$, and consider the copoint $Y = y \vee F$. If $C \nleq X$ and $C \nleq Y$ then there is a proper transversal d for (C, X, Y) and $d \wedge Y$ is an extra fixed point, since Y is an axis (being a fixed copoint which does not contain the centre C).

If $C \nleq X$, $C < Y$, and the line $C \vee F$ does not meet X , choose any point A in X but not in Y , and consider a point F' on the line $F \vee A$, with $F' \neq F, A$. Then $C \nleq (y \vee F')$ and so by the argument above there is a line, joining C to a point of X , which contains an extra fixed point.

Now let F^* be any extra fixed point such that $(C \vee F^*) \wedge X$ is a point, say B . (If θ is an elation then $B = C$.) Choose a coplane σ with $B \nleq \sigma < X$; choose distinct colines x_1 and x_2 with $B \nleq x_i$ and $\sigma < x_i < X$ $(i = 1, 2)$; and let $X_1 = x_1 \vee F^*$ and $X_2 = x_2 \vee F^*$. Then X_1 and X_2 are fixed copoints not containing C , and so they are axes. If D is a non-fixed point then, since $C \vee D$ is a fixed line and any proper transversal for (D, X_1, X_2) is a fixed line, $C \vee D$ is a proper transversal for (D, X_1, X_2) . But $C \vee F^*$ is not a proper transversal, so $F^* \nleq (C \vee D)$. Thus every point of $C \vee F^*$ is fixed.

There exists a non-fixed point D . As above, $C \vee D$ is a proper transversal for (D, X_1, X_2) . The point $(C \vee D) \wedge X_2$ is an extra fixed point for θ when θ is considered as a (C, X_1)-homology, and the line $C \vee ((C \vee D) \wedge X_2) = C \vee D$ meets X_1 in a point. So every point of $C \vee D$ is fixed, in particular D is fixed. This contradiction completes the proof.

4. A classification theorem

Let Π^* denote a subset of the set of all planes through a fixed line c^* in $PG(m, q)$. Then Π^* determines a geometric lattice $\underline{\underline{L}}(\Pi^*)$ whose elements are the intersections of the flats of $PG(m, q)$ with the set of

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all those points of $PG(m, q)$ which lie in at least one of the planes in Π^* . The following theorem will yield a characterization of the lattices $\underline{\underline{L}}(\Pi^*)$ corresponding to sets Π^* which are "non-degenerate" in a fairly mild sense.

THEOREM 2. *Let* $\underline{\underline{L}}$ *be a finite geometric lattice of rank* $n \geq 4$ *satisfying:*

(i) there exists a line c *in* $\underline{\underline{L}}$ *such that, whenever* C *is a point on* c *and* x *is a coline with* $C \nleq x$, $\underline{\underline{L}}$ *is* (C, X)*-regular for some copoint* X *containing* x *but not* C ; *and*

(ii) every line of $\underline{\underline{L}}$ *contains at least four points.*

Then $\underline{\underline{L}}$ *is isomorphic to a lattice* $\underline{\underline{L}}(\Pi^*)$ *for some set* Π^* *of planes through a line* c^* *in a projective geometry* $PG(n-1, q)$, $q \geq 3$.

The "non-degenerate" sets Π^* referred to above are those which determine a lattice $\underline{\underline{L}}(\Pi^*)$ of rank $m + 1$ in which every line contains at least four points. For such a Π^* , $\underline{\underline{L}}(\Pi^*)$ clearly satisfies the hypotheses of the theorem.

We prepare for the proof of the theorem with a sequence of seven lemmas; $\underline{\underline{L}}$ and c are assumed throughout to satisfy the hypotheses of the theorem. The automorphism group of $\underline{\underline{L}}$ generated by all the automorphisms those existence is asserted in hypothesis *(i)* is denoted by $\underline{\underline{G}}$.

LEMMA 1. *Every line which meets* c *contains at least as many points as* c .

Proof. The number of points in a flat x , or "length" of x , shall be denoted by $|x|$.

Choose a line a meeting c with $|a|$ as small as possible, and suppose $|a| < |c|$. Let $C = a \wedge c$; choose a point $A \neq C$ on a ; and let Λ be the set of all lines $\neq a$ which pass through A and meet c . If $a_1 \in \Lambda$ then $a_1 = (a \vee c) \wedge x$, for some coline x . So $|\Gamma(C, X)| = |a| - 2$ for some copoint X with $(a \vee c) \wedge X = a_1$. Also, no non-trivial element of $\Gamma(C, X)$ fixes a line in $\Lambda \backslash \{a_1\}$. Thus, if p is a prime dividing $|a| - 2$, each line in Λ is fixed by an automorphism

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of order p in $\Gamma(C)_A$ which fixes no other line in Λ, where $\Gamma(C)_A$ is the group of all automorphisms with centre C which fix A. So Λ is a complete line orbit of this group (Gleason [5], Lemma 1.7).

Let $C_1 = a_1 \wedge c$ and choose a point $A_1 \neq A, C_1$ on a_1. There exists a copoint Y with $(a \vee c) \wedge Y = A_1 \vee C$ such that $\underline{\underline{L}}$ is (C_1, Y)-regular. If $\phi \in \Gamma(C_1, Y)$ and $\phi \neq 1$, then $A' = A\phi \neq A$. Now $|A' \vee C| \geq |c|$ since:

$\Gamma(C)_A$ acts transitively on the points $\neq C$ of c;

$\Gamma(C)_A$ fixes $A, A' \vee C$, and c; and

a_1 is a transversal line for $(A, A' \vee C, c)$.

So $|a| = |A' \vee C| \geq |c|$, contradicting $|a| < |c|$.

LEMMA 2. *Every plane through c is a desarguesian projective plane.*

Proof. Let α be a plane containing c. By Gleason [5], Lemma 1.7 again, $\Gamma(C)$ is transitive on the lines in α which meet c but do not contain C, for any point C on c. Thus all lines $\neq c$ in α which meet c have the same length, say $q + 1$.

If a_1 and a_2 are lines in α with $a_1 \wedge c = C_1 \neq C_2 = a_2 \wedge c$, and A_2 is a point $\neq C_2$ on a_2, then α (or rather the lattice interval $[0, \alpha]$) is (C_1, a_2)-regular and $\Gamma(C_1, a_2)$ acts semi-regularly on the non-fixed points of both a_1 and $C_1 \vee A_2$. Since $|a_1| = |C_1 \vee A_2|$, a_1 must contain a second fixed point, that is a_1 must meet a_2.

Consider the tactical configuration whose points are the points of α not on c, and whose blocks are the lines $\neq c$ of α which meet c. In the standard notation,

$$v = q^2 , \quad b = |c|q , \quad k = q , \quad r = |c| .$$

There are q blocks through each point of c.

ASSUMPTION. Suppose there exists a line d in α which does not meet c .

By Lemma 1, $|c| \leq q + 1$. Choose a point C on c , and let $\Gamma(C, d)$ denote the group of automorphisms of $[0, \alpha]^{*}$ induced by a group $\Gamma(C, X)$, where $\alpha \wedge X = d$ and $\underline{\underline{L}}$ is (C, X)-regular. Then $\Gamma(C, d)$ acts semi-regularly on the non-fixed points of any line in α through C , and so we have $|\Gamma(C, d)|$ divides both $q - 1$ and $|c| - 1$. This shows that $|c| \neq q + 1$ and so $|\Gamma(C, d)| = |c| - 1$ and $(|c|-1)|(q-1)$.

Since all the blocks through a point C on c have the same length, and some of them meet d , they must all meet d (consider the action of $\Gamma(C, d)$). So $|d| = q$. If D is a point on d then the blocks through D contain altogether $|c|(q-1) + 1$ points of the configuration; that is the number of lines in α through D which do not meet c is

$$[q^2-|c|(q-1)-1](q-1)^{-1} = q + 1 - |c| .$$

Now consider the tactical configuration consisting of the points of α not on c and the lines of α which do not meet c . We have $v = q^2$, $k = q$, $r = q + 1 - |c|$, and so $b = q(q+1-|c|)$. Adding the blocks of this configuration to those of the previous one, we form a 2-design whose points are the points of α not on c , whose blocks are the lines $\neq c$ of α , and whose parameters are $v = q^2$, $b = q(q+1)$, $k = q$, $r = q + 1$, $\lambda = 1$. This 2-design is an affine plane of order q . We can form its projective completion $\tilde{\alpha}$ by adding $q + 1 - |c|$ points to the line c to form a new line $\tilde{c}$.

Every automorphism of the lattice $[0, \alpha]$ extends to an automorphism (collineation) of $\tilde{\alpha}$. If C and C' are distinct points on c and a_i' , $(i = 1, \ldots, q)$, are lines in α with $a_i' \wedge c = C'$, then there exists a non-trivial (C, a_i')-homology of $[0, \alpha]$ which extends to a (C, a_i')-homology of $\tilde{\alpha}$, for each i . If $i \neq j$ then (by André [1]) there exists a $(C, C\vee(a_i'\wedge a_j'))$-elation in $\langle \Gamma(C, a_i'), \Gamma(C, a_j') \rangle$ which maps a_i' to a_j' . It follows that $|\Gamma(C, \tilde{c})| = q$. Since this is true for all (and therefore at least two) points C on c , $\tilde{\alpha}$ is a translation plane with respect to $\tilde{c}$. It follows that $q = p^r$ for some prime p and positive integer r .

If C_1, C_2, C_3 are distinct points of c, and a_1, a_2 are lines in α with $a_1 \wedge c = C_1$, $a_2 \wedge c = C_2$, and $a_1 \wedge a_2 = A$ (a point), then by André [1] (again) there exists a non-trivial $(C_3, C_3 \vee A)$-elation ϕ of $\tilde{\alpha}$ in $\langle \Gamma(C_3, a_1), \Gamma(C_3, a_2) \rangle$. This elation ϕ must have order p, since the translations of $\tilde{\alpha}$ have order p. But ϕ induces a $(C_3, C_3 \vee A)$-elation in $[0, \alpha]$. So $p | (|c|-1)$, which contradicts $(|c|-1) | (q-1)$. Our assumption was therefore false, and so α is a projective plane. From the elementary results on the Lenz-Barlotti classification of projective planes (see Dembowski [4], Chapter 3) it readily follows that α is desarguesian.

DEFINITION. We call the copoint X of $\underline{\underline{L}}$ a *special copoint* if $\underline{\underline{L}}$ *is* (C, X)*-regular for some point* C *on* c *but not in* X.

LEMMA 3. *Every special copoint meets every line, including* c, *which meets* c.

Proof. With C, X as in the definition, we know that all lines through C have the same length $q + 1$, and some of them meet X. So, by (C, X)-regularity, all lines through C meet X. It follows that, if C' is any point on c,

$$
\begin{aligned}
|X| &= \text{the number of lines through } C \\
&= (l-1)q^{-1}, \text{ where } l \text{ is the number of points in } \underline{\underline{L}} \\
&= \text{the number of lines through } C',
\end{aligned}
$$

and so every line through C' meets X.

LEMMA 4. $\underline{\underline{L}}$ *is* (C, X)*-regular for all* C, X *such that* C *is a point on* c *and* X *is a copoint not containing* C.

Proof. We first show that every copoint not containing c is special. Any such copoint contains a coline y which does not meet c. Choose a point C_1 on c; then $\underline{\underline{L}}$ is (C_1, Y)-regular for some special copoint Y containing y but not C_1. Let $C_2 = Y \wedge c$ and let Z be a special copoint determined by C_2, y. If $Z \wedge c \neq C_1$ then $\langle \Gamma(C_1, Y), \Gamma(C_2, Z) \rangle$ is transitive on the points $\neq C_2$ of c and so, in this case, every copoint which joins a point of c to y is special. If, for all possible selections of C_1 on c, we always get $Z \wedge c = C_1$, then again every copoint which joins a point of c to y is special.

A simple counting argument, based on Lemma 3, shows that the $q + 1$ special copoints which contain y and meet c are all the copoints containing y. Thus all copoints not containing c are special.

Now, with C, X as in the statement of the lemma, there exists a point C_1 on c such that $C_1 \nleq X$ and $\underline{\underline{L}}$ is (C_1, X)-regular. Let $C_2 = X \wedge c$ and choose a coline w meeting c in a point $\neq C, C_1, C_2$; C_2 and w determine a copoint W such that $W \wedge c = w \wedge c$ and $\underline{\underline{L}}$ is (C_2, W)-regular. Some element of $\Gamma(C_2, W)$ maps C_1 to C, while fixing X, so $\underline{\underline{L}}$ is (C, X)-regular.

COROLLARY. $\underline{\underline{G}}$ *induces on the points of* c *a group which is permutation isomorphic to the standard representation of* PGL(2, q).

LEMMA 5. *Every coline in* $\underline{\underline{L}}$ *meets every plane containing* c.

Proof. Let x be a coline not meeting c. Then it is easily seen that

$$|x| = (l-1)q^{-2} - q^{-1}, \text{ where } l \text{ is the number of points in } \underline{\underline{L}}.$$

No plane containing c meets x in more than one point, since if it did the line joining two intersection points would be contained in x and would meet c (as every plane through c is a projective plane). So the planes joining c to points of x account for $|x|q^2 + q + 1 = l$ points of $\underline{\underline{L}}$.

LEMMA 6. *Let* $A_1, \ldots, A_n$ *be* n *points of* $\underline{\underline{L}}$ *such that* $A_1 \vee \ldots \vee A_n = 1$, $A_1 < c$ *and* $A_2 < c$; *and let* $\alpha_i = c \vee A_i$ $(i = 3, \ldots, n)$. *Then* $\underline{\underline{G}}$ *is faithfully represented by its action on the set of all points of* $\underline{\underline{L}}$ *which lie in at least one* α_i, *and the induced permutation group* $\underline{\underline{G}}^*$ *is uniquely determined (to within permutation group isomorphism) by* $q + 1 = |c|$ *and* $n = \operatorname{rank} \underline{\underline{L}}$.

Proof. We note before beginning that $A_1, \ldots, A_n$ is a basis for the combinatorial geometry associated with $\underline{\underline{L}}$, and that a basis with $A_1 < c$

and $A_2 < c$ can certainly be chosen (see Crapo and Rota [3]).

Suppose that $\phi \in \underline{\underline{G}}$ fixes every point of every plane α_i . If x is any coline not meeting c , and $B_i = x \wedge \alpha_i$, then $c \vee B_3 \vee \ldots \vee B_n \geq \alpha_3 \vee \ldots \vee \alpha_n = 1$, and so $B_3, \ldots, B_n$ are independent; that is $B_3 \vee \ldots \vee B_n = x$. It follows that ϕ fixes every coline not meeting c and so, since ϕ fixes every point of c , ϕ fixes every copoint not containing c .

Let Z be a copoint containing c . Choose a coplane σ such that $\sigma \vee c = Z$ and $\sigma \wedge c = 0$, and distinct points A and B such that c $(A \vee B) \wedge c$ is a point but $(A \vee B) \nleq Z$. Then $\sigma \vee A$ and $\sigma \vee B$ are distinct colines which do not meet c . So ϕ fixes $(\sigma \vee A) \wedge (\sigma \vee B) = \sigma$, and therefore ϕ fixes $Z = \sigma \vee c$. We have now shown that ϕ fixes every copoint, and hence that $\phi = 1$.

To prove the assertion about $\underline{\underline{G}}^*$, we first coordinatize the desarguesian projective planes α_i so that the coordinates of the points of c "match". In each α_i the coordinate system is chosen so that c is the line at infinity and the same three points of c are labelled (0), (1), (∞) . By virtue of the Corollary to Lemma 4, the coordinates (m_i) of the remaining points of c (considered as points of α_i) can be described in terms of the group induced by $\underline{\underline{G}}$ on the points of c , uniquely to within a field automorphism of $\mathrm{GF}(q)$. After suitable choices for the field automorphisms, each point of c will have the same coordinate (m) in all $n - 2$ coordinate systems.

Now the permutation group $\underline{\underline{G}}^*$ is generated by those transformations $\phi = \phi(C, a_3, \ldots, a_n; C_1, C_2)$ such that ϕ acts in each α_i as does the (C, a_i)-homology mapping C_1 to C_2 ; where C, C_1, C_2 are points on c , a_i is a line $\neq c$ in α_i , $a_i \wedge c = C'$ does not depend on i , and C, C', C_1, C_2 are distinct. Thus $\underline{\underline{G}}^*$ depends only on q and n .

LEMMA 7. *The pointwise stabilizer* $\underline{\underline{H}}$ *of* c *in* $\underline{\underline{G}}$ *acts faithfully as a Frobenius group of order* $q^{2n-4}(q-1)$ *on the* q^{2n-4} *colines of* $\underline{\underline{L}}$ *which do not meet* c .

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Proof. It suffices (Gleason [5], Lemma 1.7) to show that, for any coline x which does not meet c, the group $\underline{\underline{H}}_x$ has order $q - 1$ and fixes no other colines not meeting c. By Lemmas 5 and 6 there is no loss of generality in supposing that the planes α_i are $n - 2$ planes in $PG(n-1, q)$ which all pass through a line c in $PG(n-1, q)$ and which span $PG(n-1, q)$, and that the points and lines of $\underline{\underline{L}}$ which lie in α_i are the points and lines of $PG(n-1, q)$ which lie in α_i. The colines of $\underline{\underline{L}}$ not meeting c may be thought of as $(n-2)$-tuples $(B_3, \ldots, B_n)$, each B_i being a point in α_i not on c; and the copoints of $\underline{\underline{L}}$ not containing c may be thought of as $(n-2)$-tuples $(b_3, \ldots, b_n)$, where each b_i is a line $\neq c$ in α_i and the b_i's all meet c in the same point.

The proof of Lemma 7 now reduces to a simple exercise on the geometry of $PG(n-1, q)$.

Proof of Theorem 2. We apply the preceding lemmas to complete the proof of our theorem. From Lemma 7 and the corollary of Lemma 4 it follows that

$$\begin{aligned} |\underline{\underline{G}}| &= |\underline{\underline{H}}| . |PGL(2, q)| \\ &= q^{2n-3}(q+1)(q-1)^2 . \end{aligned}$$

Our method of proof is to show that the points and copoints of $\underline{\underline{L}}$ and the incidence (order) relation between them, can be described in terms of the permutation group $\underline{\underline{G}}^*$ (of Lemma 6) and the $n - 2$ desarguesian projective planes α_i ("joined together" along c in the appropriate way).

The elements of $\underline{\underline{G}}^*$ which are induced by the homologies of $\underline{\underline{L}}$ which generate $\underline{\underline{G}}$ may also be thought of as being induced by homologies of $PG(n-1, q)$, when the planes α_i are embedded in $PG(n-1, q)$, as described in the proof of Lemma 7. Then $\underline{\underline{G}}^*$ is the group induced (faithfully) on $\alpha_3 \cup \ldots \cup \alpha_n$ by a certain subgroup $\underline{\underline{G}}^{**}$ of $PGL(n-1, q)$.

To avoid confusion, we shall denote α_i by α_i^* and c by c^* when considered as elements of $PG(n-1, q)$. Now the copoints of $\underline{\underline{L}}$ which do not contain c may be identified with $(n-2)$-tuples $(b_3, \ldots, b_n)$ of lines b_i in α_i^* which all meet c^* in the same point. With what shall we associate the copoints of $\underline{\underline{L}}$ which contain c ?

Consider the subset $\underline{\underline{S}}^*$ of $\underline{\underline{G}}^*$ consisting of those transformations ϕ in $\underline{\underline{G}}^*$ which induce an elation ϕ_i in each α_i^* and are such that the ϕ_i's all have a common centre on c^* and the common axis c^* . (Some of the ϕ_i's may be trivial.) For a given point C^* on c^* there are at most $q^{n-2} - 1$ non-trivial $\phi \in \underline{\underline{S}}^*$ with centre C^* , since the group $\Gamma(C^*, c^*)$ for any plane α_i^* has order q , and so

$$|\underline{\underline{S}}^*| \leq (q^{n-2}-1)(q+1) + 1 .$$

We shall later see that this inequality can be replaced by equality.

Let C_1 be a point on c , Z a copoint containing c ; choose a coline $z < Z$ meeting c in a point $\neq C_1$; and choose two copoints X, Y with $X \wedge Y = z$ and $Z \neq X, Y$. Then $\underline{\underline{H}}_1 = \langle \Gamma(C_1, X), \Gamma(C_1, Y) \rangle$ acts faithfully and sharply 2-transitively on the q copoints $\neq Z$ through z , and induces a group of order $q - 1$ on the $q - 1$ points $\neq C_1$, $z \wedge c$ of c . The Frobenius kernel of $\underline{\underline{H}}_1$ is $\Gamma(C_1, Z)$ and so $|\Gamma(C_1, Z)| = q$.

Now let C_2 be a point $\neq C_1$, $z \wedge c$ on c . The Frobenius kernel of $\underline{\underline{H}}_2 = \langle \Gamma(C_2, X), \Gamma(C_2, Y) \rangle$ is $\Gamma(C_2, Z)$. We assert that $\underline{\underline{Z}} = \langle \Gamma(C_1, Z), \Gamma(C_2, Z) \rangle$, which (by the geometry of the desarguesian planes containing c) is clearly elementary abelian of order q^2 , consists entirely of elations with axis Z and centre on c . Trivially, all elements of $\underline{\underline{Z}}$ have axis Z . If $\phi \in \underline{\underline{Z}}$ induces ϕ_i in α_i , then by considering the subgroup of $\underline{\underline{G}}^{**}$ corresponding to $\underline{\underline{Z}}$ we see that the ϕ_i's all have the same centre C on c . If b is a line $\neq c$ through C and $\alpha = b \vee c$, then $b = \alpha \wedge V$ for some copoint V , and V meets every α_i in a line through C . Since ϕ fixes all the lines $\alpha_i \wedge V$, it also fixes V and therefore fixes $b = \alpha \wedge V$. Thus ϕ has centre C.

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We have shown that if Z is a copoint of $\underline{\underline{L}}$ containing c then the elations with axis Z and centre on c form a subgroup $\underline{\underline{Z}}$ of $\underline{\underline{G}}$ having order q^2. Obviously $\underline{\underline{Z}}$ is a normal subgroup of $\underline{\underline{G}}$, and corresponds to a normal subgroup $\underline{\underline{Z}}^*$ of $\underline{\underline{G}}^*$ contained in $\underline{\underline{S}}^*$ and having order q^2.

The same argument applied to $PG(n-1, q)$ and $\underline{\underline{G}}^{**}$ shows that the $\left(q^{n-2}-1\right)(q-1)^{-1}$ copoints of $PG(n-1, q)$ containing c^* give rise to $\left(q^{n-2}-1\right)(q-1)^{-1}$ "elation subgroups" (normal in $\underline{\underline{G}}^*$) of order q^2 contained in $\underline{\underline{S}}^*$. Since any two of these have trivial intersection,

$$|\underline{\underline{S}}^*| = \left(q^{n-2}-1\right)(q+1) + 1 ,$$

as was asserted earlier.

That these $\left(q^{n-2}-1\right)(q-1)^{-1}$ subgroups include all of the subgroups $\underline{\underline{Z}}$ follows from the fact that $\underline{\underline{S}}^*$ contains no other normal subgroups of $\underline{\underline{G}}^*$ having order q^2, which is easily verified by showing that each non-trivial element of $\underline{\underline{S}}^*$ has at least $q^2 - 1$ conjugates in $\underline{\underline{G}}^*$. We have now answered the question: with what shall we associate the copoints of $\underline{\underline{L}}$ which contain c ?

In order to describe the points A of $\underline{\underline{L}}$ not on c, we consider the Frobenius kernel $\underline{\underline{K}}$ of the group $\underline{\underline{H}}$ (see Lemma 7). Now $|\underline{\underline{K}}| = q^{2n-4}$ and the orbit $A^{\underline{\underline{K}}}$ clearly consists of the q^2 points of $\alpha = c \vee A$ not on c, that is $|A^{\underline{\underline{K}}}| = q^2$ and so $|\underline{\underline{K}}_A| = q^{2n-6}$. It follows that $\underline{\underline{K}}_A$ acts transitively on the set of all colines of $\underline{\underline{L}}$ which contain A but do not meet c. So the point A may be associated with a coline orbit of the group $\underline{\underline{A}} = \underline{\underline{K}}_A$. Recall that colines (not meeting c) are easily described in terms of $\alpha_3, \ldots, \alpha_n$. The subgroup $\underline{\underline{A}}^*$ of $\underline{\underline{G}}^*$ corresponding to $\underline{\underline{A}}$ is the group generated by the elation subgroups corresponding to the various copoints containing $\alpha = c \vee A$. Any subgroup of $\underline{\underline{G}}^*$ having order q^{2n-6} which is generated by elation subgroups corresponds to a plane in $PG(n-1, q)$ containing c^*. The orbits of colines (not meeting c^*) under this subgroup correspond to the points (not on c^*) of that plane. When the subgroup can be generated by elation subgroups which correspond to

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copoints of $\underline{\underline{L}}$, then the plane in $PG(n-1, q)$ also corresponds to a plane in $\underline{\underline{L}}$.

The description of the incidence relation between points and copoints of $\underline{\underline{L}}$ in terms of $\underline{\underline{G}}^*$ and $\alpha_3, \ldots, \alpha_n$ is now obvious. For example, a point A not on c lies in a copoint Z containing c if and only if the elation subgroup $\underline{\underline{Z}}^*$ is contained in $\underline{\underline{A}}^*$. The point A lies in a copoint $Y = \left(b_3, \ldots, b_n\right)$ not containing c if and only if one of the colines $\left(B_3, \ldots, B_n\right)$ in the orbit corresponding to A is contained in Y ; that is $B_i < b_i$, $i = 3, \ldots, n$.

We have completed the proof of Theorem 2.

5. A characterization of finite projective spaces

Let $\underline{\underline{L}}$ be a finite geometric lattice of rank $n \geq 4$ satisfying the hypotheses *(i)* and *(ii)* of Theorem 2. Then by Lemma 5 every coline of $\underline{\underline{L}}$ meets every plane containing c . So, if x is a coline not meeting c , then

$$|x| = \text{the number of planes through } c \text{ .}$$

Now, in $PG(n-1, q)$, the number of planes through a line is $\left(q^{n-2}-1\right)(q-1)^{-1}$. Consequently, we deduce from Theorem 2 the following characterization of finite projective spaces $PG(n-1, q)$ with $n-1 \geq 3$ and $q \geq 3$.

THEOREM 3. *Let* $\underline{\underline{L}}$ *be a finite geometric lattice of rank* $n \geq 4$. *Then* $\underline{\underline{L}}$ *is isomorphic to the lattice of a projective space* $PG(n-1, q)$ *with* $q \geq 3$ *if and only if*

(i) *there exists a line* c *in* $\underline{\underline{L}}$ *such that, whenever* C *is a point on* c *and* x *is a coline with* $C \nleq x$, $\underline{\underline{L}}$ *is* (C, X)*-regular for some copoint* X *containing* x *but not* C ;

(ii) *every line of* $\underline{\underline{L}}$ *contains at least four points; and*

(iii) *there exists a coline* y *in* $\underline{\underline{L}}$ *such that* y *does not meet* c *and* $|y| \geq \left(q^{n-2}-1\right)(q-1)^{-1}$, *where* $q + 1 = |c|$.

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