Can. J. Math., Vol. XXVIII, No. 4, 1976, pp. 673–690

SPACES IN WHICH SPECIAL SETS ARE z-EMBEDDED

ROBERT L. BLAIR

1. Introduction. A subset S of a topological space X is *z-embedded* in X in case each zero-set of S is the restriction to S of a zero-set of X. (A *zero-set* is the set of zeros of a real-valued continuous function.) For the basic theory of z-embedding, see [**3**] and [**4**] (and see [**4**] for a comprehensive bibliography of relevant papers). Here we continue the development of the theory of z-embedding, with the focus on the following three classes of spaces:

(i) *Normal spaces* (§ 2). We show that X is normal if and only if every closed subset of X is z-embedded (2.1), and that every F_σ, or, more generally, every normally placed set (in the sense of Smirnov [**23**]) in a normal space is z-embedded (2.3 and 2.5).

It is difficult to improve on the F_σ theorem just cited. This is shown in § 3, where a number of examples are given which show that a union of z- (or even C^*- or C-) embedded sets is only rarely z-embedded.

(ii) *Weakly perfectly normal spaces* (§ 4). By definition, these are the spaces X with the property that *every* subset of X is z-embedded in X. The class of weakly perfectly normal spaces is strictly included in that of completely normal spaces, and includes (strictly, if one admits measurable cardinals) the class of perfectly normal spaces (see 4.1 and 4.8). But it is an open question whether there is a weakly perfectly normal T_1-space of nonmeasurable power that is not perfectly normal. The main result here is that there is such a space if there is a (weakly) perfectly normal T_1-space of nonmeasurable power that is not realcompact (4.4). As a consequence of this and [**25**], the existence of a weakly perfectly normal compact Hausdorff space of power $\aleph_1$ that is not perfectly normal is consistent with the usual axioms of set theory (4.10).

(iii) *The class* **Oz** *of spaces in which every open set is z-embedded* (§ 5). This is a wide class of spaces which includes all (weakly) perfectly normal spaces, all extremally disconnected spaces, and (see 5.6) all products of separable metric spaces. Among other things, we show that: (a) $X \in$ **Oz** if and only if every dense subset of X is z-embedded if and only if every regular closed subset of X is a zero-set (5.1), (b) if every finite subproduct of $X = \prod_{\alpha \in I} X_\alpha$ satisfies the countable chain condition and every countable subproduct belongs to **Oz**, then $X \in$ **Oz** (5.5), (c) a pseudocompact product $X \times Y$ belongs to **Oz** only if $X, Y \in$ **Oz** (5.7), and (d) if $X \in$ **Oz** is Tychonoff and of nonmeasurable

Received May 13, 1974 and in revised form, March 30, 1976.

power, then every nonisolated point of X is contained in a nowhere dense zero-set (5.11). This last is a generalization of a theorem of Isbell, that an extremally disconnected P-space of nonmeasurable power is discrete (see 5.12).

Unless otherwise specified, X will denote an arbitrary topological space; specific separation properties will always be noted when they are needed. We follow the terminological conventions of [**16**] with respect to separation properties, but otherwise adopt the notation and terminology of [**11**] (with which familiarity is assumed). In particular, $C(X)$ (respectively, $C^*(X)$) denotes the set of all real-valued (respectively, bounded real-valued) continuous functions on X; and if $f \in C(X)$, then $Z(f)$ denotes the zero-set of f. The set of all zero-sets of X is denoted by $\mathscr{Z}(X)$. We note that $\mathscr{Z}(X)$ is closed under countable intersection [**11**, 1.14(a)].

In 1.1 we quote some results concerning z-embedding that are used several times in the sequel. Recall first that if $S \subset X$, then the *G_δ-closure* of S in X consists of all points $x \in X$ for which each G_δ-set about x meets S. (For Tychonoff X, this is precisely all $x \in X$ for which each zero-set about x meets S.) S is *G_δ-dense* (respectively, *G_δ-closed*) in case the G_δ-closure of S is X (respectively, S).

1.1. Proposition. (a) *If S is z-embedded in X and completely separated from every disjoint zero-set, then S is C-embedded in X.*

(b) *If S is z-embedded and G_δ-dense in X, then S is C-embedded in X.*

(c) *If S is z-embedded in the Tychonoff space X, then the G_δ-closure of S in υX is υS.*

For 1.1(a), see [**4**, 3.6B] or [**3**, 4.1B]. 1.1(b) (proved in [**5**, 1.1(a)]) follows immediately from 1.1(a); see also [**4**, 4.4]. For 1.1(c), see [**5**, 1.1(b)] or [**4**, 2.6(a)].

2. z-embedding of closed sets.

In this section we consider the requirement that every closed subset of X be z-embedded in X. This is easily seen to be equivalent to normality:

2.1. Theorem. *For any space X, these are equivalent:*

(a) *X is normal.*

(b) *Every closed subset of X is z-embedded in X.*

(c) *$F \cup Z$ is z-embedded in X whenever F is closed in X and Z is a zero-set in X disjoint from F.*

Proof. (a) $\Rightarrow$ (b) is a consequence of Urysohn's Extension Theorem, and (b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a): Let F be closed in X. By (c), F is z-embedded. Suppose that $Z \in \mathscr{Z}(X)$ with $F \cap Z = \emptyset$. Then $F \in \mathscr{Z}(F \cup Z)$, so by (c) there is $Z' \in \mathscr{Z}(X)$ such that $F = (F \cup Z) \cap Z'$. Thus $F \subset Z'$ and $Z \cap Z' = \emptyset$, so F and Z are completely separated. It follows from 1.1(a) that F is C-embedded, and we conclude that X is normal.

https://doi.org/10.4153/CJM-1976-068-9 Published online by Cambridge University Press

An easy consequence of 2.1 is the fact that every completely regular Lindelöf space X is normal [**11**, 3D.4] (since every closed subspace of X is Lindelöf, and a Lindelöf space is z-embedded in every superspace [**13**, 5.3]). (Of course it is known, more generally, that every regular Lindelöf space is normal [**16**, p. 113].)

2.2 PROPOSITION. *Let $S \subset X$. If every neighborhood of S contains a normal (respectively, normal and z-embedded) subspace that contains S, then S is normal (respectively, normal and z-embedded).*

Proof. Let F be closed in X and let $A \in \mathscr{Z}(F)$. Since every zero-set is a G_δ, we can write $A = \bigcap_{n\in\mathbf{N}} G_n$, where each G_n is open in F. Now $X - (\mathrm{cl}_X A \cap \mathrm{cl}_X(F - G_n))$ is a neighborhood of S, so for each n there is, by hypothesis, a normal subspace T_n of X such that

$$S \subset T_n \subset X - (\mathrm{cl}_X A \cap \mathrm{cl}_X(F - G_n)).$$

Since $\mathrm{cl}_{T_n} A$ and $\mathrm{cl}_{T_n}(F - G_n)$ are disjoint closed sets in the normal space T_n, there is $Z_n \in \mathscr{Z}(T_n)$ such that $A \subset Z_n$ and $Z_n \cap (F - G_n) = \emptyset$. If we set $Z = \bigcap_{n\in\mathbf{N}}(S \cap Z_n)$, then $Z \in \mathscr{Z}(S)$ and $A = F \cap Z$. Thus F is z-embedded in S, so S is normal by 2.1. To prove the parenthetical assertion of 2.2, we may assume that each T_n is z-embedded in X. Then (with the notation as before) $Z_n = T_n \cap Z_n'$ for some $Z_n' \in \mathscr{Z}(X)$. Set $Z' = \bigcap_{n\in\mathbf{N}} Z_n'$, take $F = S$, and note that $A = S \cap Z$. Thus S is z-embedded in X.

The nonparenthetical assertion of 2.2 is due to Smirnov [**27**].

An F_σ-set in a normal space need not be C^*-embedded (e.g., consider the open interval $(0, 1)$ in R). But, in normal spaces, F_σ's retain at least z-embeddability:

2.3. THEOREM. *Every F_σ-set in a normal space is z-embedded.*

Proof. If $(F_n)_{n\in\mathbf{N}}$ is a sequence of closed sets in the normal space X, then we must show that $S = \bigcup_{n\in\mathbf{N}} F_n$ is z-embedded. Consider any $f \in C(S)$. For each $n \in \mathbf{N}$, define a function f_n on $F_n \cup \mathrm{cl}_X Z(f)$ as follows:

$$f_n(x) = \begin{cases} f(x) & \text{if } x \in F_n, \\ 0 & \text{if } x \in \mathrm{cl}_X Z(f) - F_n. \end{cases}$$

Note that if $x \in F_n \cap \mathrm{cl}_X Z(f)$, then $x \in Z(f)$, and hence $f_n(x) = f(x) = 0$. Thus $f_n = 0$ on $\mathrm{cl}_X Z(f)$ and $f_n = f$ on F_n, so f_n is continuous. By normality of X, each f_n has an extension $g_n \in C(X)$. Let $Z = \bigcap_{n\in\mathbf{N}} Z(g_n)$ and note that $Z \in \mathscr{Z}(X)$. One easily verifies that $Z(f) = S \cap Z$, and thus S is z-embedded.

2.4. *Remarks.* (a) The countability hypothesis implicit in 2.3 is essential; see 4.1 below.

(b) We sketch an alternative proof of 2.3 (due to H. E. White, Jr.): One shows first (by induction) that any two separated F_σ's in a normal space are contained in disjoint cozero-sets. (A and B are *separated* in case $A \cap \mathrm{cl}\, B = B \cap \mathrm{cl}\, A = \emptyset$. A *cozero-set* is the complement of a zero-set.) Then, for S an

https://doi.org/10.4153/CJM-1976-068-9 Published online by Cambridge University Press

F_σ in X and $f \in C(S)$, set $Z_n = \{x \in S\colon |f(x)| \geqq 1/n\}$. $Z(f)$ and Z_n are separated F_σ's in X, so for each n there is $Z_n' \in \mathscr{Z}(X)$ with $Z(f) \subset Z_n'$ and $Z_n \cap Z_n' = \emptyset$. Then $\bigcap_{n \in \mathbf{N}} Z_n' \in \mathscr{Z}(X)$ and $Z(f) = S \cap (\bigcap_{n \in \mathbf{N}} Z_n')$.

(c) 2.3 may be viewed as a theorem concerning z-embeddability of certain unions of z-embedded sets. On general principles, one might expect such a result to have some generalization to Tychonoff spaces. However, there seems, in fact, to be no immediate generalization: the examples of § 3 show rather conclusively that a union of z- (or even C^*- or C-) embedded sets is very likely not z-embedded. (Of course, certain F_σ's are always z-embedded: A cozero-set in any space X is z-embedded in X [**5**, 1.1(c)], and any σ-compact S is Lindelöf and hence z-embedded in any superspace [**13**, 5.3].)

A subset S of X is *normally placed* in X in case for every neighborhood V of S there is an F_σ-set H in X such that $S \subset H \subset V$ (see [**27**]). It is known that every F_σ in a normal space is normal (see, e.g., [**7**, Chap. IX, § 4, Ex. 6]) and that, more generally, every normally placed subset of a normal space is normal ([**27**, Theorem 1]; see also [**9**, Problem 2G]). We sharpen the latter (and use 2.1, 2.2, and 2.3 to simplify its proof) as follows:

2.5. THEOREM. *If S is normally placed in the normal space X, then S is normal and z-embedded in X.*

Proof. By 2.2 and 2.3, it suffices to show that any F_σ-set H in X is normal. But if F is closed in H, then F is an F_σ in X, and hence z-embedded in X by 2.3. It follows that F is z-embedded in H, so H is normal by 2.1.

For completeness, we give an example of a normally placed set (in fact, a generalized cozero-set) that is not an F_σ. (S is a *generalized cozero-set* in case for every neighborhood V of S there is a cozero-set P such that $S \subset P \subset V$.)

2.6. *Example.* Let X be any regular Lindelöf T_1-space that is not σ-compact (e.g., the Sorgenfrey line). Then X is a generalized cozero-set, but not an F_σ, in βX.

Proof. If V is a neighborhood of X in βX, then X can be covered by a countable family $(P_n)_{n \in \mathbf{N}}$ of cozero-sets in βX such that $P_n \subset V$ for each n. Then $\bigcup_{n \in \mathbf{N}} P_n$ is a cozero-set in βX which lies between X and V; and since X is not σ-compact, X is not an F_σ in βX.

3. Unions of z-embedded sets. Most of this section is devoted to the examples alluded to in 2.4(c). For completeness, we phrase some of these examples in terms of v-embedding as well as z-embedding. (A subset S of the Tychonoff space X is *v-embedded* in X in case $vS \subset vX$ [**2**]. Every realcompact subspace of X is (trivially) v-embedded, and every z-embedded subset is v-embedded [**2**, 3.5].)

Recall that X is *completely normal* in case every subspace of X is normal.

3.1. *Example.* Let $X = D \cup \{\infty\}$ be the one-point compactification of an uncountable discrete space D. Then X is completely normal and D is not z-embedded in X. (Hence a union of uncountably many closed subsets of a normal space need not be z-embedded.)

Proof. Complete normality is obvious. Let N be any countably infinite subset of D and define $f \in C(D)$ as follows: $f(x) = 0$ (respectively, 1) if $x \in N$ (respectively, $x \in D - N$). Suppose there is $g \in C(X)$ such that $Z(f) = D \cap Z(g)$. Then clearly $g(\infty) = 0$, so for each $n > 0$ there is a finite subset F_n of D such that $|g(x)| < 1/n$ for all $x \in D - F_n$. Choose any $x \in D - (N \cup (\bigcup_{n \in \mathbf{N}} F_n))$. Then $f(x) = 1$ and $g(x) = 0$, a contradiction, and we conclude that D is not z-embedded in X.

3.2. *Example.* Let I be an uncountable set, let $N_\alpha = \mathbf{N}$ for each $\alpha \in I$, and let $X = \prod_{\alpha \in I} N_\alpha$. Then there are two closed disjoint C-embedded subsets F_0 and F_1 of X such that $F_0 \cup F_1$ is not z-embedded in X.

Proof. We adapt techniques of Stone [**28**], Corson [**8**], and Ross and Stone [**26**]. For $i = 0, 1$, set

$$\textstyle\sum_i = \{x \in X\colon \mathrm{pr}_\alpha(x) = i \text{ for all but countably many } \alpha \in I\}$$

and

$$F_i = \{x \in X\colon \text{for every } n \neq i,\ \mathrm{pr}_\alpha(x) = n \text{ for at most one } \alpha \in I\}.$$

(Here pr_α denotes the projection of X of index α.) Clearly F_0 and F_1 are closed in X. Since $\sum_i$ is a $\sum$-product of the family $(N_\alpha)_{\alpha \in I}$ (see [**8**]), it follows from [**8**, Theorem 1] that $\sum_i$ is normal. Moreover, by [**8**, Theorem 2], $v\sum_i = X$. Since $F_i \subset \sum_i$, we conclude that F_i is C-embedded in X.

Suppose now that $S = F_0 \cup F_1$ is z-embedded in X. Since I is uncountable, $F_0 \cap F_1 = \emptyset$, and hence $F_0 \in \mathscr{Z}(S)$. By hypothesis, therefore, there is a function $f \in C(X)$ such that $F_0 \subset Z(f)$ and $Z(f) \cap F_1 = \emptyset$. By [**26**, Theorem 4], there exists a countable subset J of I and a function $g \in C(\prod_{\alpha \in J} N_\alpha)$ such that $f = g \circ \mathrm{pr}_J$ (where pr_J is the projection of X onto $\prod_{\alpha \in J} N_\alpha$). It follows that $\mathrm{pr}_J(F_0) \subset Z(g)$ and $Z(g) \cap \mathrm{pr}_J(F_1) = \emptyset$. Let $\sigma\colon J \to \mathbf{N}$ be an arbitrary injection of J into $\mathbf{N}$, and, for $i = 0, 1$, define $x_i \in X$ by the requirements

$$\mathrm{pr}_\alpha(x_i) = \begin{cases} \sigma(\alpha) & \text{if } \alpha \in J, \\ i & \text{if } \alpha \notin J. \end{cases}$$

Then $x_i \in F_i$ and $\mathrm{pr}_J(x_0) = \mathrm{pr}_J(x_1)$, so we have $\mathrm{pr}_J(F_0) \cap \mathrm{pr}_J(F_1) \neq \emptyset$, a contradiction. Thus S is not z-embedded in X.

3.3. *Remarks.* (a) It follows from 3.2 and 2.1 that the space $X = \prod_{\alpha \in I} N_\alpha$ of 3.2 is not normal; this is due to A. H. Stone [**28**].

(b) Since realcompactness is inherited by products and closed subspaces [**11**, 8.10 and 8.11], the union $F_0 \cup F_1$ of 3.2 is realcompact, and hence v-

embedded in X. In the next two examples (which we quote from [**2**]), the unions in question are not even v-embedded.

If α is an ordinal, then $W(\alpha)$ denotes the set of all ordinals $< \alpha$, topologized with the order topology. As in [**11**], we set $\mathbf{N}^* = W(\omega + 1)$, $\mathbf{W} = W(\omega_1)$ and $\mathbf{W}^* = W(\omega_1 + 1)$.

3.4. *Example* [**2**, 6.2]. Let X be the one-point compactification of the topological sum of two copies S_1 and S_2 of $\mathbf{W}$. Then S_1 and S_2 are disjoint, open, and C-embedded in X, but $S_1 \cup S_2$ is not v-embedded (hence not v-embedded) in X.

3.5. *Example* [**2**, 6.1]. Let X be the Tychonoff plank: $X = (\mathbf{W}^* \times \mathbf{N}^*) - \{(\omega_1, \omega)\}$. Let $W = \mathbf{W} \times \{\omega\}$ and, for each $n \in \mathbf{N}$, let

$$F_n = W \cup \{(\omega_1, m): m \leqq n\}.$$

Then $(F_n)_{n \in \mathbf{N}}$ is an increasing sequence of closed C-embedded subsets of X, but $\bigcup_{n \in \mathbf{N}} F_n$ is not v-embedded (hence not z-embedded) in X. Moreover, each F_n is C-embedded in the compact space βX, but $\bigcup_{n \in \mathbf{N}} F_n$ is not v-embedded in βX.

For some positive results concerning v-embeddability of certain unions of v-embedded sets, see 6.5, 8.3, and 8.4 of [**2**].

In contrast to 3.5, the next example provides a union of an increasing sequence of z-embedded sets that is v-embedded, but not z-embedded. We are indebted to A. W. Hager for 3.6 and 3.7. (For 3.6 and related results, see [**6**].)

3.6. LEMMA (Hager). *If A is a countable (Tychonoff) space with no countable base, and if B is a discrete space with cardinality $2^{\aleph_0}$, then $A \times B$ is not z-embedded in $\beta A \times \beta B$.*

3.7. *Example* (Hager). There is a compact Hausdorff space X and an increasing sequence $(F_n)_{n \in \mathbf{N}}$ of closed C^*-embedded subsets of X such that $\bigcup_{n \in \mathbf{N}} F_n$ is v-embedded, but not z-embedded, in X.

Proof. Choose $A = \{a_1, a_2, \ldots\}$ and B as in 3.6. (For an example of such an A, see [**11**, 4M].) Let $X = \beta A \times \beta B$ and, for each n, let $F_n = \{a_1, \ldots, a_n\} \times B$. Each F_n is closed and C^*-embedded in X (for $f \in C^*(F_n)$, first extend f over $\{a_1, \ldots, a_n\} \times \beta B$, then over X). Now $\bigcup_{n \in \mathbf{N}} F_n = A \times B$ is realcompact (as the product of two realcompact spaces) and hence v-embedded in X. But $\bigcup_{n \in \mathbf{N}} F_n$ is not z-embedded by 3.6.

3.8. *Example* [**4**, 2.5(a)]. Let S be the x-axis of the tangent circle space Γ [**11**, 3K]. Then S is not z-embedded in Γ. (Hence the union of a discrete family of singletons need not be z-embedded.)

We note that the space Γ in 3.8 is (hereditarily) realcompact [**11**, 8.18]. A somewhat more interesting (pseudocompact) example involving the union of a discrete family will be given in 3.11. For convenience, we first state a simple

https://doi.org/10.4153/CJM-1976-068-9 Published online by Cambridge University Press

additivity result for realcompactness (somewhat stronger than is actually needed for 3.11). We remark that 3.9 is merely a fragment of the general theory of additivity of the Hewitt realcompactification, which will be treated elsewhere.

3.9. LEMMA. *If the Tychonoff space X is the union of a countable family of z-embedded realcompact subspaces, then X is realcompact.*

Proof. Write $X = \bigcup_{n \in \mathbf{N}} S_n$, where each S_n is realcompact and z-embedded in X, and let $\mathscr{F}$ be a real z-ultrafilter on X. Since $\mathscr{F}$ has the countable intersection property, there exists $n \in \mathbf{N}$ such that S_n meets every member of $\mathscr{F}$. Moreover, since S_n is z-embedded, the trace $\mathscr{F}|S_n$ of $\mathscr{F}$ on S_n is a real z-ultrafilter on S_n [**3**, 3.1]. By realcompactness of S_n, we have $\mathscr{F}|S_n \to x$ for some $x \in S_n$, and then clearly $\mathscr{F} \to x$. Thus X is realcompact.

3.10. *Remarks.* The preceding lemma has also been noted, independently, and with a different proof, by A. W. Hager (unpublished). 3.9 generalizes an early additivity theorem of Mrówka [**20**, Theorem 1] (the special case of 3.9 for which X is normal and the countable family consists of closed realcompact subspaces). We note in passing that 3.9 fails if the countability hypothesis is omitted (consider any nonrealcompact space regarded as the union of its points) or if the requirement that the subspaces by z-embedded is omitted (Mrówka has given an example of a nonrealcompact space that is the union of two closed realcompact subspaces (see [**21**] and [**22**])).

3.11. *Example.* There is a (nonnormal, nonrealcompact, pseudocompact) space X and a discrete family $(S_\alpha)_{\alpha \in I}$ of compact subsets of X (with $|I| = 2^{\aleph_0}$) such that $\bigcup_{\alpha \in I} S_\alpha$ is not z-embedded in X.

Proof. As is well known, $\beta\mathbf{N} - \mathbf{N}$ has exactly $2^{\aleph_0}$ open-and-closed subsets, and in fact there exist $2^{\aleph_0}$ mutually disjoint such sets (see [**11**, 6QS]). Consequently, there exists, by Zorn's lemma, a maximal pairwise disjoint family $(S_\alpha)_{\alpha \in I}$ of open-and-closed subsets of $\beta\mathbf{N} - \mathbf{N}$ such that $|I| = 2^{\aleph_0}$. Let $S = \bigcup_{\alpha \in I} S_\alpha$ and let $X = \mathbf{N} \cup S$. It is clear that each S_α is compact and that the family $(S_\alpha)_{\alpha \in I}$ is discrete in X. Moreover, since S is the topological sum of the compact spaces S_α, S is realcompact [**11**, 12G].

Now consider any nonempty zero-set $Z \in \mathscr{Z}(\beta\mathbf{N})$. If $Z \cap \mathbf{N} = \emptyset$, then the interior of Z in $\beta\mathbf{N} - \mathbf{N}$ is nonempty [**11**, 6S.8], and hence Z contains a nonempty open-and-closed subset of $\beta\mathbf{N} - \mathbf{N}$ (since $\beta\mathbf{N}$ is extremally disconnected). By the maximality of $(S_\alpha)_{\alpha \in I}$, Z must therefore meet S. Thus every nonempty zero-set in $\beta\mathbf{N}$ meets X, and hence (since $\beta X = \beta\mathbf{N}$) X is pseudocompact. On the other hand, S is noncompact and closed in X, so X is noncompact and hence nonrealcompact. But $\mathbf{N}$ is C^*-embedded, and hence z-embedded, in X, so it follows from 3.9 that S is not z-embedded in X. Finally, since X contains a closed set which is not z-embedded, X is nonnormal by 2.1.

https://doi.org/10.4153/CJM-1976-068-9 Published online by Cambridge University Press

By now it is clear that rather stringent conditions must be imposed in order that the union of a family of z-embedded sets be z-embedded. The only general sufficient condition of which we are aware is as follows:

3.12. THEOREM. *Let $(S_\alpha)_{\alpha \in I}$ be a family of subsets of X and assume there is a locally finite pairwise disjoint family $(P_\alpha)_{\alpha \in I}$ of cozero-sets of X such that $S_\alpha \subset P_\alpha$ for every $\alpha \in I$. If each S_α is z-embedded in X (or even just in P_α), then $\bigcup_{\alpha \in I} S_\alpha$ is z-embedded in X.*

Proof. Let $S = \bigcup_{\alpha \in I} S_\alpha$ and let $A \in \mathscr{Z}(S)$. For each α there is, by hypothesis, $Z_\alpha \in \mathscr{Z}(X)$ such that $A \cap S_\alpha = S_\alpha \cap Z_\alpha$. (Since the cozero-set P_α is z-embedded in X, this is so even if S_α is just z-embedded in P_α.) Set

$$Q_\alpha = X - ((X - P_\alpha) \cup Z_\alpha)$$

and let $Q = \bigcup_{\alpha \in I} Q_\alpha$. Clearly $Q_\alpha = \operatorname{coz} f_\alpha$ for some $f_\alpha \in C(X)$ with $f_\alpha \geqq 0$. Moreover, the family $(Q_\alpha)_{\alpha \in I}$ is locally finite in X, so $f = \sum_{\alpha \in I} f_\alpha$ is a (well-defined) continuous function on X. Clearly $Z(f) = X - Q$. It is now easy to verify that $A = S \cap Z(f)$, and thus S is z-embedded.

3.13. COROLLARY. *Assume that X is the topological sum of a family $(X_\alpha)_{\alpha \in I}$ of subspaces of X, and let $S_\alpha \subset X_\alpha$. If each S_α is z-embedded in X_α, then $\bigcup_{\alpha \in I} S_\alpha$ is z-embedded in X.*

3.14. COROLLARY. *If $(S_i)_{1 \leqq i \leqq n}$ is a finite sequence of pairwise completely separated z-embedded subsets of X, then $\bigcup_{i=1}^n S_i$ is z-embedded in X.*

4. z-embedding of every subset. Recall that X is *perfectly normal* in case X is normal and every closed subset of X is a G_δ (equivalently: every closed subset is a zero-set). We shall say that X is *weakly perfectly normal* in case every subset of X is z-embedded in X. Our terminology is motivated by the following:

4.1. PROPOSITION. *Every perfectly normal space is weakly perfectly normal, and every weakly perfectly normal space is completely normal.*

Proof. The first assertion is obvious, and the second is an immediate consequence of 2.1. (Thus 2.1 makes completely transparent the (well-known) fact that perfectly normal spaces are completely normal.)

Example 3.1 shows that a completely normal space need not be weakly perfectly normal; and (see 4.8) if there is a measurable cardinal, then there is a weakly perfectly normal space (of measurable power) that is not perfectly normal. But, barring measurable cardinals, the precise relationship between perfect normality and weak perfect normality is open. In this connection, consider the following problems (all open):

4.2. *Problems.* (a) Does there exist a weakly perfectly normal (T_1-) space of nonmeasurable power that is not perfectly normal?

(b) Does there exist a weakly perfectly normal T_1-space of nonmeasurable power that is not realcompact?

(c) Does there exist a perfectly normal T_1-space of nonmeasurable power that is not realcompact?

The apparently difficult 4.2(c) was first posed by the author in 1962 (unpublished), and later by R. M. Stephensen, Jr. (see [**12**, p. 140]). An affirmative answer to 4.2(c) obviously implies one to 4.2(b). What we show here is that an affirmative answer to 4.2(b) implies one to 4.2(a) (see 4.4), and that, in fact, 4.2(b) is equivalent to the modified form of 4.2(a) obtained by replacing "perfectly normal" by the requirement that every point be a G_δ (see 4.3). As a consequence of 4.4 and [**25**] (which came to the author's attention after the original version of this paper had been accepted for publication), we note also that affirmative answers to 4.2(a), (b), and (c) are consistent with the usual axioms of set theory (see 4.9(a) and 4.10).

4.3. THEOREM. *These are equivalent:*

(a) *There is a weakly perfectly normal T_1-space (of nonmeasurable power) with a non-G_δ point.*

(b) *There is a weakly perfectly normal T_1-space (of nonmeasurable power) that is not realcompact.*

4.4. COROLLARY. *If there is a weakly perfectly normal T_1-space of nonmeasurable power that is not realcompact, then there is a weakly perfectly normal T_1-space of nonmeasurable power that is not perfectly normal.*

We first prove a couple of lemmas:

4.5. LEMMA. *Let S be a z-embedded subset of X and let $A \subset \operatorname{cl} S$. If $A \in \mathscr{Z}(S \cup A)$, then there is $Z \in \mathscr{Z}(X)$ such that $A = (S \cup A) \cap Z$.*

Proof. By hypothesis, there is a function $f \in C(S \cup A)$ such that $A = Z(f)$. For each $n > 0$, set

$$A_n = \{x \in S : |f(x)| \leqq 1/n\}.$$

Then $A_n \in \mathscr{Z}(S)$, so there is $Z_n \in \mathscr{Z}(X)$ such that $A_n = S \cap Z_n$. Let $Z = \bigcap_n Z_n$ and note that $Z \in \mathscr{Z}(X)$.

Now suppose that $x \in A$. Consider any $n > 0$ and let V be an arbitrary neighborhood of x in X. Since $f(x) = 0$, there is a neighborhood W of x in X such that $|f(y)| < 1/n$ for every $y \in S \cap W$. By hypothesis, $x \in \operatorname{cl} S$, and hence $V \cap W$ meets S. It follows that V meets A_n, and we conclude that $x \in \operatorname{cl}_X A_n \subset Z_n$. Thus $x \in Z$, so we have $A \subset (S \cup A) \cap Z$. On the other hand,

$$(S \cup A) \cap Z = (\bigcap_n A_n) \cup (A \cap Z) \subset A,$$

and the proof is complete.

https://doi.org/10.4153/CJM-1976-068-9 Published online by Cambridge University Press

4.6. LEMMA. *Let Y be completely regular and assume that $Y = X \cup \{p\}$ with X z-embedded in Y. If $S \subset Y$ and if $S \cap X$ is z-embedded in X, then S is z-embedded in Y.*

Proof. If $S \subset X$, then S is z-embedded in Y by transitivity of z-embedding. We may therefore assume that $p \in S$. Let $A \in \mathscr{Z}(S)$ and set $T = S \cap X$. By hypothesis, T is z-embedded in X, and hence also in Y.

Case 1. $p \in A$. Suppose first that $p \in \mathrm{cl}_Y T$ so that $A \subset T \cup \{p\} \subset \mathrm{cl}_Y T$. Since $p \in A$, we have $S = T \cup A$, and thus $A \in \mathscr{Z}(T \cup A)$. It follows from 4.5 that there is $Z \in \mathscr{Z}(Y)$ such that $A = (T \cup A) \cap Z = S \cap Z$.

Suppose, on the other hand, that $p \notin \mathrm{cl}_Y T$. Then there is $f \in C(Y)$ such that $f(p) = 0$ and $f = \mathbf{1}$ on $\mathrm{cl}_Y T$. Moreover, $A \cap T \in \mathscr{Z}(T)$, so there is $g \in C(Y)$ such that $A \cap T = T \cap Z(g)$. Then (as one easily verifies)

$$A = S \cap Z(|f| \wedge |g|).$$

Case 2. $p \notin A$. In this case, $A \subset T$ so $A \in \mathscr{Z}(T)$. Then there is $f \in C(Y)$ such that $A = T \cap Z(f)$. Moreover, since $p \notin A$ and $p \in S$, we have $p \notin \mathrm{cl}_Y A$. Hence there is $g \in C(Y)$ such that $g(p) = 1$ and $g = \mathbf{0}$ on $\mathrm{cl}_Y A$. In this case, one can verify that $A = S \cap Z(|f| \vee |g|)$, so the proof is complete.

Proof of 4.3. (a) $\Rightarrow$ (b): Assume that Y is a weakly perfectly normal T_1-space with a non-G_δ point p. Let $X = Y - \{p\}$ and note that X is weakly perfectly normal. Moreover, X is both z-embedded and G_δ-dense in Y, so, by 1.1(b), X is C-embedded in Y. It follows that X is not realcompact.

(b) $\Rightarrow$ (a): Let X be a weakly perfectly normal T_1-space (of nonmeasurable power) that is not realcompact. Pick any $p \in \upsilon X - X$ and set $Y = X \cup \{p\}$. By 4.6, Y is weakly perfectly normal (and of nonmeasurable power). Now if there is $f \in C(Y)$ with $Z(f) = \{p\}$, extend f to $g \in C(\upsilon X)$. But then $Z(g) \cap X = \emptyset$, contrary to the fact that every nonempty zero-set in υX meets X [**11**, 8.8(b)]. Thus $\{p\} \notin \mathscr{Z}(Y)$, so $\{p\}$ is not a G_δ in Y.

4.7. *Remark.* It seems unlikely that (a) $\Rightarrow$ (b) of 4.3 can be improved by replacing, in 4.3(b), the phrase "with a non-G_δ point" by "that is not perfectly normal." (Consider the "obvious" modification of the preceding proof of (a) $\Rightarrow$ (b): "Let F be a closed set in Y which is not a G_δ and let Y/F be the quotient space obtained by collapsing F to a point." The difficulty is that the image of $X - F$ in Y/F under the natural map does not appear to be z-embedded in Y/F).

If D is a discrete space of measurable power, then D is not realcompact [**11**, 12.2]. The proof of 4.3 therefore shows:

4.8. THEOREM. *If D is discrete of measurable power, then there is $p \in \upsilon D - D$, and $D \cup \{p\}$ is weakly perfectly normal but not perfectly normal.*

4.9. *Remarks.* (a) In [**25**], Ostaszewski shows that Jensen's $\diamond$ implies the existence of a perfectly normal, countably compact, almost compact [**11**, 6J]

T_1-space of power $\aleph_1$ that is not compact (and hence not realcompact). Thus (in view of 4.4) affirmative answers to 4.2(a), (b), and (c) are consistent with the usual axioms of set theory (see also 4.10 below).

(b) If υX is weakly perfectly normal, then clearly X is also, but the converse is an open question (obviously related to 4.2(a), (b), and (c)). (A plausible conjecture is that the converse fails under $\diamond$.)

(c) In [**32**], Weiss shows that $MA + \neg CH$ implies that every perfectly normal, countably compact T_1-space is compact. (MA and CH denote Martin's axiom and the continuum hypothesis, respectively.) This (together with (a)) suggests the question: Does $MA + \neg CH$ imply a negative answer to any one of 4.2(a), (b), or (c)?

(d) Denote by $\mathscr{B}o(X)$ (respectively, $\mathscr{B}a(X)$) the set of all Borel (respectively, Baire) subsets of X (see, e.g., [**12**, p. 136]). If X is perfectly normal, then clearly $\mathscr{B}o(X) = \mathscr{B}a(X)$, but the converse is an open question (due to Katětov [**15**, p. 74]). In [**12**, p. 140], Hager, Reynolds, and Rice conjecture that if $\mathscr{B}o(X) = \mathscr{B}a(X)$, and if X has no closed discrete subspace of measurable power, then X is realcompact. By 4.9(a), this conjecture fails in ordinary set theory.

(e) Every perfectly normal realcompact space is hereditarily realcompact [**11**, 8.15]. However, by 4.10, the existence of a weakly perfectly normal realcompact (in fact, compact Hausdorff) space that is not hereditarily realcompact is consistent with the usual axioms of set theory.

4.10. THEOREM [$\diamond$]. *There is a weakly perfectly normal compact Hausdorff space Y of power $\aleph_1$ such that $\mathscr{B}o(Y) \neq \mathscr{B}a(Y)$ (whence Y is not perfectly normal) and such that Y is not hereditarily realcompact.*

Proof. Let X be the Ostaszewski space described in 4.9(a), and let $Y = \beta X$. Since X is countably compact but not realcompact, Y is not Borel-complete [**12**, 3.2 and 2.3]; hence $\mathscr{B}o(Y) \neq \mathscr{B}a(Y)$ by [**12**, 3.1], and obviously Y is not hereditarily realcompact. Moreover, since X is perfectly normal and almost compact, $|Y| = \aleph_1$ and Y is weakly perfectly normal (by 4.6).

We conclude this section with some simple characterizations of weak perfect normality.

Recall that a subset A of X is *locally closed* in case for each $x \in A$ there is a neighborhood V of x in X such that $A \cap V$ is closed in V. It is easy to show that A is locally closed if and only if A is the intersection of an open set and a closed set (see [**7**, Chap. I, § 3.3]).

4.11. PROPOSITION. *For any space X, these are equivalent:*

(a) *X is weakly perfectly normal.*

(b) *Every locally closed subset of X is z-embedded in X.*

(c) *Every open subset of X is normal and z-embedded in X.*

(d) *X is completely normal and every open subset of X is z-embedded in X.*

https://doi.org/10.4153/CJM-1976-068-9 Published online by Cambridge University Press

Proof. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c): If G is open in X, then G is locally closed and hence z-embedded. Moreover, if F is closed in G, then F is locally closed in X. Then F is z-embedded in X, and hence in G, so G is normal by 2.1.

The implications (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (a) follow immediately from 2.2.

A subset of a locally compact Hausdorff space is locally closed if and only if it is locally compact [**7**, Chap. I, § 9.7], so we have:

4.12. COROLLARY. *A locally compact Hausdorff space X is weakly perfectly normal if and only if every locally compact subset of X is z-embedded in X.*

5. z-embedding of open sets. In this final section we study the class **Oz** of spaces X for which every open subset of X is z-embedded in X. This class includes (a) all (weakly) perfectly normal spaces (this is trivial), (b) all extremally disconnected spaces (since X is extremally disconnected (if and) only if every open subset of X is C^*-embedded in X [**11**, 1H.6]), and (c) all products of separable metric spaces (5.6). Moreover, membership in **Oz** is inherited by open subspaces, dense subspaces, regular closed subspaces, by retracts, and by arbitrary topological sums (5.3).

We first give several characterizations of **Oz**. Recall that a subset A of X is *regular closed* in case A is the closure of an open set (equivalently: $A =$ cl int A). A *regular open* set is the complement of a regular closed set.

5.1. THEOREM. *For any space X, these are equivalent:*

(a) $X \in$ **Oz** (*i.e., every open subset of X is z-embedded in X*).

(b) *Every dense open subset of X is z-embedded in X.*

(c) *Every regular closed subset of X is a zero-set in X.*

(d) *Every dense subset of X is z-embedded in X.*

Proof. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c): Let G be open in X, let $S = G \cup (X - \operatorname{cl} G)$, and note that S is a dense open set (hence z-embedded by (b)). Define f on S by:

$$f(x) = \begin{cases} 0 & \text{if } x \in G, \\ 1 & \text{if } x \in X - \operatorname{cl} G. \end{cases}$$

Then $f \in C(S)$, so we have $Z(f) = S \cap Z$ for some $Z \in \mathscr{Z}(X)$. It follows that $\operatorname{cl} G = Z$, so the regular closed set $\operatorname{cl} G$ is a zero-set in X.

(c) $\Rightarrow$ (d): Let S be dense in X and let A_1 and A_2 be subsets of S that are completely separated in S. To prove that S is z-embedded, it suffices to find $Z_1, Z_2 \in \mathscr{Z}(X)$ such that $A_i \subset Z_i (i = 1, 2)$ and $Z_1 \cap Z_2 \cap S = \emptyset$ (see [**3**, 3.1]). By hypothesis, there is $f \in C(S)$ with $f = \mathbf{0}$ on A_1 and $f = \mathbf{1}$ on A_2. Let

$$G_1 = \{x \in S: f(x) < 1/3\}, \quad G_2 = \{x \in S: f(x) > 2/3\}.$$

Now $S - \operatorname{cl}_S G_i$ is a regular open set in S, so by [**30**, Prop. 1.2] $X - \operatorname{cl}_X(\operatorname{cl}_S G_i)$

$= X - \mathrm{cl}_X G_i$ is a regular open set in X. Hence $Z_i = \mathrm{cl}_X G_i$ is regular closed in X, so $Z_i \in \mathscr{Z}(X)$ by (c). But clearly $Z_1 \cap Z_2 \cap S = \emptyset$.

Since (d) $\Rightarrow$ (b) is trivial, it suffices now to show that (c) $\Rightarrow$ (a). Let S be open in X and let $f \in C(S)$. For each $n > 0$, let

$$G_n = \{x \in S: |f(x)| < 1/n\}.$$

Then, by (c), $\mathrm{cl}_X G_n \in \mathscr{Z}(X)$, so $Z = \bigcap_n \mathrm{cl}_X G_n \in \mathscr{Z}(X)$. But $Z(f) = S \cap Z$, so S is z-embedded in X. The proof is now complete.

The implication (a) $\Rightarrow$ (c) is applied in [**6**] (where it is shown that if X has a countable base and $Y \in \mathbf{Oz}$, then $X \times Y$ is z-embedded in $\beta X \times \beta Y$).

5.2. Corollary. *If $X \in \mathbf{Oz}$, then X is weak* cb.

Proof. (For the definition of "weak cb", see [**18**, § 3].) By [**18**, 3.1], X is weak cb if (and only if), given a decreasing sequence $(F_n)_{n \in \mathbf{N}}$ of regular closed sets in X with $\bigcap_{n \in \mathbf{N}} F_n = \emptyset$, there exist $Z_n \in \mathscr{Z}(X)$ with $\bigcap_{n \in \mathbf{N}} Z_n = \emptyset$ and $Z_n \supset F_n$ for each n. The result is therefore immediate from 5.1.

The converse of 5.2 is false (even with "weak cb" replaced by "compact"); 3.1 and 3.4 provide easy counter-examples.

Remark. Let us call a space X *regularly normal* in case X is normal and every regular closed subset of X is a G_δ in X. It follows easily from 5.1 and 2.1 that X is regularly normal if and only if every open subset of X is z-embedded in X and also every closed subset of X is z-embedded in X. The class of regularly normal spaces is strictly included in that of normal spaces, and strictly includes that of weakly perfectly normal spaces. (The space of 3.1 is (completely) normal but not regularly normal. On the other hand, $\beta\mathbf{N}$ is normal and extremally disconnected, and hence regularly normal; but $\beta\mathbf{N}$ is not completely normal [**11**, 6Q.6], and hence not weakly perfectly normal (4.1).)

5.3. Proposition. (a) *Let $X \in \mathbf{Oz}$ and let $S \subset X$. If S is either open, dense, or regular closed in X, or if S is a retract of X, then $S \in \mathbf{Oz}$.*

(b) $\mathbf{Oz}$ *is closed under the formation of arbitrary topological sums.*

Proof. (a): The result is trivial for S open. If S is dense, let T be dense in S. Then T is dense in X, hence z-embedded in X (5.1). But then T is z-embedded in S, so $S \in \mathbf{Oz}$ by 5.1. The regular closed case follows from 5.1 and the fact that if F is regular closed in a regular closed subset of X, then F is regular closed in X. Finally, if $f: X \to S$ is a retraction onto S, G is open in S, and $Z \in \mathscr{Z}(G)$, then there is $Z' \in \mathscr{Z}(X)$ with $f^{-1}(Z) = f^{-1}(G) \cap Z'$. Then $Z' \cap S \in \mathscr{Z}(S)$ and $Z = G \cap (Z' \cap S)$.

5.13 below shows that membership in $\mathbf{Oz}$ is not, in general, inherited by closed subspaces.

5.4. Proposition. *Let S be dense and C-embedded in the Tychonoff space X.*

Then $S \in \mathbf{Oz}$ *if and only if* $X \in \mathbf{Oz}$. (*In particular,* $X \in \mathbf{Oz}$ *if and only if* $\upsilon X \in \mathbf{Oz}$.)

Proof. Let $S \in \mathbf{Oz}$. By 5.3(a), it suffices to show that $\upsilon X \in \mathbf{Oz}$. Let G be open in υX. By 5.1, $\mathrm{cl}_S(G \cap S) \in \mathscr{Z}(S)$, and, since $\upsilon S = \upsilon X$, it follows that $\mathrm{cl}_{\upsilon X}(G \cap S) \in \mathscr{Z}(\upsilon X)$ [**11**, 8.8(b)]. But since S is dense in υX, $\mathrm{cl}_{\upsilon X}(G \cap S) = \mathrm{cl}_{\upsilon X} G$, so the result follows from 5.1.

We do not know whether υX can be replaced by βX in the parenthetical assertion of 5.4. (In particular, we do not know whether any of these spaces are in $\mathbf{Oz}$: $\beta\mathbf{R}$, $\beta\mathbf{Q}$, or $\beta\mathbf{Q} - \mathbf{Q}$).

Questions about z-embedding in products are usually elusive, so it is not surprising that the behavior of $\mathbf{Oz}$ with respect to products is not at all transparent. The information we have is summarized in 5.5, 5.6, 5.7, 5.8, and 5.9.

5.5. THEOREM. *Assume that each finite subproduct of* $X = \prod_{\alpha \in I} X_\alpha$ *satisfies the countable chain condition. If every countable subproduct of* X *belongs to* $\mathbf{Oz}$, *then* $X \in \mathbf{Oz}$.

Proof. Let F be a regular closed subset of X. By [**24**, 1.4(i)], X satisfies the countable chain condition, so by (the proof of) [**26**, Theorem 3] there is a countable subset J of I such that if $x \in F$, $y \in X$, and $\mathrm{pr}_\alpha(x) = \mathrm{pr}_\alpha(y)$ for every $\alpha \in J$, then $x = y$ (cf. [**24**, 2.2]). It follows that $F = \mathrm{pr}_J^{-1}(\mathrm{pr}_J(F))$ (where pr_J is the projection of X onto $Y = \prod_{\alpha \in J} X_\alpha$), which implies that $\mathrm{pr}_J(F)$ is closed and $\mathrm{pr}_J(F) = \mathrm{cl}\ \mathrm{pr}_J(\mathrm{int}\ F)$. By hypothesis, $Y \in \mathbf{Oz}$, so $\mathrm{pr}_J(F) \in \mathscr{Z}(Y)$ by 5.1. Hence $F \in \mathscr{Z}(X)$, so $X \in \mathbf{Oz}$.

The following corollary is essentially due to Noble (who shows in [**23**] that a product of separable metric spaces satisfies 5.1(d) above).

5.6. COROLLARY (Noble). *Every product of separable metric spaces belongs to* $\mathbf{Oz}$.

We do not know whether "countable subproduct" can be replaced by "finite subproduct" in 5.5. At any rate, it does not suffice merely to assume that each factor X_α is in $\mathbf{Oz}$ (see 5.8).

5.7. THEOREM. *Assume* $X \times Y$ *is pseudocompact. If* $X \times Y \in \mathbf{Oz}$, *then* $X \in \mathbf{Oz}$ *and* $Y \in \mathbf{Oz}$.

Proof. We show that $X \in \mathbf{Oz}$. Let π be the projection map from $X \times Y$ onto X and let F be a regular closed set in X. Since π is open, $\pi^{-1}(F)$ is regular closed in $X \times Y$, so by 5.1 we have $\pi^{-1}(F) = Z(f)$ for some $f \in C(X \times Y)$. For each $n > 0$, set

$$U_n = \{p \in X \times Y : |f(p)| < 1/n\}.$$

Then $\pi^{-1}(F)$ and $(X \times Y) - U_n$ are completely separated, so there is $g_n \in C(X \times Y)$ such that $g_n = -\mathbf{1}$ on $\pi^{-1}(F)$ and $g_n = \mathbf{1}$ on $(X \times Y) - U_n$. For each $r \in [0, 1]$, set

$$V_{nr} = \{p \in X \times Y: g_n(p) < r\},$$

and for each $r \in \mathbf{R}$, define W_{nr} by:

$$W_{nr} = \begin{cases} \emptyset & \text{if } r < 0, \\ \pi(V_{nr}) & \text{if } 0 \leqq r \leqq 1, \\ X & \text{if } 1 < r. \end{cases}$$

We claim that if $r < s$, then cl $W_{nr} \subset W_{ns}$. This is trivial if $r < 0$ or $1 < s$, so assume that $0 \leqq r < s \leqq 1$. By a theorem of Tamano [**29**], pseudocompactness of $X \times Y$ implies that π is z-closed (i.e., $\pi(Z)$ is closed in X whenever $Z \in \mathscr{Z}(X \times Y)$), and we therefore have

$$\text{cl } W_{nr} = \text{cl } \pi(V_{nr}) \subset \pi(\{p \in X \times Y: g_n(p) \leqq r\}) \subset W_{ns}.$$

It follows from [**11**, 3.12] that the formula

$$h_n(x) = \inf \{r \in \mathbf{R}: x \in W_{nr}\} \quad (x \in X)$$

defines h_n as a continuous real-valued function on X. Since $F = \pi(\pi^{-1}(F)) \subset W_{n0}$ and $W_{n1} \subset \pi(U_n)$, it is easy to see that $h_n \leqq \mathbf{0}$ on F and $h_n \geqq \mathbf{1}$ on $X - \pi(U_n)$. Set $k_n = (h_n \vee \mathbf{0}) \wedge \mathbf{1}$, and let $k = \sum_n 2^{-n} k_n$.

Suppose that $x \in Z(k)$, but that $\pi^{-1}(x) \cap Z(f) = \emptyset$. Then $1/f^* \in C(\pi^{-1}(x))$, where $f^* = f|\pi^{-1}(x)$. Consider any $m > 0$: Then $k_m(x) = 0$, so $h_m(x) \leqq 0$, whence $x \in \pi(U_m)$. Thus $x = \pi(p)$ for some $p \in U_m$, and we have $1/|f^*(p)| = 1/|f(p)| > m$. We conclude that $1/f^*$ is unbounded on $\pi^{-1}(x)$. But $\pi^{-1}(x)$ is pseudocompact, a contradiction. Thus $\pi^{-1}(x)$ meets $Z(f) = \bigcap_n U_n$, so $x \in \pi(\bigcap_n U_n)$, and it follows that $Z(k) \subset F$. But clearly $F \subset Z(k)$, so $F = Z(k)$ is a zero-set in X. By 5.1, $X \in \mathbf{Oz}$.

It seems unlikely that the hypothesis of pseudocompactness can simply be omitted in 5.7, but we have no counter-example.

The converse of 5.7 fails (even if $X \times Y$ is compact):

5.8. THEOREM. *If D is an infinite discrete space, then $\beta D \in$ **Oz**, but $\beta D \times \beta D \notin$ **Oz**.*

Proof. Since βD is extremally disconnected [**11**, 6M.1], $\beta D \in \mathbf{Oz}$. Now since D is infinite, there is a point $p \in \beta D - \upsilon D$. Let $X = \upsilon D \cup \{p\}$ and $Y = \beta D - \{p\}$, and note that X is realcompact [**11**, 8.16] and Y is pseudocompact [**11**, 6I.1 and 9.6]. Since $\Delta = \{(x, x): x \in D\}$ is open-and-closed in $X \times Y$, the characteristic function f of Δ on $X \times Y$ can be extended to $f^* \in C(\upsilon(X \times Y))$. Now if $\beta D \times \beta D \in \mathbf{Oz}$, then, by 5.1, $X \times Y$ is z-embedded in $\beta D \times \beta D$, and hence in $X \times \beta D = \upsilon X \times \upsilon Y$, Then $X \times Y$ is υ-embedded in $\upsilon X \times \upsilon Y$ [**2**, 3.5] (cf. 1.1(c)), so $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ [**2**, 7.6(a)]. But then $0 = f^*(p, p) = 1$, a contradiction.

5.9. *Remarks*. (a) 5.8 improves the known result that $\beta D \times \beta D$ is not extremally disconnected [**33**, 19I] (see also [**11**, 6N.1]). Part of the construction of the preceding proof generalizes [**19**, 5.3].

(b) None of the familiar spaces of ordinals is in **Oz**. Note first that $\mathbf{W} \notin \mathbf{Oz}$.

https://doi.org/10.4153/CJM-1976-068-9 Published online by Cambridge University Press

(It suffices to show that the open (discrete) subset

$$S = \{\alpha \in \mathbf{W}: \alpha \text{ is not a limit ordinal}\}$$

of $\mathbf{W}$ is not z-embedded in $\mathbf{W}$. Define $f \in C(S)$ by $f(\alpha) = 0$ (respectively, 1) if α is even (respectively, odd). Since each $g \in C(\mathbf{W})$ is constant on some tail of $\mathbf{W}$, we have $Z(f) \neq S \cap Z(g)$ for every $g \in C(\mathbf{W})$.) Since $\mathbf{W} \times \mathbf{W}$ is pseudocompact [**11**, 8M.3], it follows from 5.7 that $\mathbf{W} \times \mathbf{W} \notin \mathbf{Oz}$. Hence, by 5.3(a), none of the spaces $\mathbf{W}^*$, $\mathbf{W} \times \mathbf{W}^*$, $\mathbf{W}^* \times \mathbf{W}^*$, $\mathbf{W} \times \mathbf{N}$, $\mathbf{W} \times \mathbf{N}^*$, $\mathbf{W}^* \times \mathbf{N}$, and $\mathbf{W}^* \times \mathbf{W}^*$ is in $\mathbf{Oz}$. Hence also, by 5.4, the Tychonoff plank is not in $\mathbf{Oz}$.

(c) There are several results in [**5**] to the effect that the relation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds if it holds "uniformly locally" in some appropriate sense (see also [**2**, 8.11]). The following is another result of this kind: *If X, $Y \in \mathbf{Oz}$, and if $X \times Y$ has a normal closed cover $\mathscr{U} = (U_\alpha \times V_\alpha)_{\alpha \in I}$ of nonmeasurable power such that $\upsilon(U_\alpha \times V_\alpha) = \upsilon U_\alpha \times \upsilon V_\alpha$ for every $\alpha \in I$, then $\upsilon(X \times Y) = \nu X \times \upsilon Y$.* (In view of 5.1, $\mathscr{P} = (\text{int } U_\alpha \times \text{int } V_\alpha)_{\alpha \in I}$ is a (nonmeasurable) refinement of $\mathscr{U}$ by cozero-rectangles. Moreover, $\mathscr{P}$ is normal by [**5**, 2.4 and 0.(ii)], so the result follows from [**5**, 3.3].)

A point x of a Tychonoff space X is a *P'-point* of X in case $x \in Z \in \mathscr{Z}(X)$ implies that $x \in \text{cl int } Z$ (equivalently: every zero-set of X which contains x has nonempty interior); and X is a *P'-space* in case every $x \in X$ is a P'-point (see [**31**] and [**17**]). (Obviously every P-point of X [**11**] is a P'-point, and every P-space is a P'-space. For examples of (compact) P'-spaces, see [**31**]. I am indebted to R. Atalla for calling my attention to [**31**] and [**17**].) Isbell has shown that every extremally disconnected P-space of nonmeasurable power is discrete [**14**, 2.4] (for other proofs, see [**11**, 12H] and [**1**]). We show next that Isbell's result can be generalized to the class of P'-spaces in $\mathbf{Oz}$ (see 5.12 below).

5.10. LEMMA. *Assume X is Tychonoff. If Z is a nowhere dense zero-set in υX, then $X \cap Z$ is a nowhere dense zero-set in X.*

Proof. Obviously $X \cap Z \in \mathscr{Z}(X)$. Suppose that $\text{int}_X(X \cap Z) \neq \emptyset$. Then there is a nonempty cozero-set P in X such that $P \subset X \cap Z$. By [**2**, 5.1], $\upsilon P = \upsilon X - \text{cl}_{\upsilon X}(X - P)$, so υP is a nonempty open set in υX such that $\upsilon P \subset \text{cl}_{\upsilon X} P \subset Z$. But then Z is not nowhere dense.

5.11. THEOREM. *Assume X is Tychonoff of nonmeasurable power. If $X \in \mathbf{Oz}$, then every nonisolated point of X is contained in a nowhere dense zero-set.*

Proof. We adapt the proof of [**1**]. By 5.4 and 5.10, we may assume that X is realcompact. Let p be a nonisolated point of X. By Zorn's lemma, there is a maximal set $\mathscr{S}$ of mutually disjoint cozero-sets of X such that $p \notin \bigcup \mathscr{S}$. Since p is nonisolated, it follows easily that $S = \bigcup \mathscr{S}$ is dense in X, and hence z-embedded in X by 5.1. Now each member of $\mathscr{S}$ is cozero in X and hence realcompact [**11**, 8.14], and S is the topological sum of the members of $\mathscr{S}$. Since $|\mathscr{S}|$ is nonmeasurable, it follows that S is realcompact [**11**, 12G]. But a

z-embedded realcompact subspace must be G_δ-closed (1.1(c)), so there is $Z \in \mathscr{Z}(X)$ such that $p \in Z$ and $Z \cap S = \emptyset$. Obviously Z is nowhere dense.

5.12. COROLLARY. *Assume X is Tychonoff of nonmeasurable power and that $X \in$ **Oz**. Then every P'-point of X is isolated. Hence if X is a P'-space, then X is discrete.*

5.13. COROLLARY. *Assume that X is of nonmeasurable power. If X is locally compact and realcompact, but not compact (e.g., if X is infinite and discrete), then $\beta X - X \notin$ **Oz**.*

Proof. By [**10**, 3.1], each zero-set in $\beta X - X$ is the closure of its interior, so $\beta X - X$ is a P'-space. Now if $\beta X - X \in$ **Oz**, then, by 5.12, $\beta X - X$ is discrete (as well as compact), so $\beta X - X$ is finite. But then X is pseudocompact [**11**, 9D.3], and hence compact, a contradiction.

5.14. *Remarks.* (a) The hypothesis of nonmeasurability cannot be omitted in 5.11 or 5.12: If D is discrete of measurable power, then (i) $vD \in$ **Oz** (5.4), (ii) vD is a P-space [**11**, 8A.5], and (iii) $D \neq vD$ [**11**, 12.2] (so vD is not discrete).

(b) By 5.12, any nondiscrete P-space of nonmeasurable power (see, e.g., [**11**, 4N]) is an example of a basically disconnected space that is not in **Oz**.

(c) We do not know whether the nonmeasurability hypothesis of 5.13 can be omitted. (Without it, the remaining hypotheses imply that $\beta X - X$ is not basically disconnected [**10**, 3.2].)

We conclude with an F-space analogue of 5.12. (For the definition of "F-space", see [**11**, 14.25].)

5.15. PROPOSITION. *Every (Tychonoff) F-space in* **Oz** *is extremally disconnected.*

Proof. We need only note that X is an F-space (if and) only if every z-embedded subset of X is C^*-embedded [**4**, 4.5].

Added in proof (June 2, 1976). 2.1, 2.3, and 2.5 appear, without attribution, in § 7 of R. A. Alò and H. L. Shapiro, *Normal topological spaces*, Cambridge Tracts in Math. *65* (Cambridge Univ. Press, London, 1974). These results are due to the present author. For some additional results concerning z-embedding of every subset and of every open subset, see T. Terada, *Note on z-, C*-, and C-embedding*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. *A13* (1975), 129–132. (Among other things, Terada notes, independently, the equivalence of (a) and (d) of 4.11 and of (a), (b), and (d) of 5.1.)

REFERENCES

1. R. L. Blair, *On a theorem of Isbell concerning extremally disconnected P-spaces*, Arch. Math. *12* (1961), 434.

2. ——— *On v-embedded sets in topological spaces*, TOPO 72—General Topology and its Applications (Second Pittsburgh International Conference, December 18-22, 1972), Lecture Notes in Math., Vol. 378 (Springer-Verlag, Berlin-Heidelberg-New York, 1974), pp. 46-79.

3. ——— *Filter characterizations* of z-, C^*-, *and C-embeddings*, Fund. Math. *90* (1976), 285–300.

4. R. L. Blair and A. W. Hager, *Extensions of zero-sets and of real-valued functions*, Math. Zeit. *136* (1974), 41-52.
5. ——— *Notes on the Hewitt realcompactification of a product*, Gen. Top. Appl. *5* (1975), 1–8.
6. ——— *z-embedding* in $\beta X \times \beta Y$, to appear.
7. N. Bourbaki, *Elements of mathematics: General topology* (Hermann, Paris; Addison-Wesley, Reading, Mass., 1966).
8. H. H. Corson, *Normality in subsets of product spaces*, Amer. J. Math. *81* (1959), 785–796.
9. R. Engelking, *Outline of general topology* (PWN, Warsaw, 1965; English transl., North-Holland, Amsterdam; Interscience, New York, 1968).
10. N. J. Fine and L. Gillman, *Extension of continuous functions in* $\beta\mathbf{N}$, Bull. Amer. Math. Soc. *66* (1960), 376–381.
11. L. Gillman and M. Jerison, *Rings of continuous functions* (Van Nostrand, Princeton, 1960).
12. A. W. Hager, G. D. Reynolds, and M. D. Rice, *Borel-complete topological spaces*, Fund. Math. *75* (1972), 135–143.
13. M. Henriksen and D. G. Johnson, *On the structure of a class of archimedean lattice-ordered algebras*, Fund. Math. *50* (1961), 73–94.
14. J. R. Isbell, *Zero-dimensional spaces*, Tôhoku Math. J., Ser. II *7* (1955), 1–8.
15. M. Katětov, *Measures in fully normal spaces*, Fund. Math. *38* (1951), 73–84.
16. J. L. Kelley, *General topology* (Van Nostrand, New York, 1955).
17. R. F. Levy, *Almost-P-spaces and Lusin's hypothesis*, Notices Amer. Math. Soc. *22* (1975), A-582.
18. J. E. Mack and D. G. Johnson, *The Dedekind completion of* $C(X)$, Pacific J. Math. *20* (1967), 231–243.
19. W. G. McArthur, *Hewitt realcompactifications of products*, Can. J. Math. *22* (1970), 645–655.
20. S. Mrówka, *Some properties of Q-spaces*, Bull. Acad. Polon. Sci. Cl. III *5* (1957), 947–950.
21. ——— *On the unions of Q-spaces*, Bull. Acad. Polon. Sci. Cl. III *6* (1958), 365–368.
22. ——— *Some comments on the author's example of a non-*$\mathscr{R}$*-compact space*, Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys. *18* (1970), 443–448.
23. N. Noble, *C-embedded subsets of products*, Proc. Amer. Math. Soc. *31* (1972), 613–614.
24. N. Noble and M. Ulmer, *Factoring functions on cartesian products*, Trans. Amer. Math. Soc. *163* (1972), 329–339.
25. A. J. Ostaszewski, *On countably compact, perfectly normal spaces*, J. London Math. Soc., to appear.
26. K. A. Ross and A. H. Stone, *Products of separable metric spaces*, Amer. Math. Monthly *71* (1964), 398–403.
27. Yu. M. Smirnov, *On normally disposed sets of normal spaces*, Mat. Sb. (N. S.) *29* (1951), 173–176. (Russian)
28. A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. *54* (1948), 977–982.
29. H. Tamano, *A note on the pseudocompactness of the product of two spaces*, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. *33* (1960/61), 225–230.
30. ——— *On compactifications*, J. Math. Kyoto Univ. *1–2* (1962), 161–193.
31. A. I. Veksler, *P′-points, P′-sets, P′-spaces. A new class of order-continuous measures and functionals*, Dokl. Akad. Nauk SSSR *212* (1973), 789–792; Soviet Math. Dokl. *14* (1973), 1445–1450.
32. W. A. R. Weiss, *Some applications of set theory to topology*, Ph.D thesis, University of Toronto, 1975.
33. S. Willard, *General topology* (Addison Wesley, Reading, Mass., 1970).

Ohio University,
Athens, Ohio

https://doi.org/10.4153/CJM-1976-068-9 Published online by Cambridge University Press