A CHARACTERIZATION OF THE ORDERS OF REGRESSIVE ω-GROUPS

MATTHEW J. HASSETT

(Received 26 February 1970, revised 21 April 1971)

Communicated by J. N. Crossley

1. Introduction

Let E, Λ and Λ_R denote the collections of all non-negative integers, isols and regressive isols respectively. An *ω-group* is a pair (α, p) where (1) $\alpha \subseteq E$, (2) $p(x,y)$ is a partial recursive group multiplication for α and (3) the function which maps each element of α to its inverse under p has a partial recursive extension. If $G = (\alpha, p)$ is an ω-group, we call the recursive equivalence type of α the *RET* or *order* of G (written $o(G)$). Let $G_R = \{T \in \Lambda_R \mid T = o(G)$ for some ω-group $G\}$. It follows from the version of the Lagrange Theorem given in [4] that $\Lambda_R - G_R$ is non-empty and has cardinality c. In this paper we characterise the isols in G_R as follows: *A regressive isol T belongs to G_R if and only if $T \in E$ or T is infinite and there exist a regressive isol $U \leqq T$ and a function a_n from E into $E - \{0\}$ such that $U \leqq_* a_n$ and $T = \Pi_U a_n$.* (The "$\leqq_*$" relation is defined in [2]). In presenting the proof of this result, we shall assume that the reader is familiar with either [3] or [4]. The proof that, given a_n and $U \leqq_* a_n$, a group of order $\Pi_U a_n$ exists is based on the natural trick—one constructs a direct product of disjoint cyclic groups of order $a_0, a_1, \cdots$ indexed by elements of a set of RET U. The proof that any regressive group G has order of the form $\Pi_U a_n$ is trivial for finite groups; the proof for infinite regressive groups is based upon the construction of an ascending chain of finite subgroups G_i of G such that $\bigcup_{i=0}^{\infty} G_i = G$ and

$$o(G) = o(G_1) \cdot \underbrace{\frac{o(G_2)}{o(G_1)} \cdot \frac{o(G_3)}{o(G_2)} \cdots \frac{o(G_{n+1})}{o(G_n)} \cdots}_{U}.$$

2. Characterization of G_R

In the following we shall write ρf to denote the range of a function f.

THEOREM 1. *Let $G = (\tau, p)$ be an ω-group of order T, where $T \in \Lambda_R$. Then*

https://doi.org/10.1017/S1446788700010867 Published online by Cambridge University Press

there exist a regressive isol $U \leqq T$ and a function a_n from E to $E - \{0\}$ such that $U \leqq_* a_n$ and $T = \Pi_U a_n$.

PROOF. We need consider only $T \in \Lambda_R - E$. Let t_n be a regressive function such that $\tau = \rho t_n$. Let $q(x)$ be a regressing function for t_n. (We assume without loss of generality that t_0 is the identity element of G.) We shall give simultaneous inductive definitions of an increasing function $d(n)$, an increasing sequence $\{G_n\}$ of finite subgroups of G such that $\bigcup_{n=0}^{\infty} G_n = G$ and a function a_n such that $t_{d(n)} \leqq_* a_n$. We will then show that $\nu \mid \tau - \nu$, where $\nu = \rho t_{d(n)}$, and complete the proof by showing that $T = \Pi_U a_n$, where $U = \text{Req}(\nu)$.

We first note that given a finite subset σ of τ if we alternately apply q and take the group closure we can effectively enumerate the smallest subgroup $G(\sigma)$ of G containing σ and closed under q and the group operation. Since τ is isolated $G(\sigma)$ is finite and we can decide when $G(\sigma)$ has been obtained. We now proceed with the necessary definitions.

$$G_{-1} = \phi$$

$i = 0$

(1) $d(o) = 1$

(2) $G_0 = G(\{t_1\})$

(3) $a_0 = o(G_0)$

$i = n + 1$

(1) $d(n+1) = (\mu y)[t_y \notin G_n]$

(2) $G_{n+1} = G(\{t_{d(n+1)}\})$

(3) $a_{n+1} = \dfrac{o(G_{n+1})}{o(G_n)}$.

It follows immediately from the definition above that both the function $d(n)$ and the sequence of sets $\{G_n\}$ are increasing. Given $t_{d(n)}$ one can find $t_0, t_1, \cdots, t_{d(n)}$ and determine all elements of each G_i, $0 \leqq i \leqq n$. This information is sufficient for the computation of $a_0, a_1, \cdots, a_n$. Hence $t_{d(n)} \leqq_* a_n$. Similarly given $x = t_k \in \tau$, there is a number m such that $x \in G_m - G_{m-1}$ and one can use the definition above to compute $t_{d(o)}, \cdots, t_{d(m)}$. This information is sufficient to determine whether or not $x \in \nu$. Hence $\nu \mid \tau - \nu$.

We now prove that $o(G) = \Pi_U a_n$. We use the notation of [3]. Let γ denote the set of all indices of finite functions f such that $\delta_e f \subseteq \nu$ and $(\forall n)[f(t_{d(n)}) < a_n]$. Since $\text{Req}(\gamma) = \Pi_U a_n$, we need only show that $\gamma \simeq \tau$. Using Proposition 1 of [1], we shall prove this by describing a one-to-one function $\alpha(x)$ mapping τ onto γ such that $\alpha(x)$ and $\alpha^{-1}(x)$ have partial recursive extensions. $\alpha(x)$ is defined as follows: For each $m \in E$, let

$$\gamma_m = \{n \mid n \in \gamma \text{ and } t_{d(m)} \in \delta_e r_n \text{ and } [t_{d(k)} \in \delta_e r_n \Rightarrow k \leqq m]\}.$$

We write G and γ as disjoint unions of finite sets as follows:

$$\gamma = \gamma_0 \cup \gamma_1 \cup \cdots \cup \gamma_n \cup \cdots$$

$$G = \{t_0, t_1, \cdots, t_{d(1)-1}\} \cup \{t_{d(1)}, \cdots, t_{d(2)-1}\} \cup \cdots \cup \{t_{d(n)}, \cdots, t_{d(n+1)-1}\} \cup \cdots$$

We observe that the nth finite set in the decomposition of G is merely $G_n - G_{n-1}$. It is easily seen that the cardinality of the nth set in each decomposition above is $(a_0 a_1 \cdots a_n) - (a_0 a_1 \cdots a_{n-1})$ for $n > 1$ and a_0 for $n = 0$. Furthermore, given any element of either γ_n or $G_n - G_{n-1}$, we can (uniformly) effectively recover $t_{d(n)}$ and list all elements of both γ_n and $G_n - G_{n-1}$ in increasing order. Let $\alpha(x)$ be the function from τ to γ which pairs the elements of each set $G_n - G_{n-1}$ with the elements of γ_n in increasing order. The preceding discussion shows that both $\alpha(x)$ and $\alpha^{-1}(x)$ have partial recursive extensions. This completes the proof.

Let a_n be a sequence of positive integers.

PROPOSITION 1. *If* $T \leqq_* a_n$ *and* $T \in \Lambda_R - E$, *then* $\Pi_T a_n \in \Lambda_R$.

PROOF. Left to the reader.

THEOREM 2. *Let* $T \in \Lambda_R - E$ *and let* a_n *be a sequence of positive integers such that* $T \leqq_* a_n$. *Then there exists a regressive* ω*-group of order* $\Pi_T a_n$.

PROOF. Let B_E be the group of all permutations of E which leave all but finitely many numbers fixed, and let $f \leftrightarrow f^*$ be any Gödel numbering of B_E which is one-to-one and bi-effective. It was shown in [5] that $P(E) = \{f^* | f \in B_E\}$ is an ω-group under the induced multiplication $f^* \cdot g^* = (f \circ g)^*$. We shall construct a subgroup of $P(E)$ of order $\Pi_T a_n$. We will use the recursive pairing function $j(x, y)$ and associated projection functions $k(x)$ and $l(x)$ defined in [1]. Let t_n be any regressive function such that range $(t_n) \in T$. Let γ_n denote the cyclic permutation

$$(j(t_n,0), j(t_n,1), \cdots, j(t_n, a_n - 1)).$$

Let $[\gamma_n^*]$ denote the cyclic subgroup of $P(E)$ generated by γ_n^*. Let G be the subgroup of $P(E)$ generated by $\{\gamma_n^* | n \subseteq E\}$.

Since the permutations γ_n move disjoint sets of numbers, G is the weak direct product of the cyclic groups $[\gamma_n^*]$. Thus G is an ω-group with non-trivial elements of the form $\prod_{i=1}^k \gamma_{n_i}^{*l_i}$, where $0 < l_i < a_{n_i}$, $i = 1, \cdots, k$. Let $\alpha(x)$ be the function with domain G which maps $x = \prod_{i=1}^k \gamma_{n_i}^{*l_i}$ to the index of the finite function f defined by

$$f(x) = \begin{cases} l_i, & x = t_{n_i} \text{ for some } i,\ 1 \leqq i \leqq k \\ 0, & x \notin \{t_{n_1}, \cdots, t_{n_k}\} \end{cases}$$

and maps the identity permutation to the constant function $f(x) \equiv 0$. It is readily seen that α maps G one-to-one onto the set

$$\beta = \{n \mid \delta_e r_n \subseteq \rho t_k \text{ and } (\forall k)[r_n(t_k) < a_k]\}.$$

Since the RET of β is $\Pi_{\gamma} a_n$, we need only show that $\alpha(x)$ and $\alpha^{-1}(x)$ have partial recursive extensions to complete the proof. We leave this straightforward verification to the reader.

References

[1] J. C. E. Dekker, *Infinite series of Isols* (Proc. Sympos. Pure Math., V (1962), Amer. Math. Soc., Providence, R. I.).

[2] J. C. E. Dekker, 'The Minimum of Two Regressive Isols', *Math. Zeitschr.* 83 (1964), 345–366.

[3] E. Ellentuck, 'Infinite Products of Isols', *Pacific J. Math.* 14 (1964), 49–52.

[4] D. Ferguson, 'Infinite Products Recursive Equivalence Types', *J. Symbolic Logic* 33 (1968), 221–230.

[5] M. Hassett, 'Recursive Equivalence Types and Groups', *J. Symbolic Logic* 34 (1969), 13–20.

[6] J. Myhill, 'Recursive Equivalence Types and Combinatorial Function', *Bull. Am. Math. Soc.* 64 (1958), 373–376.

[7] A. Nerode, 'Extensions to Isols', *Annals of Math.* 73 (1961), 362–403.

Arizona State University
Tempe, Arizona
U.S.A.