BULL. AUSTRAL. MATH. SOC. 34G20
VOL. 56 (1997) [149–156]

ON THE IMPLICIT DARBOUX PROBLEM IN BANACH SPACES

DARIA WÓJTOWICZ

In this paper we prove the existence theorem for the implicit Darboux problem on the quarterplane $x \geqslant 0$, $y \geqslant 0$. Moreover, we study the topological structure of the solution set of this problem.

1. INTRODUCTION

In this paper we shall consider the following implicit Darboux problem

$$(1)\qquad \begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= g\left(x, y, z, \frac{\partial^2 z}{\partial x \partial y}\right), \\ z(x,0) &= 0, \qquad 0 \leqslant x < +\infty, \\ z(0,y) &= 0, \qquad 0 \leqslant y < +\infty, \end{aligned}$$

in a Banach space, where $\dfrac{\partial^2 z}{\partial x \partial y}$ denotes the mixed derivative of z. We shall give sufficient conditions for the existence of a solution of (1). Moreover, under the same assumptions we shall prove an Aronszajn type theorem for this problem.

In our considerations we shall apply the following two theorems.

THEOREM 1 [3] *Let D be a closed and convex subset of a Hausdorff locally convex space such that $0 \in D$, and let G be a continuous mapping of D into itself. If the implication*

$$(2)\qquad (V = \mathit{conv}G(V) \quad \text{or} \quad V = G(V) \cup \{0\}) \Longrightarrow V \text{ is relatively compact}$$

holds for every subset V of D, then G has a fixed point.

THEOREM 2 [7] *Let X, Y be metric spaces. Assume that y is a point of Y with a neighbourhood homeomorphic to a closed convex subset of a Fréchet space. Let $T : X \to Y$ be a continuous y-closed mapping, and $T_n : X \to Y$ a homeomorphism into. If y is an interior point of $\bigcap_{n=1}^{\infty} T_n(X)$ and $T^{-1}(y)$ is compact and nonempty, then $T^{-1}(y)$ is an R_δ*

Received 30th September, 1996
I would like to thank to dr D. Bugajewski for his valuable suggestions.

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9729/97 $A2.00+0.00.

https://doi.org/10.1017/S0004972700030811 Published online by Cambridge University Press

whenever $\lim_{n\to\infty} T_n = T$ *uniformly on* $T^{-1}(y)$ *and all sets of the form* $\bigcup_{n=1}^{\infty} T_n^{-1}(C)$, *where* C *is a compact subset of* $\bigcap_{n=1}^{\infty} T_n(X)$.

Recall that a subset of a metric space is an R_δ if it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.

Our main condition that guarantee the existence of the solution of (1) will be formulated in terms of the measure of noncompactness α introduced by Kuratowski (see [**2**] for the definition and basic properties).

2. An existence theorem

Let $I = [0, +\infty)$ and let E be a Banach space. Assume that:

1^0 $g : I \times I \times E \times E \to E$ is a continuous mapping;

2^0 there exists a number $k \in [0, 1)$ such that

$$\|g(x, y, u, v_1) - g(x, y, u, v_2)\| \leqslant k\,\|v_1 - v_2\|$$

for every $(x, y, u) \in I \times I \times E$ and $v_1,\ v_2 \in E$;

3^0 for every $a, b > 0$ there exists $m(a, b) \in \mathbb{R}_+$ such that

$$\|g(x, y, u, 0)\| \leqslant m(a, b) \quad \text{whenever} \quad |x| < a,\ |y| < b.$$

First we shall show that (1) is equivalent to some Darboux problem in the explicit form. Indeed, consider the sequence of functions $f_n : I \times I \times E \to E$ such that $f_0(x, y, u) = 0$, $f_{n+1}(x, y, u) = g(x, y, u, f_n(x, y, u))$ for every $(x, y, u) \in I \times I \times E$ and $n \in \{0, 1, 2, \ldots\}$. By 2^0, in view of the Banach contraction principle, for every $(x, y, u) \in I \times I \times E$ there exists exactly one element $f(x, y, u) \in E$ such that $f(x, y, u) = g(x, y, u, f(x, y, u))$ and $f(x, y, u) = \lim_{n\to\infty} f_n(x, y, u)$. Hence the mapping $(x, y, u) \to f(x, y, u)$ satisfies the equation

$$f(x, y, u) = g(x, y, u, f(x, y, u)).$$

Moreover, for every $n, p \in \mathbb{N}$ we have

$$\|f_{n+p}(x, y, u) - f_n(x, y, u)\| \leqslant \frac{k^n}{1-k} m(a, b),$$

whenever $|x| < a$, $|y| < b$. Thus $f_n \to f$ as $n \to \infty$, uniformly on every bounded subset of $I \times I \times E$. Hence the mapping $f : I \times I \times E \to E$ is continuous and

$$\|f(x, y, u)\| \leqslant M(a, b) \quad \text{for} \quad |x| < a,\ |y| < b,$$

where $M(a, b) = 1/(1-k)m(a, b)$.

https://doi.org/10.1017/S0004972700030811 Published online by Cambridge University Press

We note that the mapping $z : I \times I \to E$ is a solution of (1) if and only if it is a solution of the following Darboux problem

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= f(x,y,z), \\ z(x,0) &= 0, \qquad 0 \leqslant x < +\infty, \\ z(0,y) &= 0, \qquad 0 \leqslant y < +\infty. \end{aligned} \tag{3}$$

Now, we shall prove the following

Lemma *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that*

$$\alpha(g(A \times X \times Y)) \leqslant \max\left(h(\alpha(X)), \alpha(Y)\right) \tag{4}$$

for all bounded subsets $A \subset I \times I$ and $X \times Y \subset E \times E$. Then

$$\alpha(f(A \times Z)) \leqslant h(\alpha(Z)) \tag{5}$$

for all bounded subsets $A \subset I \times I$ and $Z \subset E$ (see [**5**]*).*

Proof: From the definition of the sequence (f_n) and (4), by mathematical induction we have

$$\alpha(f_n(A \times Z)) \leqslant h(\alpha(Z))$$

for all bounded subsets $A \subset I \times I$ and $Z \subset E$.

Fix $\varepsilon > 0$. Since $f_n \to f$ uniformly on every bounded subset of $I \times I \times E$, as $n \to \infty$,

$$f(A \times Z) \subset f_n(A \times Z) + K(0, \varepsilon)$$

for all bounded subsets $A \subset I \times I$ and $Z \subset E$, and for sufficiently large $n \in \mathbb{N}$, where $K(0,\varepsilon)$ denotes the open ball of center 0 and radius ε in E.

Hence

$$\alpha(f(A \times Z)) \leqslant \alpha(f_n(A \times Z)) + 2\varepsilon \leqslant h(\alpha(Z)) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we receive (4). □

Our first main result is given by the following

Theorem 3 *If g satisfies $1^0 - 3^0$ and (4), where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, nondecreasing function such that the inequality*

$$0 \leqslant u(x,y) \leqslant \int_0^x \int_0^y h(u(t,s))\, dt ds, \quad (x,y) \in I \times I \tag{6}$$

has only a trivial solution, then the problem (1) has a solution.

The above theorem extends the main result from [**6**].

https://doi.org/10.1017/S0004972700030811 Published online by Cambridge University Press

PROOF: Let $C = C(I \times I, E)$ be the space of all continuous functions $I \times I \to E$ with the topology of uniform convergence on each compact subset of $I \times I$. Set

$$F(z)(x,y) = \int_{D(x,y)} f(t,s,z(t,s))\,dtds,$$

$(x,y) \in I \times I$, $z \in C$ and $D(x,y) = \{(t,s) \in I \times I:\ 0 \leqslant t \leqslant x,\ 0 \leqslant s \leqslant y\}$. Obviously the operator F maps C into C and is continuous. Let $D = \overline{\text{conv}}(F(C) \cup \{0\})$. It is clear that F maps D into itself. We shall show now that F satisfies (2).

Indeed, let V be a subset of D such that $V \subset \overline{\text{conv}}(F(V) \cup \{0\})$. First, we verify that V is equicontinuous on every compact subset of $I \times I$. Since

$$\begin{aligned} \|F(z)(x_1,y_1) - F(z)(x_2,y_2)\| &= \left\| \int_{D(x_1,y_1)} f(t,s,z(t,s))\,dtds - \int_{D(x_2,y_2)} f(t,s,z(t,s))\,dtds \right\| \\ &\leqslant \mu(D(x_1,y_1) - D(x_2,y_2))M(a,b) \end{aligned}$$

where $|x_1| < a$, $|x_2| < a$, $|y_1| < b$, $|y_2| < b$, $z \in C$ the family $F(C)$ is equicontinuous on every compact subset of $I \times I$. Hence V is equicontinuous on every compact subset of $I \times I$.

Let $W = F(V)$, $v(x,y) = \alpha(V(x,y))$ and $w(x,y) = \alpha(W(x,y))$ for $(x,y) \in I \times I$. From the basic properties of α we obtain

$$(7) \qquad \begin{aligned} v(x,y) &= \alpha(V(x,y)) \leqslant \alpha(\overline{\text{conv}}(F(V)(x,y) \cup \{0\})) \\ &= \alpha(F(V)(x,y) \cup \{0\}) = \max\left(\alpha(F(V)(x,y)), \alpha(\{0\})\right) \\ &= \alpha(F(V)(x,y)) = w(x,y), \quad (x,y) \in I \times I \end{aligned}$$

and, similarly,

$$\alpha(V(T)) \leqslant \alpha(W(T)) \quad \text{for each compact subset} \quad T \text{ for } I \times I.$$

Further, we have

$$\begin{aligned} &|w(x_1,y_1) - w(x_2,y_2)| = |\alpha(W(x_1,y_1)) - \alpha(W(x_2,y_2))| \\ &\quad = |\alpha(F(V)(x_1,y_1)) - \alpha(F(V)(x_2,y_2))| \\ &\quad \leqslant \sup_{u,v \in V} \|F(u)(x_1,y_1) - F(u)(x_2,y_2) - F(v)(x_1,y_1) + F(v)(x_2,y_2)\| \\ &\quad \leqslant 2 \sup_{u,v \in V} \|F(u)(x_1,y_1) - F(u)(x_2,y_2)\|, \quad (x_1,y_1),(x_2,y_2) \in I \times I. \end{aligned}$$

By the above inequality and the equicontinuity $F(V)$ on every compact subset of $I \times I$, we deduce that w is continuous on every compact subset of $I \times I$. Hence w is continuous on $I \times I$.

Divide the rectangle $D(x,y)$ into n^2 parts: $0 = x_0 < x_1 < \ldots < x_n = x$, $0 = y_0 < y_1 < \ldots < y_n = y$ in such a way that $x_i - x_{i-1} < 1/n$ and $y_j - y_{j-1} < 1/n$

https://doi.org/10.1017/S0004972700030811 Published online by Cambridge University Press

for $i,j = 1,\dots,n$. Put $D_{ij}(x,y) = [x_{i-1},x_i] \times [y_{j-1},y_j]$, $i,j = 1,\dots,n$. Since W is equicontinuous and uniformly bounded on every compact subset of $I \times I$, by Ambrosetti's Lemma [**1**] and the continuity of w there exists $(p_i,q_j) \in D_{ij}(x,y)$ such that

(8)
$$\begin{aligned}\alpha(W(D_{ij}(x,y))) &= \sup_{(t,s)\in D_{ij}(x,y)} \alpha(W(t,s)) = \sup_{(t,s)\in D_{ij}(x,y)} w(t,s)\\ &= w(p_i,q_j), \quad (x,y) \in I \times I.\end{aligned}$$

From the mean value theorem, for every $z \in V$ we obtain

$$\begin{aligned}F(z)(x,y) &= \int_{D(x,y)} f(t,s,z(t,s))\,dtds\\ &= \sum_{i,j=1}^{n} \int_{D_{ij}(x,y)} f(t,s,z(t,s))\,dtds\\ &\in \sum_{i,j=1}^{n} \mu(D_{ij}(x,y))\overline{\text{conv}} f(D_{ij}(x,y) \times z(D_{ij}(x,y))), \quad (x,y) \in I \times I.\end{aligned}$$

Thus
$$F(V)(x,y) \subset \sum_{i,j=1}^{n} \mu(D_{ij}(x,y))\overline{\text{conv}} f(D_{ij}(x,y) \times V(D_{ij}(x,y))),$$
$(x,y) \in I \times I$.

Hence, by the Lemma, the properties of α, (8) and (7) we have

$$\begin{aligned}w(x,y) &\leqslant \sum_{i,j=1}^{n} \mu(D_{ij}(x,y))\alpha(f(D_{ij}(x,y)) \times V(D_{ij}(x,y)))\\ &\leqslant \sum_{i,j=1}^{n} \mu(D_{ij}(x,y))h(\alpha(V(D_{ij}(x,y))))\\ &\leqslant \sum_{i,j=1}^{n} \mu(D_{ij}(x,y))h(\alpha(W(D_{ij}(x,y))))\\ &= \sum_{i,j=1}^{n} \mu(D_{ij}(x,y))h(w(p_i,q_j)) \quad (x,y) \in I \times I.\end{aligned}$$

If $n \to \infty$, by the continuity of h and w we obtain

$$w(x,y) \leqslant \int_{D(x,y)} h(w(t,s))\,dtds, \quad (x,y) \in I \times I.$$

Thus, by (6) $w(x,y) = 0$ and, therefore by (7), $v(x,y) = 0$ for every $(x,y) \in I \times I$. Hence $V(x,y)$ is relatively compact for every $(x,y) \in I \times I$. In view of the generalisation of Ascoli's Theorem [**4**, p.81], V is relatively compact.

The operator F satisfies all the assumptions of Theorem 1 and, therefore, there exists $z \in D$ such that $z = F(z)$. This completes the proof of Theorem 3. ∎

https://doi.org/10.1017/S0004972700030811 Published online by Cambridge University Press

3. An Aronszajn property

The aim of this Section is to prove the following Aronszajn type theorem.

Theorem 4 *Under the assumptions of Theorem 3 the set S of all solutions of* (1) *on $I \times I$ is an R_δ.*

Proof: Let $F : C \to C$ be the operator defined in the proof of Theorem 3 and let $T = I - F$, where I denotes the identity map. Obviously T is continuous mapping of C into itself. Now, we verify that T is 0-closed, that is, the following implication

$$0 \in \overline{T(V)} \Longrightarrow 0 \in T(V)$$

holds for every closed subset $V \subset C$. It is enough to verify that T is a proper map, that is, if Z is relatively compact, then $T^{-1}(Z)$ is relatively compact.

Let $Z \subset C$ be a relatively compact set and put $U = T^{-1}(Z)$. Consider the sequence (u_n), where $u_n \in U$ for $n \in \mathbb{N}$. Set $V = \{u_n : n \in \mathbb{N}\}$. Since $V(x,y) \subset (I - F)(V)(x,y) + F(V)(x,y) \subset \overline{Z}(x,y) + F(V)(x,y)$, $\alpha(V(x,y)) \leqslant \alpha\big(\overline{Z}(x,y)\big) + \alpha(F(V)(x,y)) = \alpha(F(V)(x,y))$, $(x,y) \in I \times I$.

By arguing similarly as in the proof of Theorem 3, we infer that V is relatively compact. Hence there exists a convergent subsequence (u_{n_k}) of (u_n), so U is relatively compact.

Define

$$F_n(z)(x,y) = \int\limits_{D(r_n(x,y))} f(t,s,z(t,s))\,dtds, \quad (x,y) \in I \times I,\ z \in C,\ n \in \mathbb{N},$$

where

$$r_n(x,y) = \begin{cases} 0, & (x,y) \in K(1/n), \\ \left(1 - 1/(\|(x,y)\|)n\right)(x,y), & (x,y) \in (I \times I) \setminus K(1/n), \end{cases}$$

$K(1/n) = \{(x,y) : x \geqslant 0, y \geqslant 0, \sqrt{x^2 + y^2} \leqslant 1/n\}$.

Obviously, the operators F_n map C into itself and are continuous. Put $T_n = I - F_n$, $n \in \mathbb{N}$. Now, we shall prove that T_n is a homeomorphism of C into itself for every $n \in \mathbb{N}$. Obviously the mappings T_n are continuous. Fix $n \in \mathbb{N}$. It is easy to see that for any $z_1, z_2 \in C$

$$T_n(z_1) = T_n(z_2) \Longrightarrow z_1 = z_2. \tag{9}$$

It is enough to prove the continuity of T_n^{-1}. Assume that $z_i, z_0 \in C$, $T_n(z_i) \to T_n(z_0)$, as $i \to \infty$. We have $F_n(z_i)(x,y) = F_n(z_0)(x,y) = 0$ for $(x,y) \in K(1/n)$, so $z_i \to z_0$ uniformly on $K(1/n)$, as $i \to \infty$. Since $f(t,s,z_i(t,s)) \to f(t,s,z_0(t,s))$ uniformly on $K(1/n)$, as $i \to \infty$,

$$\int\limits_{D(r_n(x,y))} f(t,s,z_i(t,s))\,dtds \to \int\limits_{D(r_n(x,y))} f(t,s,z_0(t,s))\,dtds$$

https://doi.org/10.1017/S0004972700030811 Published online by Cambridge University Press

for $(x,y) \in \overline{K(2/n) \setminus K(1/n)}$ (that is, to the closure of $K(2/n) \setminus K(1/n)$), as $i \to \infty$. Hence, it is clear that $z_i \to z_0$ uniformly on $\overline{K(2/n) \setminus K(1/n)}$. By arguing similarly to the above, we infer that $z_i \to z_0$ uniformly on every compact subset of $I \times I$, as $i \to \infty$. This proves the continuity of T_n^{-1}.

Now, we shall show that $\lim_{n\to\infty} T_n = T$ uniformly. Fix a set $K(r)$, $r > 0$. Choose $n \in \mathbb{N}$ such that $K(1/n) \subset K(r)$. From the inequalities

$$\begin{aligned} \|F_n(z)(x,y) - F(z)(x,y)\| &= \left\| \int_{D(x,y)} f(t,s,z(t,s))\,dtds \right\| \\ &\leqslant M\left(\frac{1}{n},\frac{1}{n}\right)\frac{1}{n^2}, \quad \text{for } (x,y) \in K\left(\frac{1}{n}\right),\ z \in C, \end{aligned}$$

and

$$\begin{aligned} &\|F_n(z)(x,y) - F(z)(x,y)\| \\ &\quad = \left\| \int_{D(r_n(x,y))} f(t,s,z(t,s))\,dtds - \int_{D(x,y)} f(t,s,z(t,s))\,dtds \right\| \\ &\quad \leqslant \frac{1}{n}\left(2r - \frac{1}{n}\right)M(r,r) \quad \text{for } (x,y) \in K(r) \setminus K\left(\frac{1}{n}\right),\ z \in C, \end{aligned}$$

it is clear that $F_n(z) \to F(z)$ uniformly in z, on every compact subset of $I \times I$.

Further, since $T^{-1}(0)$ is the set of all fixed points of F, by Theorem 3 it is nonempty. Let (z_k) be sequence such that $z_k \in T^{-1}(0)$ for $k \in \mathbb{N}$. Put $V = \{z_k : k \in \mathbb{N}\}$. Obviously $V = F(V)$. By arguing similarly as in the proof of Theorem 3, we deduce that V is relatively compact. Hence $T^{-1}(0)$ is relatively compact. Since it is closed, it is compact.

To complete our proof, it is enough to show that 0 is an interior point of $\bigcap_{n=1}^{\infty} T_n(C)$. We shall prove that $C \subset (I - F_k)(C)$ for every $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and $z \in C$. Define a sequence (u_i), $u_i \in C$ in the following way:

$$u_1(x,y) = z(x,y), \qquad (x,y) \in K\left(\frac{1}{k}\right)$$

$\widetilde{u_i}(x,y)$ is a continuous extension of $u_i(x,y)$ from $K\left(\frac{i}{k}\right)$ to $I \times I$,

$$u_{i+1}(x,y) = u_i(x,y) \qquad \text{for } (x,y) \in K\left(\frac{i}{k}\right),$$

$$u_{i+1}(x,y) = z(x,y) + F_k(\widetilde{u_i})(x,y) \quad \text{for } (x,y) \in K\left(\frac{i+1}{k}\right) \setminus K\left(\frac{i}{k}\right).$$

Put $u(x,y) = \lim_{i\to\infty} u_i(x,y)$. This convergence is uniform on every compact subset of $I \times I$. Hence in view of the continuity of F_k, we obtain $u = z + F_k(u)$, so $z \in (I - F_k)(C)$.

In view of Theorem 2 the set $T^{-1}(0)$ is an R_δ, which completes our proof. $\square$

References

[1] A. Ambrosetti, 'Un teorema di esistenza per le equazioni differenziali negli spazi di Banach', *Rend. Sem. Math. Univ. Padova* **39** (1967), 349-360.

https://doi.org/10.1017/S0004972700030811 Published online by Cambridge University Press

[2] J. Banaś and K. Goebel, 'Measures of noncompactnes in Banach spaces', in *Lecture Notes in Pure and Appl. Math.* **60** (Marcel Dekker, New York and Basel, 1980).

[3] D. Bugajewski, 'Weak solutions of integral equations with weakly singular kernel in Banach spaces', *Comm. Math.* **34** (1994), 49–58.

[4] J.L. Kelley and I. Namioka, *Linear topological spaces* (Van Nostrand, Princeton, 1963).

[5] D. Ozdarska, 'On the equation $x' = g(t, x, x')$ in Banach spaces', *Rad. Mat.* **2** (1991), 363–370.

[6] B. Rzepecki, 'On the existence of solutions of the Darboux problem for the hyperbolic partial differential equations in Banach spaces', *Rend. Sem. Math. Univ. Padova* **76** (1986), 201–206.

[7] S. Szufla, 'Solution sets on nonlinear equation', *Bull. Acad. Polon. Scien.* **11** (1973), 971–976.

Faculty of Mathematics and Computer Science
A. Mickiewicz University
Matejki 48/49
60-769 Poznań
Poland
e-mail: dbw@math.amu.edu.pl

https://doi.org/10.1017/S0004972700030811 Published online by Cambridge University Press