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THE DIMENSION OF A PRIMITIVE INTERIOR *G*-ALGEBRA

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Abstract. We give the residue class, modulo a certain power of p, for the dimension of a primitive interior G-algebra in terms of the dimension of the source algebra. To illustrate, we improve a theorem of Brauer on the dimension of a block algebra.

Almost always, the G-algebras arising in group representation theory have been interior. Both in applications and in the general theory, it often suffices to consider primitive interior G-algebras. One of the themes of the theory is the characterisation of a primitive interior G-algebra in terms of its source algebra S. Stories revolving around this theme are told in the two books devoted to G-algebra theory, namely Külshammer [**8**], Thévenaz [**15**] and in the papers listed in their bibliographies. We mention particularly Puig [**11**], [**12**]. These stories focus on rich algebraic relationships between A and S; for a start, [**11**, 3.5] tells us that A and S are Morita equivalent. However, many outstanding conjectures, some old and some new, hark back to Brauer's more arithmetical approach to group representation theory. See, for instance, conjectures in Alperin [**1**], Dade [**4**], Feit [**6**, Section 4.6] and Robinson [**13**]. In this note, we point out an arithmetical relationship between A and S. As an illustration, we shall discuss a theorem of Knörr on the dimension of a simply defective module, and shall improve a theorem of Brauer on the dimension of a block algebra. See also Ellers [**5**].

Our notation is as in Thévenaz [**15**]; we repeat a little of it to set the scene, and extend it slightly. Let $\mathcal{O}$ be a complete local noetherian ring with an algebraically closed residue field k of prime characteristic p. Let G be a finite group, and let A be an interior G-algebra; as usual, we assume that A is finitely generated over $\mathcal{O}$, and either free over $\mathcal{O}$ or annihilated by $J(\mathcal{O})$. Given a pointed group H_β on A, we choose an element $j \in \beta$, and define $A_\beta := jAj$ as an interior H-algebra. Now let X be an A-module; again we assume that X is finitely generated over $\mathcal{O}$, and either free over $\mathcal{O}$ or annihilated by $J(\mathcal{O})$. We define $X_\beta := jX$ as an A_β-module. It is easy to extend the use of embeddings in Puig [**12**, 2.13.1] to show that X_β is unique up to a natural isomorphism of A_β-modules.

Henceforth, let us assume that A is primitive. Let P_γ be a defect pointed group on A. The source algebra A associated with P_γ is an interior P-algebra. The multiplicity module $V(\gamma)$ associated with P_γ is a projective indecomposable $k_*\hat{N}(P_\gamma)$-module. By the construction of $V(\gamma)$, if $1_A = \sum_{t \in \mathcal{T}} t$ as a sum of mutually orthogonal primitive idempotents of A^P, then $\dim_k V(\gamma) = |\gamma \cap \mathcal{T}|$.

When $V(\gamma)$ is simple, we say that A is *simply defective*. This notion has its origins in Knörr [**7**], and was introduced explicitly in Picaronny-Puig [**10**]. Necessary and sufficient conditions for A to be simply defective are to be found in [**2**, 1.3], [**10**, Proposition 1], and Thévenaz [**14**, 15, 9.3]. We recall that any block algebra of G over $\mathcal{O}$ or over k is simply defective. Also, the linear endomorphism algebras of certain $\mathcal{O}G$-modules are simply defective (see below). Whenever A is simply defective, the p-part of the dimension of the multiplicity module is

$$(\dim_k V(\gamma))_p = |N_G(P_\gamma) : P|_p.$$

We shall give a formula for the residue class, modulo a certain power of p, for the $\mathcal{O}$-rank $\mathrm{rk}_{\mathcal{O}}A$ (interpreted as the k-dimension $\dim_k A$ when $J(\mathcal{O})$ annihilates A). The terms of the formula are $\dim_k V(\gamma)$, some group-theoretic invariants of A, and a residue class of $\mathrm{rk}_{\mathcal{O}}A_\gamma$. Information about $\dim_k V(\gamma)$ and the group-theoretic invariants is usually much easier to obtain than information about $\mathrm{rk}_{\mathcal{O}}A_\gamma$, so the formula may be seen as a congruence relation between $\mathrm{rk}_{\mathcal{O}}A$ and $\mathrm{rk}_{\mathcal{O}}A_\gamma$. Since A_γ and $V(\gamma)$ are uniquely determined up to a G-conjugacy condition, $\dim_k V(\gamma)$ and $\mathrm{rk}_{\mathcal{O}}A_\gamma$ are isomorphism invariants of A. Similarly, given an A-module X, then $\mathrm{rk}_{\mathcal{O}}X_\gamma$ is an isomorphism invariant of X.

For a p-subgroup $P \leq G$, we define the *spire* of P in G by the formulae

$$\mathrm{spr}_G(P) := \begin{cases} \min\{|P : P \cap {}^g P|\} & \text{if } P \ntrianglelefteq G, \\ 0 & \text{if } P \trianglelefteq G. \end{cases}$$

We interpret congruences modulo zero as equalities; this convention will apply to our results when $P \trianglelefteq G$.

PROPOSITION 1. *Let A be a primitive interior G-algebra, let $P\gamma$ be a defect pointed group on A, and let X be an A-module. Then*

$$\mathrm{rk}_{\mathcal{O}}X \equiv |G : N_G(P_\gamma)|.\dim_k V(\gamma).\mathrm{rk}_{\mathcal{O}}X_\gamma \textit{ modulo } |G : P|_p \mathrm{spr}_G(P).$$

In particular, if A is simply defective, then

$$(\mathrm{rk}_{\mathcal{O}}X)_p \equiv (|G : P|.\mathrm{rk}_{\mathcal{O}}X_\gamma)_p. \textit{ modulo } |G : P|_p \mathrm{spr}_G(P).$$

Proof. If $P \trianglelefteq G$, then the points of P on A are precisely the G-conjugates of γ. Writing $1_A = \sum_{t \in \mathcal{T}} t$ as above, we have

$$\mathrm{rk}_{\mathcal{O}}X = \sum_{gN_G(P_\gamma) \subseteq G} |\mathcal{T} \cap {}^g\gamma|.\mathrm{rk}_{\mathcal{O}}X_{({}^g\gamma)} = |G : N_G(P_\gamma)|.\dim_k V(\gamma).\mathrm{rk}_{\mathcal{O}}X_\gamma.$$

Now suppose that $P \ntrianglelefteq G$. Let $H := N_G(P)$. By the Green Correspondence Theorem in Thévenaz [**15**, 20.1], there exists a unique point β of H on A such that $P_\gamma \leq H_\beta$. Furthermore, β has multiplicity unity; that is to say, if $1_A = \sum_{s \in \mathcal{S}} s$ as a sum of mutually orthogonal primitive idempotents of A^H, then precisely one element of $\mathcal{S}$ belongs to β.

Consider the induced interior G-algebra $A' := \mathrm{Ind}_H^G(A_\beta)$. Recall that $A' = \mathcal{O}G \otimes_{\mathcal{O}H} A_\beta \otimes_{\mathcal{O}H} \mathcal{O}G$ as $\mathcal{O}G$—$\mathcal{O}G$-bimodules, and $A' \cong \mathrm{Mat}_{|G:H|}(A_\beta)$ as algebras. Let $X' := \mathcal{O}G \otimes_{\mathcal{O}H} X_\beta$ as an A'-module. Let γ' and β' be the points of P and H on A' corresponding to γ and β, respectively. Since $P_{\gamma'}$ is a defect pointed subgroup of $H_{\beta'}$, the Green Correspondence Theorem implies that there exists a unique point α' of G on A satisfying $P_{\gamma'} \leq G_{\alpha'}$. Furthermore, α' has multiplicity unity. By Puig [**11**, 3.6], $A'_{\alpha'} \cong A$ as interior G-algebras, and via this isomorphism, $X'_{\alpha'} \cong X$ as A-modules. A routine application of Mackey Decomposition and Rosenberg's Lemma shows that if $Q_{\delta'}$ is a local pointed group on A' not G-conjugate to $P_{\gamma'}$ then Q is contained in the intersection of two distinct G-conjugates of P. Therefore, every point of G on A' distinct from α' has a defect group contained in $P \cap {}^gP$ for some $g \in G - H$. By Green's Indecomposibility Criterion, $|G:P|_p \mathrm{spr}_G(P)$ divides $\mathrm{rk}_{\mathcal{O}} X' - \mathrm{rk}_{\mathcal{O}} X$. We also have $\mathrm{rk}_{\mathcal{O}} X' = |G:H| \mathrm{rk}_{\mathcal{O}} X_\beta$ and, by the first paragraph of the argument,

$$\mathrm{rk}_{\mathcal{O}} X_\beta = |H : N_G(P_\gamma)|. \dim_k V(\gamma).\mathrm{rk}_{\mathcal{O}} X_\gamma. \qquad \square$$

To illustrate Proposition 1, let us consider an indecomposable $\mathcal{O}G$-module M (finitely generated over $\mathcal{O}$, and either free over $\mathcal{O}$ or annihilated by $J(\mathcal{O})$). Let P be a vertex of M, let U be a source $\mathcal{O}P$-module of M, let F be the inertia group of U in $N_G(P)$, and let m be the multiplicity of U as a direct factor of the restricted $\mathcal{O}P$-module of M. The linear endomorphism algebra $\mathrm{End}_{\mathcal{O}}(M)$ (interpreted as $\mathrm{End}_k(M)$ when $J(\mathcal{O})$ annihilates M) is a primitive interior G-algebra with a defect pointed group P_γ such that $M_\gamma \cong U$. Also, $N_G(P_\gamma) = F$, and $\dim_k(V(\gamma)) = m$. By [**2**, 1.4], $\mathrm{End}_{\mathcal{O}}(M)$ is simply defective if and only if m is the multiplicity of M in the induced $\mathcal{O}G$-module of U. When these equivalent conditions hold, we say that M is *simply defective*. If M satisfies the hypothesis of Knörr [**7**, 4.5] (in particular, if M is an irreducible $\mathcal{O}G$-module or a simple kG-module), then by Picaronny-Puig [**10**, Proposition 1] M is simply defective. Proposition 1 implies the following result.

Corollary 2. *Let M be an indecomposable $\mathcal{O}G$-module. With the notation above, we have*

$$\mathrm{rk}_{\mathcal{O}} M \equiv |G:F|.m.\mathrm{rk}_{\mathcal{O}} U \ \textit{modulo} \ |G:P|_p \mathrm{spr}_G(P).$$

In particular, if M is simply defective, then

$$(\mathrm{rk}_{\mathcal{O}} M)_p \equiv (|G:P|.\mathrm{rk}_{\mathcal{O}} U)_p. \ \textit{modulo} \ |G:P|_p \mathrm{spr}_G(P).$$

The rider to Corollary 2 relates to [**7**, 4.5] and [**10**, Proposition 3], but has slightly weaker hypothesis and conclusion.

Lemma 3. *Let G and H be finite groups. Let P_γ and Q_δ be defect pointed groups on, respectively, a primitive G-algebra A and a primitive H-algebra B. Then $\gamma \otimes \delta$ is contained in a local point ε of $P \times Q$ on $A \otimes_{\mathcal{O}} B$, and $(P \times Q)_\varepsilon$ is a defect pointed group on the primitive $G \times H$-algebra $A \otimes B$.*

Proof. It is easy to check that $A \otimes B$ is primitive, and that $\gamma \otimes \delta$ is contained in a point ε of $P \times Q$. By considering the evident isomorphism of Brauer quotients

$$\overline{A}(P) \otimes \overline{B}(Q) \cong \overline{A \otimes B}(P \times Q)$$

we see that ε is local. On the other hand,

$$1_{A \otimes B} \in \mathrm{Tr}_{P \times Q}^{G \times H}(A^P \otimes B^Q.\varepsilon.A^P \otimes B^Q)$$

so that $(P \times Q)_\varepsilon$ is a defect pointed group. $\square$

Theorem 4. *Given a defect pointed group P_γ on a primitive interior G-algebra A, then*

$$\mathrm{rk}_{\mathcal{O}} A \equiv (|G : N_G(P_\gamma)|.\dim_k V(\gamma))^2 \mathrm{rk}_{\mathcal{O}} A_\gamma \ \textit{modulo}\ |G : P|_p^2 \mathrm{spr}_G(P).$$

In particular, if A is simply defective, then

$$(\mathrm{rk}_{\mathcal{O}} A)_p \equiv (|G : P|^2.\mathrm{rk}_{\mathcal{O}} A_\gamma)_p \ \textit{modulo}\ |G : P|_p^2 \mathrm{spr}_G(P).$$

Proof. This follows from Proposition 1 and Lemma 3 upon considering A as an $A \otimes_{\mathcal{O}} A^{op}$-module by left-right translation. □

Let us consider a block idempotent b of $\mathcal{O}G$ with defect group P. Brauer [**3**, Theorem 1] used character theory to prove that the block algebra $\mathcal{O}Gb$ satisfies

$$(\mathrm{rk}_{\mathcal{O}} \mathcal{O}Gb)_p = (|G||G : P|)_p.$$

A module-theoretic demonstration was later given by Michler [**9**, 2.1], and the result is generalised in Picaronny-Puig [**10**, Proposition 3]. Since $\mathcal{O}Gb$ is simply defective, Theorem 4 gives, more precisely, the following result.

Corollary 5. *Let b be a block idempotent of $\mathcal{O}G$. Let (P, e) be a maximal Brauer pair associated with b, let T denote the inertia group of e in $N_G(P)$, and let W be a copy of the isomorphically unique simple $kC_G(P)e$-module. Then*

$$\mathrm{rk}_{\mathcal{O}} \mathcal{O}Gb \equiv (|G| \dim_k W)^2 |Z(P)|/|T||C_G(P)| \textit{modulo} (|G||G : P|)_p \mathrm{spr}_G(P).$$

Proof. By an easy adaptation of part of the argument in Michler [**9**, 2.1], we may and shall assume that $P \trianglelefteq G$. Thévenaz [**15**, 40.13] describes a defect pointed group P_γ on $\mathcal{O}Gb$ associated with (P, e), and also informs us that $T = N_G(P_\gamma)$ and $\dim_k W = \dim_k V(\gamma)$. By Puig [**12**, 6.6, 14.6], we have

$$\mathrm{rk}_{\mathcal{O}} (\mathcal{O}Gb)_\gamma = |N_G(P_\gamma) : PC_G(P)||P| = |T||Z(P)|/|C_G(P)|. \qquad \square$$

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