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ON m-COVERS AND m-SYSTEMS

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Abstract

Let $\mathcal{A} = \{a_s (\mathrm{mod}\ n_s)\}_{s=0}^{k}$ be a system of residue classes. With the help of cyclotomic fields we obtain a theorem which unifies several previously known results related to the covering multiplicity of $\mathcal{A}$. In particular, we show that if every integer lies in more than $m_0 = \lfloor \sum_{s=1}^{k} 1/n_s \rfloor$ members of $\mathcal{A}$, then for any $a = 0, 1, 2, \ldots$ there are at least $\binom{m_0}{\lfloor a/n_0 \rfloor}$ subsets I of $\{1, \ldots, k\}$ with $\sum_{s \in I} 1/n_s = a/n_0$. We also characterize when any integer lies in at most m members of $\mathcal{A}$, where m is a fixed positive integer.

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1. Main results

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, we simply denote the residue class

$$\{x \in \mathbb{Z} \mid x \equiv a \pmod{n}\}$$

by $a(n)$. For a finite system

$$A = \{a_s(n_s)\}_{s=1}^{k} \tag{1.1}$$

of residue classes, the function $w_A : \mathbb{Z} \to \mathbb{N} = \{0, 1, 2, \ldots\}$ given by

$$w_A(x) = |\{1 \leqslant s \leqslant k \mid x \in a_s(n_s)\}| \tag{1.2}$$

is called the *covering function* of A. Obviously $w_A(x)$ is periodic modulo the least common multiple N of the moduli $n_1, \ldots, n_k$, and it is easy to see that the average $\sum_{x=0}^{N-1} w_A(x)/N$ equals $\sum_{s=1}^{k} 1/n_s$. As in [13] we call $m(A) = \min_{x \in \mathbb{Z}} w_A(x)$ the *covering multiplicity* of system (1.1).

Let m be any positive integer. If $w_A(x) \geqslant m$ for all $x \in \mathbb{Z}$ (that is, $m(A) \geqslant m$), then (1.1) is said to be an *m-cover* of $\mathbb{Z}$ as in [11, 12], and in this case $\sum_{s=1}^{k} 1/n_s \geqslant m$. Covers (that is, 1-covers) of $\mathbb{Z}$ were first introduced by Erdős [2] and they are also called covering systems. If $w_A(x) = m$ for all $x \in \mathbb{Z}$, then we call (1.1) an *exact*

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m-cover of $\mathbb{Z}$ as in [12, 13] (and in this case $\sum_{s=1}^{k} 1/n_s = m$). By [8, Theorem 1.3], when $m \geqslant 2$ there are exact m-covers of $\mathbb{Z}$ that cannot split into two covers of $\mathbb{Z}$. If $w_A(x) \leqslant m$ for all $x \in \mathbb{Z}$, then we call (1.1) an *m-system*, and in this case $\sum_{s=1}^{k} 1/n_s \leqslant m$; any 1-system is said to be *disjoint*.

The reader may consult Guy [5, pp. 383–390] and Simpson [9] for some problems and results in covering theory. Covers of $\mathbb{Z}$ have many surprising applications; see, for example [1], [5, Sections A19 and B21], [14, 20, 21]. Sun [19] showed that m-covers of $\mathbb{Z}$ are related to zero-sum problems for abelian groups. Also, the topic of covering systems stimulated the birth of some new algebraic results (see [15, 17]).

Throughout this paper, for $a, b \in \mathbb{Z}$ we set $[a, b] = \{x \in \mathbb{Z} \mid a \leqslant x \leqslant b\}$ and define $[a, b)$ and $(a, b]$ similarly. As usual, the integral part and the fractional part of a real number α are denoted by $\lfloor \alpha \rfloor$ and $\{\alpha\}$, respectively.

For system (1.1) we define its *dual system* A^* by

$$A^* = \{a_s + r(n_s) \mid 1 \leqslant r < n_s,\ 1 \leqslant s \leqslant k\}. \tag{1.3}$$

As $\{a_s + r(n_s)\}_{r=0}^{n_s-1}$ is a partition of $\mathbb{Z}$ for any $s \in [1, k]$, we have $w_A(x) + w_{A^*}(x) = k$ for all $x \in \mathbb{Z}$. Thus $w_A(x) \leqslant m$ for all $x \in \mathbb{Z}$ if and only if $w_{A^*}(x) \geqslant k - m$ for all $x \in \mathbb{Z}$. This simple and new observation shows that we can study m-systems via covers of $\mathbb{Z}$, and construct covers of $\mathbb{Z}$ via m-systems.

By a result in [12], if (1.1) is an m-cover of $\mathbb{Z}$ then for any $m_1, \ldots, m_k \in \mathbb{Z}^+$ there are at least m positive integers in the form $\sum_{s\in I} m_s/n_s$ with $I \subseteq [1, k]$. Applying this result to the dual A^* of an m-system (1.1), we obtain that there are more than $k - m$ integers in the form $\sum_{s=1}^{k} x_s/n_s$ with $x_s \in [0, n_s)$; equivalently, at most $m - 1$ of the numbers in $[1, k]$ cannot be written in the form $\sum_{s=1}^{k} m_s/n_s = k - \sum_{s=1}^{k}(n_s - m_s)/n_s$ with $m_s \in [1, n_s]$. This implies the following result stated in [16, Remark 1.3]: if (1.1) is an m-system, then there are $m_1, \ldots, m_k \in \mathbb{Z}^+$ such that $\sum_{s=1}^{k} m_s/n_s = m$.

The following theorem unifies and generalizes several known results.

THEOREM 1.1. *Let $\mathcal{A} = \{a_s(n_s)\}_{s=0}^{k}$ be a finite system of residue classes with $m(\mathcal{A}) > m = \lfloor \sum_{s=1}^{k} m_s/n_s \rfloor$, where $m_1, \ldots, m_k \in \mathbb{Z}^+$. Then, for any $0 \leqslant \alpha < 1$, either*

$$\sum_{\substack{I\subseteq[1,k] \\ \sum_{s\in I} m_s/n_s=(\alpha+a)/n_0}} (-1)^{|I|} \exp\left(2\pi i \sum_{s\in I} \frac{a_s m_s}{n_s}\right) = 0 \tag{1.4}$$

for any $a \in \mathbb{N}$, or

$$\left|\left\{I \subseteq [1, k] : \sum_{s\in I} \frac{m_s}{n_s} = \frac{\alpha + a}{n_0}\right\}\right| \geqslant \binom{m}{\lfloor a/n_0 \rfloor} \tag{1.5}$$

for all $a = 0, 1, 2, \ldots$.

EXAMPLE 1.2. Erdős observed that $\{0(2), 0(3), 1(4), 5(6), 7(12)\}$ is a cover of $\mathbb{Z}$ with the moduli

$$n_0 = 2, \quad n_1 = 3, \quad n_2 = 4, \quad n_3 = 6, \quad n_4 = 12$$

distinct. As $\lfloor\sum_{s=1}^{4} 1/n_s\rfloor = 0$, by Theorem 1.1 in the case $\alpha = 0$ we have $\sum_{s\in I} 1/n_s = 1/n_0 = 1/2$ for some $I \subseteq [1, 4]$; we can actually take $I = \{1, 3\}$. Since $\sum_{s=1}^{4} 1/n_s < (5/6 + 1)/n_0 = 11/12$, by Theorem 1.1 in the case $\alpha = 5/6$ the set $\mathcal{I} = \{I \subseteq [1, 4] : \sum_{s\in I} 1/n_s = 5/12\}$ cannot have a single element; in fact, $\mathcal{I} = \{\{1, 4\}, \{2, 3\}\}$ and

$$\begin{aligned}(-1)^{|\{1,4\}|} &\exp(2\pi i(0/n_1 + 7/n_4)) + (-1)^{|\{2,3\}|} \exp(2\pi i(1/n_2 + 5/n_3))\\ &= -\exp(\pi i/6) + \exp(\pi i/6) = 0.\end{aligned}$$

COROLLARY 1.3. *If $\mathcal{A} = \{a_s(n_s)\}_{s=0}^{k}$ is a finite system of residue classes with $w_{\mathcal{A}}(x) > m = \lfloor\sum_{s=1}^{k} 1/n_s\rfloor$ for all $x \in \mathbb{Z}$, then*

$$\left|\left\{I \subseteq [1, k] : \sum_{s\in I} \frac{1}{n_s} = \frac{a}{n_0}\right\}\right| \geqslant \binom{m}{\lfloor a/n_0\rfloor} \quad \text{for all } a \in \mathbb{N}. \tag{1.6}$$

In particular, if (1.1) has covering multiplicity $m(A) = \lfloor\sum_{s=1}^{k} 1/n_s\rfloor$, then

$$\left|\left\{I \subseteq [1, k] : \sum_{s\in I} \frac{1}{n_s} = n\right\}\right| \geqslant \binom{m(A)}{n} \quad \text{for each } n \in \mathbb{N}. \tag{1.7}$$

PROOF. Observe that the left-hand side of (1.4) is nonzero in the case $\alpha = a = 0$. So (1.6) follows from Theorem 1.1 immediately. In the case $n_0 = 1$ this yields the latter result in Corollary 1.3. $\square$

REMARK 1.4. Let (1.1) be an exact m-cover of $\mathbb{Z}$. Then $\sum_{s=1}^{k} 1/n_s = m$ and $\lfloor\sum_{s\in[1,k]\setminus\{t\}} 1/n_s\rfloor = m - 1$ for any $t = 1, \ldots, k$. So Corollary 1.3 implies the following result in [13]: for any $t \in [1, k]$ and $a \in \mathbb{N}$,

$$\left|\left\{I \subseteq [1, k] \setminus \{t\} : \sum_{s\in I} \frac{1}{n_s} = \frac{a}{n_t}\right\}\right| \geqslant \binom{m-1}{\lfloor a/n_t\rfloor}.$$

As $m(A) = \sum_{s=1}^{k} 1/n_s$, the inequality $|\{I \subseteq [1, k] : \sum_{s\in I} 1/n_s = n\}| \geqslant \binom{m}{n}$ also holds for all $n = 0, 1, \ldots, m$, which was first established in [10] by means of the Riemann zeta function.

COROLLARY 1.5. *Let (1.1) be an m-system with $m = \lceil\sum_{s=1}^{k} 1/n_s\rceil$, where $\lceil\alpha\rceil$ denotes the least integer not smaller than a real number α. Then*

$$\left|\left\{\langle m_1, \ldots, m_k\rangle \in \mathbb{Z}^k : m_s \in [1, n_s], \sum_{s=1}^{k} \frac{m_s}{n_s} = n\right\}\right| \geqslant \binom{k-m}{n-m} \tag{1.8}$$

for every $n = m, \ldots, k$.

PROOF. Let $n \in [m, k]$. Clearly the left-hand side of (1.8) coincides with

$$L := \left|\left\{\langle x_1, \ldots, x_k\rangle : x_s \in [0, n_s - 1],\ \sum_{s=1}^{k} \frac{x_s}{n_s} = \sum_{s=1}^{k} \frac{n_s}{n_s} - n = k - n\right\}\right|.$$

Since $\sum_{s=1}^{k} 1/n_s > m - 1$, $w_A(x) = m$ for some $x \in \mathbb{Z}$. As the dual A^* of (1.1) has covering multiplicity $m(A^*) = k - m$, applying Corollary 1.3 to A^* leads to $L \geqslant \binom{k-m}{k-n} = \binom{k-m}{n-m}$. This concludes the proof. □

REMARK 1.6. When (1.1) is an exact m-cover of $\mathbb{Z}$, it was proved in [13] (by a different approach) that for each $n \in \mathbb{N}$ the equation $\sum_{s=1}^{k} x_s/n_s = n$ with $x_s \in [0, n_s)$ has at least $\binom{k-m}{n}$ solutions.

COROLLARY 1.7. *Let $\mathcal{A} = \{a_s(n_s)\}_{s=0}^{k}$ be a finite system of residue classes with $m(\mathcal{A}) > m = \lfloor \sum_{s=1}^{k} m_s/n_s \rfloor$, where $m_1, \ldots, m_k \in \mathbb{Z}^+$. Suppose that $J \subseteq [1, k]$ and $\sum_{s\in I} m_s/n_s = \sum_{s\in J} m_s/n_s$ for no $I \subseteq [1, k]$ with $I \neq J$. Then*

$$\left\{n_0 \sum_{s\in J} \frac{m_s}{n_s}\right\} + \left\{n_0 \sum_{s\in \bar{J}} \frac{m_s}{n_s}\right\} < 1, \tag{1.9}$$

where $\bar{J} = [1, k] \setminus J$. Also

$$\sum_{s\in J} \frac{m_s}{n_s} \geqslant m \quad \text{or} \quad \sum_{s\in \bar{J}} \frac{m_s}{n_s} \geqslant m. \tag{1.10}$$

PROOF. Let $v = \sum_{s\in J} m_s/n_s$, $\alpha = \{n_0 v\}$ and $b = \lfloor n_0 v \rfloor$. Then $(\alpha + b)/n_0 = v$ and

$$\sum_{\substack{I\subseteq[1,k] \\ \sum_{s\in I} m_s/n_s = v}} (-1)^{|I|} \exp\left(2\pi i \sum_{s\in I} \frac{a_s m_s}{n_s}\right) = (-1)^{|J|} \exp\left(2\pi i \sum_{s\in J} \frac{a_s m_s}{n_s}\right) \neq 0.$$

By Theorem 1.1, inequality (1.5) holds for any $a \in \mathbb{N}$. Applying (1.5) with $a = mn_0 + n_0 - 1$ we find that $\sum_{s\in I} m_s/n_s = (\alpha + mn_0 + n_0 - 1)/n_0$ for some $I \subseteq [1, k]$, therefore $\sum_{s=1}^{k} m_s/n_s \geqslant m + (\alpha + n_0 - 1)/n_0$. As $\lfloor \sum_{s=1}^{k} m_s/n_s \rfloor = m$, we must have

$$\left\{\sum_{s=1}^{k} \frac{m_s}{n_s}\right\} \geqslant \frac{\alpha + n_0 - 1}{n_0}, \quad \text{that is, } n_0 - 1 + \alpha \leqslant n_0 \left\{\sum_{s=1}^{k} \frac{m_s}{n_s}\right\} < n_0.$$

Therefore $\alpha \leqslant \{n_0\{\sum_{s=1}^{k} m_s/n_s\}\} = \{n_0 \sum_{s=1}^{k} m_s/n_s\}$, which is equivalent to (1.9).

Inequality (1.5) in the case $a = b$ gives that $\binom{m}{\lfloor b/n_0 \rfloor} \leqslant 1$, thus $\lfloor v \rfloor \in \{0, m\}$. As $n_0\{v\} - \alpha = \lfloor n_0\{v\} \rfloor \leqslant n_0 - 1$, $\{v\} \leqslant (\alpha + n_0 - 1)/n_0 \leqslant \{\sum_{s=1}^{k} m_s/n_s\}$. If $\lfloor v \rfloor = 0$, then $m + v \leqslant m + \{\sum_{s=1}^{k} m_s/n_s\} = \sum_{s=1}^{k} m_s/n_s$ and hence $\sum_{s\in \bar{J}} m_s/n_s \geqslant m$. Therefore (1.10) is valid. We are done. □

REMARK 1.8. Let (1.1) be an exact m-cover of $\mathbb{Z}$. [11, Theorem 4(ii)] asserts that if $\emptyset \neq J \subset [1, k]$ then $\sum_{s\in I} 1/n_s = \sum_{s\in J} 1/n_s$ for some $I \subseteq [1, k]$ with $I \neq J$. This follows from Corollary 1.7, for, $\mathcal{A} = \{a_s(n_s)\}_{s=0}^k$ (where $a_0 = 0$ and $n_0 = 1$) is an $(m+1)$-cover of $\mathbb{Z}$ with $\sum_{s\in J\cup\bar{J}} 1/n_s = \sum_{s=1}^k 1/n_s = m$.

In the 1960s Erdős made the following conjecture: for any system (1.1) with $1 < n_1 < \cdots < n_k$, if it is a cover of $\mathbb{Z}$ then $\sum_{s=1}^k 1/n_s > 1$, in other words it cannot be a disjoint cover of $\mathbb{Z}$. This was later confirmed by H. Davenport, L. Mirsky, D. Newman and R. Radó who proved that if (1.1) is a disjoint cover of $\mathbb{Z}$ with $1 < n_1 \leqslant \cdots \leqslant n_{k-1} \leqslant n_k$ then $n_{k-1} = n_k$.

COROLLARY 1.9. *Let (1.1) be an m-cover of $\mathbb{Z}$ with*

$$n_1 \leqslant \cdots \leqslant n_{k-l} < n_{k-l+1} = \cdots = n_k \quad (0 < l < k). \tag{1.11}$$

Then, for any $r \in [0, l]$ with $r < n_k/n_{k-l}$, either $\sum_{s=1}^{k-r} 1/n_s \geqslant m$ or

$$\binom{l}{r} \in D(n_k) = \left\{ \sum_{p\mid n_k} p x_p \mid x_p \in \mathbb{N} \text{ for any prime divisor } p \text{ of } n_k \right\}.$$

PROOF. Set $\mathcal{A} = \{a_s(n_s)\}_{s=0}^k$ where $a_0 = 0$ and $n_0 = 1$. Suppose that $\sum_{s=1}^{k-r} 1/n_s < m$. Then $\sum_{s=1}^k 1/n_s < m + r/n_k < m + 1 \leqslant m(\mathcal{A})$. Since

$$\left|\left\{ I \subseteq [1, k] : \sum_{s\in I} \frac{1}{n_s} = m + \frac{r}{n_k} \right\}\right| = 0 < \binom{m}{m},$$

by Theorem 1.1 we must have

$$\sum_{\substack{I\subseteq[1,k]\\ \sum_{s\in I} 1/n_s = r/n_k}} (-1)^{|I|} \exp\left(2\pi i \sum_{s\in I} \frac{a_s}{n_s}\right) = 0.$$

Observe that $r/n_k < 1/n_{k-l} = \min\{1/n_s \mid 1 \leqslant s \leqslant k-l\}$. Therefore,

$$0 = \sum_{\substack{I\subseteq(k-l,k]\\ \sum_{s\in I} 1/n_s = r/n_k}} (-1)^{|I|} \exp\left(2\pi i \sum_{s\in I} \frac{a_s}{n_s}\right) = (-1)^r \Sigma_r,$$

where

$$\Sigma_r = \sum_{\substack{I\subseteq(k-l,k]\\ |I|=r}} \exp\left(2\pi i \sum_{s\in I} \frac{a_s}{n_k}\right).$$

By [16, Lemma 3.1], $\Sigma_r = 0$ implies that

$$\binom{l}{r} = |\{I \subseteq (k-l, k] : |I| = r\}| \in D(n_k).$$

This concludes the proof. □

REMARK 1.10. Let (1.1) be an m-cover of $\mathbb{Z}$ with (1.11). By Corollary 1.9 in the case $r = l$, either $l \geqslant n_k/n_{k-l} > 1$ or $\sum_{s=1}^{k-l} 1/n_s \geqslant m$; this is one of the main results in [12]. Corollary 1.9 in the case $r = 1$ yields that either $\sum_{s=1}^{k-1} 1/n_s \geqslant m$ or $l \in D(n_k)$; this implies the extended Newman–Znám result (see [7]) which asserts that if (1.1) is an exact m-cover of $\mathbb{Z}$ (and hence $\sum_{s=1}^{k-1} 1/n_s < \sum_{s=1}^{k} 1/n_s = m$) then l is not smaller than the least prime divisor of n_k.

Let (1.1) be an m-system with (1.11), and let $r \in \mathbb{N}$ and $r < n_k/n_{k-l}$. With the help of the dual system of (1.1), we can also show that either $\sum_{s=1}^{k} 1/n_s \leqslant m - r/n_k$ or

$$\binom{l+r-1}{r} = |\{\langle x_{k-l+1}, \dots, x_k\rangle \in \mathbb{N}^l \mid x_{k-l+1} + \cdots + x_k = r\}| \in D(n_k).$$

If (1.1) is disjoint with $1 < n_1 < \cdots < n_k$, then $\sum_{s=1}^{k} 1/n_s < 1$ since (1.1) is not a disjoint cover of $\mathbb{Z}$; Erdős [3] showed further that $\sum_{s=1}^{k} 1/n_s \leqslant 1 - 1/2^k$. Now we give a generalization of this result.

THEOREM 1.11. *Let (1.1) be an m-system with $k \geqslant m$, $\sum_{s=1}^{k} 1/n_s \neq m$ and $n_1 \leqslant \cdots \leqslant n_k$. Then*

$$\sum_{s=1}^{k} \frac{1}{n_s} \leqslant m - \frac{1}{2^{k-m+1}}, \tag{1.12}$$

and equality holds if and only if $n_s = 2^{\max\{s-m+1,0\}}$ for all $s = 1, \dots, k$.

REMARK 1.12. Let $k \geqslant m \geqslant 1$ be integers. Then $m - 1$ copies of $0(1)$, together with the $k - m + 1$ residue classes

$$1(2),\ 2(2^2), \dots, 2^{k-m}(2^{k-m+1}),$$

form an m-system with the moduli $2^{\max\{s-m+1,0\}}$ ($s = 1, \dots, k$).

We will prove Theorems 1.1 and 1.11 in the next section. Section 3 deals with two characterizations of m-systems one of which is as follows.

THEOREM 1.13. *System (1.1) is an m-system if and only if, for any $n \in [m, k)$,*

$$S(n, \alpha) = \begin{cases} (-1)^k & \text{if } \alpha = 0, \\ 0 & \text{if } 0 < \alpha < 1, \end{cases} \tag{1.13}$$

where $S(n, \alpha)$ represents the sum

$$\sum_{\substack{m_1,\dots,m_k \in \mathbb{Z}^+ \\ \{\sum_{s=1}^{k} m_s/n_s\} = \alpha}} (-1)^{\lfloor \sum_{s=1}^{k} m_s/n_s \rfloor} \binom{n}{\lfloor \sum_{s=1}^{k} m_s/n_s \rfloor} \exp\left(2\pi i \sum_{s=1}^{k} \frac{a_s m_s}{n_s}\right).$$

Theorem 1.13 in the case $m = 1$ yields the following result.

COROLLARY 1.14. *If (1.1) is disjoint, then*

$$\sum_{\substack{m_1,\ldots,m_k\in\mathbb{Z}^+\\ \sum_{s=1}^k m_s/n_s=1}} \exp\left(2\pi i\sum_{s=1}^k \frac{a_s m_s}{n_s}\right) = (-1)^{k-1}. \tag{1.14}$$

A residue class $a(n) = a + n\mathbb{Z}$ is a coset of $n\mathbb{Z}$ in the additive group $\mathbb{Z}$ with $[\mathbb{Z} : n\mathbb{Z}] = n$. In [18] the author conjectured that if $\{a_s G_s\}_{s=1}^k$ $(1 < k < \infty)$ is a disjoint system of left cosets in a group G with all the indices $n_s = [G : G_s]$ finite, then $\gcd(n_s, n_t) \geqslant k$ for some $1 \leqslant s < t \leqslant k$.

2. Proofs of Theorems 1.1 and 1.11

LEMMA 2.1. *Let $N \in \mathbb{Z}^+$ be a common multiple of the moduli $n_1, \ldots, n_k$ in (1.1). And let $m, m_1, \ldots, m_k \in \mathbb{Z}^+$. If (1.1) is an m-cover of $\mathbb{Z}$, then $(1 - z^N)^m$ divides the polynomial $\prod_{s=1}^k (1 - z^{Nm_s/n_s}\exp(2\pi i a_s m_s/n_s))$. When $m_1, \ldots, m_k$ are relatively prime to $n_1, \ldots, n_k$ respectively, the converse also holds.*

PROOF. For any $r = 0, 1, \ldots, N-1$, clearly $\exp(2\pi i r/N)$ is a zero of the polynomial $\prod_{s=1}^k (1 - z^{Nm_s/n_s}\exp(2\pi i a_s m_s/n_s))$ with multiplicity $M_r = |\{s \in [1, k] : n_s \mid m_s(r + a_s)\}|$. Observe that $M_r \geqslant w_A(-r)$. If m_s is relatively prime to n_s for each $s \in [1, k]$, then $M_r = w_A(-r)$. As $(1 - z^N)^m = \prod_{r=0}^{N-1}(1 - z\exp(-2\pi i r/N))^m$, the desired result follows from the above. □

PROOF OF THEOREM 1.1. Set $m_0 = 1$, and let N_0 be the least common multiple of $n_0, n_1, \ldots, n_k$. In light of Lemma 2.1, we can write

$$P(z) = \prod_{s=0}^k \left(1 - z^{N_0 m_s/n_s}\exp\left(2\pi i\frac{a_s m_s}{n_s}\right)\right)$$

in the form $(1 - z^{N_0})^{m+1}Q(z)$ where $Q(z) \in \mathbb{C}[z]$. Clearly

$$\deg Q = \deg P - (m+1)N_0 = N_0\left(\sum_{s=0}^k \frac{m_s}{n_s} - m - 1\right) < \frac{N_0}{n_0}.$$

Also

$$\prod_{s=1}^k \left(1 - z^{N_0 m_s/n_s}\exp\left(2\pi i\frac{a_s m_s}{n_s}\right)\right) = \sum_{n=0}^m (-1)^n \binom{m}{n} z^{nN_0} \sum_{r=0}^{n_0-1} z^{rN_0/n_0}\exp\left(2\pi i r\frac{a_0}{n_0}\right) Q(z) \tag{2.1}$$

since

$$\frac{1 - z^{N_0}}{1 - z^{N_0/n_0}\exp(2\pi i a_0/n_0)} = \sum_{r=0}^{n_0-1} z^{rN_0/n_0}\exp\left(2\pi i r\frac{a_0}{n_0}\right).$$

Let $a \in \mathbb{N}$ and

$$C_a = (-1)^{\lfloor a/n_0 \rfloor} \sum_{\substack{I \subseteq [1,k] \\ \sum_{s\in I} m_s/n_s = (\alpha+a)/n_0}} (-1)^{|I|} \exp\bigg(2\pi i \sum_{s\in I} (a_s - a_0)\frac{m_s}{n_s}\bigg).$$

By comparing the coefficients of $z^{N_0(\alpha+a)/n_0}$ on both sides of (2.1) we obtain that

$$\sum_{\substack{I \subseteq [1,k] \\ \sum_{s\in I} m_s/n_s = (\alpha+a)/n_0}} (-1)^{|I|} \exp\bigg(2\pi i \sum_{s\in I} \frac{a_s m_s}{n_s}\bigg)$$
$$= (-1)^{\lfloor a/n_0 \rfloor} \binom{m}{\lfloor a/n_0 \rfloor} \exp(2\pi i a_0 \{a/n_0\}) [z^{\alpha N_0/n_0}] Q(z),$$

where $[z^{\alpha N_0/n_0}]Q(z)$ denotes the coefficient of $z^{\alpha N_0/n_0}$ in $Q(z)$. Therefore

$$C_a = \exp\bigg(-2\pi i \alpha \frac{a_0}{n_0}\bigg) \binom{m}{\lfloor a/n_0 \rfloor} [z^{\alpha N_0/n_0}] Q(z) = \binom{m}{\lfloor a/n_0 \rfloor} C_0. \tag{2.2}$$

For an algebraic integer ω in the field $K = \mathbb{Q}(\exp(2\pi i/N_0))$, the norm

$$N(\omega) = \prod_{1 \leqslant r \leqslant N_0,\ \gcd(r, N_0)=1} \sigma_r(\omega)$$

(with respect to the field extension $K/\mathbb{Q}$) is a rational integer, where σ_r is the automorphism of K (in the Galois group $\mathrm{Gal}(K/\mathbb{Q})$) induced by $\sigma_r(\exp(2\pi i/N_0)) = \exp(2\pi i r/N_0)$. (See, for example, [6, Ch. 1].) As $N((-1)^{\lfloor a/n_0 \rfloor} C_a)$ equals

$$\prod_{\substack{1 \leqslant r \leqslant N_0 \\ \gcd(r,N_0)=1}} \sum_{\substack{I \subseteq [1,k] \\ \sum_{s\in I} m_s/n_s = (\alpha+a)/n_0}} (-1)^{|I|} \exp\bigg(2\pi i r \sum_{s\in I} (a_s - a_0)\frac{m_s}{n_s}\bigg),$$

we have

$$|N(C_a)| = \prod_{\substack{1 \leqslant r \leqslant N_0 \\ \gcd(r,N_0)=1}} \bigg| \sum_{\substack{I \subseteq [1,k] \\ \sum_{s\in I} m_s/n_s = (\alpha+a)/n_0}} (-1)^{|I|} \exp\bigg(2\pi i r \sum_{s\in I} (a_s - a_0)\frac{m_s}{n_s}\bigg) \bigg|$$
$$\leqslant \bigg| \bigg\{ I \subseteq [1, k] : \sum_{s\in I} \frac{m_s}{n_s} = \frac{\alpha + a}{n_0} \bigg\} \bigg|^{\varphi(N_0)},$$

where φ is Euler's totient function. Also

$$|N(C_a)| = \bigg| N\bigg(\binom{m}{\lfloor a/n_0 \rfloor}\bigg) \bigg| \times |N(C_0)| = \binom{m}{\lfloor a/n_0 \rfloor}^{\varphi(N_0)} |N(C_0)|.$$

Suppose that $C_b \neq 0$ for some $b \in \mathbb{N}$. Then $N(C_b) \neq 0$, and hence $N(C_0) \in \mathbb{Z}$ is nonzero. For any $a \in \mathbb{N}$,

$$\left|\left\{I \subseteq [1, k] : \sum_{s\in I} \frac{m_s}{n_s} = \frac{\alpha + a}{n_0}\right\}\right|^{\varphi(N_0)} \geqslant |N(C_a)| \geqslant \binom{m}{\lfloor a/n_0 \rfloor}^{\varphi(N_0)},$$

and hence (1.5) holds. This concludes the proof. □

PROOF OF THEOREM 1.11. We use induction on k.

In the case $k = m$, we have $n_k > 1$ and hence

$$\sum_{s=1}^{k} \frac{1}{n_s} \leqslant k - 1 + \frac{1}{n_k} \leqslant m - \frac{1}{2} = m - \frac{1}{2^{k-m+1}};$$

also $\sum_{s=1}^{k} 1/n_s = m - 1/2$ if and only if $n_1 = \cdots = n_{k-1} = 1$ and $n_k = 2$.

Now let $k > m$. Clearly $\sum_{s=1}^{k-1} 1/n_s < \sum_{s=1}^{k} 1/n_s < m$. Assume that

$$\sum_{s=1}^{k-1} \frac{1}{n_s} \leqslant m - \frac{1}{2^{(k-1)-m+1}} = m - \frac{1}{2^{k-m}}$$

and that equality holds if and only if $n_s = 2^{\max\{s-m+1,0\}}$ for all $s \in [1, k-1]$. When $n_k > 2^{k-m+1}$,

$$\sum_{s=1}^{k} \frac{1}{n_s} = \sum_{s=1}^{k-1} \frac{1}{n_s} + \frac{1}{n_k} < \left(m - \frac{1}{2^{k-m}}\right) + \frac{1}{2^{k-m+1}} = m - \frac{1}{2^{k-m+1}}.$$

If $\sum_{s=1}^{k} 1/n_s > m - 1/n_k$, then $\lceil \sum_{s=1}^{k} 1/n_s \rceil = m$, thus $\sum_{s=1}^{k} m_s/n_s = m$ for some $m_1, \ldots, m_k \in \mathbb{Z}^+$ (by Corollary 1.5), and hence

$$m - \sum_{s=1}^{k} \frac{1}{n_s} \geqslant \min\left\{\frac{1}{n_s} \mid 1 \leqslant s \leqslant k\right\} = \frac{1}{n_k}.$$

This shows that indeed $\sum_{s=1}^{k} 1/n_s \leqslant m - 1/n_k$. Provided that $n_k \leqslant 2^{k-m+1}$, inequality (1.12) holds, and also

$$\sum_{s=1}^{k} \frac{1}{n_s} = m - \frac{1}{2^{k-m+1}} \iff n_k = 2^{k-m+1} \quad \text{and} \quad \sum_{s=1}^{k-1} \frac{1}{n_s} = m - \frac{1}{2^{k-m}}$$
$$\iff n_s = 2^{\max\{s-m+1,0\}} \quad \text{for } s = 1, \ldots, k-1, k.$$

This concludes the induction step and we are done. □

3. Characterizations of m-systems

Proof of Theorem 1.13. Like Lemma 2.1, system (1.1) is an m-system if and only if $f(z)=(1-z^N)^m/\prod_{s=1}^k(1-z^{N/n_s}\exp(2\pi i a_s/n_s))$ is a polynomial, where N is the least common multiple of $n_1,\ldots,n_k$.

Set $c=m-\sum_{s=1}^k 1/n_s$. If $f(z)$ is a polynomial, then $\deg f=cN$ and $[z^{cN}]f(z)=(-1)^{k-m}\exp(-2\pi i\sum_{s=1}^k a_s/n_s)$.

For $|z|<1$,

$$f(z)=\sum_{n=0}^m\binom{m}{n}(-1)^n z^{nN}\prod_{s=1}^k\sum_{x_s=0}^\infty \exp\Big(2\pi i\frac{a_sx_s}{n_s}\Big)z^{Nx_s/n_s}.$$

Let $\alpha\geqslant 0$. Then

$$\begin{aligned}[z^{(c+\alpha)N}]f(z)&=\sum_{n=0}^m(-1)^n\binom{m}{n}\sum_{\substack{x_1,\ldots,x_k\in\mathbb{N}\\ \sum_{s=1}^k x_s/n_s=c+\alpha-n}}\exp\Big(2\pi i\sum_{s=1}^k\frac{a_sx_s}{n_s}\Big)\\ &=\sum_{n=0}^m(-1)^n\binom{m}{n}\sum_{\substack{m_1,\ldots,m_k\in\mathbb{Z}^+\\ \sum_{s=1}^k m_s/n_s=\alpha+m-n}}\exp\Big(2\pi i\sum_{s=1}^k\frac{a_s(m_s-1)}{n_s}\Big)\\ &=(-1)^m\exp\Big(-2\pi i\sum_{s=1}^k\frac{a_s}{n_s}\Big)S(m,\alpha),\end{aligned}$$

where $S(n,\alpha)$ $(n\in\mathbb{N})$ represents the sum

$$\sum_{\substack{m_1,\ldots,m_k\in\mathbb{Z}^+\\ \sum_{s=1}^k m_s/n_s-\alpha\in\mathbb{N}}}(-1)^{\sum_{s=1}^k m_s/n_s-\alpha}\binom{n}{\sum_{s=1}^k m_s/n_s-\alpha}\exp\Big(2\pi i\sum_{s=1}^k\frac{a_sm_s}{n_s}\Big)$$

which agrees with its definition in the case $0\leqslant\alpha<1$ given in Theorem 1.13.

(i) Suppose that (1.1) is an m-system. Then $f(z)$ is a polynomial of degree cN and hence

$$S(m,\alpha)=(-1)^m\exp\Big(2\pi i\sum_{s=1}^k\frac{a_s}{n_s}\Big)[z^{(c+\alpha)N}]f(z)=\begin{cases}(-1)^k & \text{if } \alpha=0,\\ 0 & \text{if } \alpha>0.\end{cases}$$

For any integer $n\geqslant m$, (1.1) is also an n-system and so (1.13) holds.

(ii) Now assume that (1.13) holds for all $n\in[m,k)$. For any $n\geqslant k$, we also have (1.13) by (i) because (1.1) is a k-system. If $0<\alpha<1$ then $S(n,\alpha)=0$ for any integer $n\geqslant m$. Fix $\alpha>0$. If $S(n,\alpha)=0$ for all integers $n\geqslant m$, then for any integer $n\geqslant m$,

$$S(n,\alpha+1)=S(n,\alpha)-S(n+1,\alpha)=0$$

because $\binom{n}{j-1}=\binom{n+1}{j}-\binom{n}{j}$ for $j=1,2,\ldots$. Thus, by induction, $S(n,\alpha)=0$ for all $\alpha > 0$ and $n = m, m+1, \ldots$. It follows that $[z^{(c+\alpha)N}]f(z) = 0$ for any $\alpha > 0$. So $f(z)$ is a polynomial and (1.1) is an m-system.

The proof of Theorem 1.13 is now complete. □

The following characterization of m-covers plays important roles in [11, 12].

LEMMA 3.1 (Sun [11]). *Let $m, m_1, \ldots, m_k \in \mathbb{Z}^+$. If (1.1) forms an m-cover of $\mathbb{Z}$, then*

$$\sum_{\substack{I \subseteq [1,k] \\ \{\sum_{s\in I} m_s/n_s\}=\theta}} (-1)^{|I|} \binom{\lfloor \sum_{s\in I} m_s/n_s \rfloor}{n} \exp\Big(2\pi i \sum_{s\in I} \frac{a_s m_s}{n_s}\Big) = 0 \tag{3.1}$$

for all $0 \leqslant \theta < 1$ and $n = 0, 1, \ldots, m-1$. The converse also holds if $m_1, \ldots, m_k$ are relatively prime to $n_1, \ldots, n_k$, respectively.

We can provide a new proof of Lemma 3.1 in a way similar to the proof of Theorem 1.13.

LEMMA 3.2. *Let $n \in \mathbb{Z}^+$ and $l \in [0, n-1]$. Then*

$$\sum_{\substack{J \subseteq [1,n) \\ |J|=l}} \exp\Big(2\pi i \sum_{j\in J} \frac{j}{n}\Big) = (-1)^l. \tag{3.2}$$

PROOF. Clearly we have the identity

$$\prod_{0<j<n} (1 - z e^{2\pi i j/n}) = \frac{1-z^n}{1-z} = 1 + z + \cdots + z^{n-1}.$$

Comparing the coefficients of z^l we then obtain (3.2). □

Using Lemmas 3.1 and 3.2 we can deduce another characterization of m-systems.

THEOREM 3.3. *System (1.1) is an m-system if and only if*

$$\sum_{\substack{x_s\in[0,n_s) \text{ for } s\in[1,k] \\ \{\sum_{s=1}^k x_s/n_s\}=\theta}} \binom{\lfloor \sum_{s=1}^k x_s/n_s \rfloor}{n} \exp\Big(2\pi i \sum_{s=1}^k \frac{a_s x_s}{n_s}\Big) = 0 \tag{3.3}$$

for all $0 \leqslant \theta < 1$ and $n \in [0, k-m)$.

PROOF. The case $k \leqslant m$ is trivial, so we just let $k > m$. Recall that (1.1) is an m-system if and only if its dual A^* is a $(k-m)$-cover of $\mathbb{Z}$.

By Lemma 3.1 in the case $m_1 = \cdots = m_k = 1$, A^* forms a $(k-m)$-cover of $\mathbb{Z}$ if and only if for any $0 \leqslant \theta < 1$ and $n \in [0, k-m)$ the sum

$$\sum_{\substack{x_s\in[0,n_s) \text{ for } s\in[1,k] \\ \{\sum_{s=1}^k x_s/n_s\}=\theta}} (-1)^{\sum_{s=1}^k x_s} \binom{\lfloor \sum_{s=1}^k x_s/n_s \rfloor}{n} \exp\Big(2\pi i \sum_{s=1}^k \frac{a_s x_s}{n_s}\Big) \prod_{s=1}^k f_s(x_s)$$

vanishes, where

$$f_s(x_s)=\sum_{\substack{J\subseteq[1,n_s)\\ |J|=x_s}}\exp\Big(2\pi i\sum_{j\in J}\frac{j}{n_s}\Big)=(-1)^{x_s}$$

by Lemma 3.2. This concludes the proof. □

The following consequence extends Corollary 1.14.

COROLLARY 3.4. *Let (1.1) be an m-system. Then we have*

$$\sum_{\substack{m_s\in[1,n_s]\ for\ s\in[1,k]\\ m-\sum_{s=1}^k m_s/n_s\in\mathbb{N}}}\binom{k-1-\sum_{s=1}^k m_s/n_s}{m-\sum_{s=1}^k m_s/n_s}\exp\Big(2\pi i\sum_{s=1}^k\frac{a_sm_s}{n_s}\Big)=(-1)^{k-m}.$$

PROOF. If $k\leqslant m$, then the left-hand side of the last equality coincides with

$$\binom{k-1-\sum_{s=1}^k n_s/n_s}{m-\sum_{s=1}^k n_s/n_s}\exp\Big(2\pi i\sum_{s=1}^k\frac{a_sn_s}{n_s}\Big)=\binom{-1}{m-k}=(-1)^{m-k}.$$

Now let $k>m$. As $\{-a_s(n_s)\}_{s=1}^k$ is an m-system, by Theorem 3.3 and the identity

$$(-1)^{k-m-1}\binom{x-1}{k-m-1}=\sum_{n=0}^{k-m-1}(-1)^n\binom{x}{n}$$

(see [4, (5.16)]) we have

$$\begin{aligned}0&=\sum_{\substack{x_s\in[0,n_s)\text{ for }s\in[1,k]\\ \{\sum_{s=1}^k x_s/n_s\}=0}}\binom{\lfloor\sum_{s=1}^k x_s/n_s\rfloor-1}{k-m-1}\exp\Big(2\pi i\sum_{s=1}^k\frac{-a_sx_s}{n_s}\Big)\\ &=\sum_{\substack{m_s\in[1,n_s]\text{ for }s\in[1,k]\\ \sum_{s=1}^k (n_s-m_s)/n_s\in\mathbb{N}}}\binom{\sum_{s=1}^k(n_s-m_s)/n_s-1}{k-m-1}\exp\Big(-2\pi i\sum_{s=1}^k\frac{a_s(n_s-m_s)}{n_s}\Big)\\ &=\sum_{\substack{m_s\in[1,n_s]\text{ for }s\in[1,k]\\ \sum_{s=1}^k m_s/n_s\in[0,k-1]}}\binom{k-1-\sum_{s=1}^k m_s/n_s}{k-1-m}\exp\Big(2\pi i\sum_{s=1}^k\frac{a_sm_s}{n_s}\Big)\\ &\quad+\binom{k-1-\sum_{s=1}^k n_s/n_s}{k-1-m}\exp\Big(2\pi i\sum_{s=1}^k\frac{a_sn_s}{n_s}\Big).\end{aligned}$$

So the desired equality follows. □

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