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# ON 4-ENGEL GROUPS

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*Abstract*

In this note we prove that a group $G$ is a 4-Engel group if and only if the normal closure of every element $g \in G$ is a 3-Engel group.

## 1. *Introduction*

A group $G$ is said to be an $n$-Engel group if

$$[x, \underbrace{y, y, \ldots, y}_{n}] = 1$$

for all $x, y \in G$. (Here $[x, y]$ denotes $x^{-1}y^{-1}xy$, and $[x, y, \ldots, y]$ denotes the left-normed commutator $[[\ldots[x, y], \ldots], y]$.) The structure of 2-Engel groups is straightforward, as they are known to be nilpotent of class at most 3 — see Levi [**9**]. It is easy to see that the normal closure of an element in a 2-Engel group is abelian. Heineken [**7**] proved that 3-Engel groups are locally nilpotent, and Kappe and Kappe [**8**] proved that a group is a 3-Engel group if and only if the normal closure of every element is nilpotent of class at most 2.

In 2005 Havas and Vaughan-Lee [**6**] proved that 4-Engel groups are locally nilpotent. Traustason [**11**] had earlier proved that if $G$ is a locally nilpotent 4-Engel group, and if $g \in G$, then the normal closure of $g$ in $G$ is nilpotent of class at most 4. So Traustason's result, together with Havas and Vaughan-Lee's work, implies that the normal closure of an element in a 4-Engel group is nilpotent of class at most 4. This bound is sharp — Nickel [**10**] showed that if $G$ is the class 6 quotient of the free 4-Engel group of rank three, and if $a, b, c$ generate $G$, then the commutator $[a, b, b, b, c, b]$ has order 10. So the normal closure of an element in a 4-Engel group is nilpotent, but the nilpotency class can be as big as 4. However we are able to show that the normal closure of an element in a 4-Engel group is a 3-Engel group. Conversely, it is easy to see that if the normal closure of every element in a group $G$ is a 3-Engel group, then $G$ is a 4-Engel group. For suppose that $a$ and $b$ are two elements in a group $G$ with the property that the normal closure of every element is a 3-Engel group. Then

$$[b, a, a, a, a] = [[b, a], a, a, a] = 1,$$

since $[b, a]$ is in the normal closure of $a$. So we are able to prove the following theorem.

THEOREM 1. *A group $G$ is a 4-Engel group if and only if the normal closure of every element in $G$ is a 3-Engel group.*

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Note that the results on the normal closure of an element in 2-Engel and 3-Engel groups imply that analogues of this theorem hold in 2-Engel and 3-Engel groups. It is still an open question whether or not 5-Engel groups are locally nilpotent, but the normal closure of an element in a 5-Engel group need not be 4-Engel. If we let $G$ be the largest nilpotent 5-Engel group generated by two elements $a, b$ of order 3, then $G$ is nilpotent of class 9 and has order $3^{17}$. And in $G$

$$[a, [b, a], [b, a], [b, a], [b, a]] \neq 1.$$

Gupta and Levin [3] showed that the normal closure of an element in a 5-Engel group need not be nilpotent. Actually, this is quite easy to see. Let $B$ be the free group of countably infinite rank in the variety of groups which are nilpotent of class 2 and of exponent 3, and let $A$ be the cyclic group of order 3. Let the free generators of $B$ be $b_1, b_2, \ldots$, and let $A$ be generated by $a$. Let $G$ be the wreath product of $A$ with $B$. Then

$$[a, [b_1, b_2], [b_1, b_3], \ldots, [b_1, b_k]]$$

is a non-trivial element of $G$ for all $k \geqslant 2$. So the normal closure of $b_1$ in $G$ is not nilpotent. But it is quite easy to see that $G$ is 5-Engel — in fact $G$ satisfies the identity

$$[w, x, y, z, z, z] = 1.$$

To see this, let $N$ be the normal closure of $A$ in $G$. Then $N$ is an elementary abelian 3-group, and $G/N \cong B$, which is nilpotent of class 2 and has exponent 3. So if $w, x, y, z \in G$ then $[w, x, y]$ and $z^3$ both lie in $N$, and

$$1 = \big[[w, x, y], z^3\big] = [w, x, y, z]^3 [w, x, y, z, z]^3 [w, x, y, z, z, z] = [w, x, y, z, z, z].$$

We apply a series of reductions to prove that the normal closure of an element in a 4-Engel group is a 3-Engel group. First we show that it is sufficient to prove the result for four generator groups. Using the results of Havas and Vaughan-Lee mentioned above, and the results of Traustason [11], the problem reduces to the cases of 4-Engel 5-groups and 4-Engel 2-groups. The case of 4-Engel 5-groups is quite straightforward, and the bulk of the difficulty in the theorem arises from the case of 4-Engel 2-groups.

The key to the proof is Traustason's theorem that the normal closure of an element in a locally nilpotent 4-Engel group is nilpotent of class 4, together with Havas and Vaughan-Lee's theorem that all 4-Engel groups are locally nilpotent. We also use Traustason's theorem from [11] that if $G$ is a locally nilpotent 4-Engel group without elements of order 2 or 5, then the normal closure of an element in $G$ is nilpotent of class at most 3. We shall use these results without further comment.

## 2. *Reduction to four generator groups*

Let $G$ be a 4-Engel group and let $a \in G$. We need to show that $[x, y, y, y] = 1$ for all $x, y$ in the normal closure of $a$ in $G$. We can write $x = a_1 a_2 \ldots a_m$ and write $y = b_1 b_2 \ldots b_n$ where $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ are conjugates of $a$ or inverses of conjugates of $a$. Using the fact that the normal closure of $a$ is nilpotent of class at most 4 we see that

$$[x, y, y, y] = \prod_{1 \leqslant i \leqslant m} [a_i, y, y, y],$$

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and so it is sufficient to show that $[b, y, y, y] = 1$ whenever $b$ is a conjugate of $a$ or the inverse of a conjugate of $a$, and whenever $y$ is in the normal closure of $a$. If $b = a^g$ then

$$[b, y, y, y] = [a^g, y, y, y] = [a, y^{g^{-1}}, y^{g^{-1}}, y^{g^{-1}}]^g,$$

and if $b = a^{-g}$ then

$$[b, y, y, y] = [a^{-g}, y, y, y] = [a, y^{g^{-1}}, y^{g^{-1}}, y^{g^{-1}}]^{-g}.$$

So it is sufficient to show that $[a, y, y, y] = 1$ whenever $y$ is in the normal closure of $a$. Writing $y = b_1 b_2 \ldots b_n$, as above, we have

$$[a, y, y, y] = \prod_{1 \leqslant i,j,k \leqslant n} [a, b_i, b_j, b_k]. \tag{1}$$

It follows that $[a, y, y, y]$ is a product of terms of the following forms (where $b, c, d$ are conjugates of $a$ or inverses of conjugates of $a$):

1. $[a, b, b, b]$,
2. $[a, b, a, b][a, b, b, a]$,
3. $[a, b, a, a]$,
4. $[a, b, b, c][a, b, c, b][a, c, b, b]$,
5. $[a, b, a, c][a, c, a, b][a, b, c, a][a, c, b, a]$,
6. $[a, b, c, d][a, b, d, c][a, c, b, d][a, c, d, b][a, d, b, c][a, d, c, b]$.

There is some redundancy in this list, but this is deliberate. Cases 1 to 5 cover the situation when $i, j, k$ in (1) are not all distinct, or when one of $b_i$, $b_j$, $b_k$ equals $a$. So to show that $[a, y, y, y] = 1$ we need to show that elements of the form 1 to 6 above are all trivial in 4-Engel groups. Note that we need only check that elements of form 1, 2 or 3 are trivial in two generator 4-Engel groups, that elements of form 4 or 5 are trivial in three generator 4-Engel groups, and that elements of form 6 are trivial in four generator 4-Engel groups. It is an easy computation in either GAP [**2**] or Magma [**1**] to calculate the free 4-Engel group of rank two, and it is then a trivial matter to check that elements of form 1 or 2 or 3 are trivial. It is a rather more substantial calculation to compute the free 4-Engel group of rank three — it has class 9 and Hirsch length 88. In fact Magma v2.13-1 gets an arithmetic overflow while computing the class 9 quotient of the group, and fails. But GAP performs the calculation satisfactorily using Werner Nickel's package nq [**10**], and it is then easy to verify that elements of form 4 or 5 are trivial. So we need only show that elements of form 6 are trivial in four generator 4-Engel groups. Unfortunately the free 4-Engel group of rank four is out of reach of both GAP and Magma, and so we need an indirect proof. But we will use the fact that elements of form 1 to 5 are trivial.

## 3. *Reduction to 2-groups and 5-groups*

As we showed in the last section, we need to show that

$$[a, b, c, d][a, b, d, c][a, c, b, d][a, c, d, b][a, d, b, c][a, d, c, b] = 1 \tag{2}$$

whenever $a, b, c, d$ are elements of the free 4-Engel group of rank four with $b, c, d$ either conjugates of $a$ or inverses of conjugates of $a$.

Using the fact that

$$[a,b^{-1},c,d][a,b^{-1},d,c][a,c,b^{-1},d][a,c,d,b^{-1}][a,d,b^{-1},c][a,d,c,b^{-1}]$$
$$= ([a,b,c,d][a,b,d,c][a,c,b,d][a,c,d,b][a,d,b,c][a,d,c,b])^{-1},$$

we see that it is sufficient to deal with the case when $b,c,d$ are conjugates of $a$. Now the free 4-Engel group of rank four is a nilpotent group, and so is residually a finite $p$-group. So it is sufficient to establish (2) whenever $a,b,c,d$ are elements of a finite 4-Engel $p$-group, with $b,c,d$ conjugates of $a$. By Traustason's result, we know that the normal closure of an element in a finite 4-Engel $p$-group is nilpotent of class at most 3 unless $p=2$ or $p=5$. So this reduces the problem to the case of 2-groups and 5-groups.

## 4. 4-*Engel* 5-*groups*

We need to show that

$$[a,b,c,d][a,b,d,c][a,c,b,d][a,c,d,b][a,d,b,c][a,d,c,b]=1$$

whenever $a$ is an element of a finite 4-Engel 5-group $G$ and $b,c,d$ are conjugates of $a$ in $G$. We can write $b=a^x=a[a,x]$ for some $x\in G$, and we can similarly write $c=a[a,y]$ and $d=a[a,z]$. Then, expanding, we obtain

$$\begin{aligned}
&[a,b,c,d][a,b,d,c][a,c,b,d][a,c,d,b][a,d,b,c][a,d,c,b]\\
&\quad= [a,[a,x],a,a]^2\,[a,[a,y],a,a]^2\,[a,[a,z],a,a]^2\\
&\qquad\cdot [a,[a,x],[a,y],a]\,[a,[a,y],[a,x],a]\,[a,[a,x],a,[a,y]]\,[a,[a,y],a,[a,x]]\\
&\qquad\cdot [a,[a,x],[a,z],a]\,[a,[a,z],[a,x],a]\,[a,[a,x],a,[a,z]]\,[a,[a,z],a,[a,x]]\\
&\qquad\cdot [a,[a,y],[a,z],a]\,[a,[a,z],[a,y],a]\,[a,[a,y],a,[a,z]]\,[a,[a,z],a,[a,y]]\\
&\qquad\cdot [a,[a,x],[a,y],[a,z]]\,[a,[a,x],[a,z],[a,y]]\,[a,[a,y],[a,x],[a,z]]\\
&\qquad\cdot [a,[a,y],[a,z],[a,x]]\,[a,[a,z],[a,x],[a,y]]\,[a,[a,z],[a,y],[a,x]]\,.
\end{aligned}$$

We now use the fact that elements of form 1 to 5 above are trivial to simplify this expression. Writing $[a,x]=a^{-1}a^x$ we see that $[a,[a,x],a,a]=[a,a^x,a,a]=1$. Similarly $[a,[a,y],a,a]=[a,[a,z],a,a]=1$. Also

$$\begin{aligned}
&[a,[a,x],[a,y],a]\,[a,[a,y],[a,x],a]\,[a,[a,x],a,[a,y]]\,[a,[a,y],a,[a,x]]\\
&\quad= \left[a,a^x,a^{-1},a\right]\,\left[a,a^y,a^{-1},a\right]\,\left[a,a^x,a,a^{-1}\right]\,\left[a,a^y,a,a^{-1}\right]\\
&\qquad\cdot [a,a^x,a^y,a]\,[a,a^y,a^x,a]\,[a,a^x,a,a^y]\,[a,a^y,a,a^x]\\
&\quad= [a,a^x,a,a]^{-2}\,[a,a^y,a,a]^{-2}\,[a,a^x,a^y,a]\,[a,a^y,a^x,a]\,[a,a^x,a,a^y]\,[a,a^y,a,a^x]\\
&\quad= 1,
\end{aligned}$$

and we similarly see that

$$[a,[a,y],[a,z],a]\,[a,[a,z],[a,y],a]\,[a,[a,y],a,[a,z]]\,[a,[a,z],a,[a,y]]=1$$

and

$$[a,[a,y],[a,z],a]\,[a,[a,z],[a,y],a]\,[a,[a,y],a,[a,z]]\,[a,[a,z],a,[a,y]]=1.$$

So

$$\begin{aligned}&[a,b,c,d][a,b,d,c][a,c,b,d][a,c,d,b][a,d,b,c][a,d,c,b]\\&\qquad=[a,[a,x],[a,y],[a,z]]\,[a,[a,x],[a,z],[a,y]]\,[a,[a,y],[a,x],[a,z]]\\&\qquad\quad\cdot[a,[a,y],[a,z],[a,x]]\,[a,[a,z],[a,x],[a,y]]\,[a,[a,z],[a,y],[a,x]]\,.\end{aligned}$$

Now we set $t=[a,x][a,y][a,z]$, so that $[a,t,t,t]$ is a product of terms of the form $[a,u,v,w]$ where $u,v,w$ lie in the set $\{[a,x],[a,y],[a,z]\}$. Again using the fact that elements of form 1 to 5 above are trivial, we see that in this product all the terms where $\{u,v,w\}\neq\{[a,x],[a,y],[a,z]\}$ cancel with each other. For example, if we pick out the terms where two of $u,v,w$ equal $[a,x]$ and the third equals $[a,y]$, then we obtain

$$[a,[a,x],[a,x],[a,y]]\,[a,[a,x],[a,y],[a,x]]\,[a,[a,y],[a,x],[a,x]]\,.$$

Writing $[a,x]=a^{-1}a^x$, $[a,y]=a^{-1}a^y$, and expanding, this gives

$$\begin{aligned}&[a,a^x,a^{-1},a^{-1}]^2[a,a^y,a^{-1},a^{-1}][a,a^x,a^x,a^{-1}][a,a^x,a^{-1},a^x]\\&\cdot[a,a^x,a^{-1},a^y][a,a^y,a^{-1},a^x][a,a^x,a^y,a^{-1}][a,a^y,a^x,a^{-1}]\\&\cdot[a,a^x,a^x,a^y][a,a^x,a^y,a^x][a,a^y,a^x,a^x]\\&\qquad=[a,a^x,a,a]^2[a,a^y,a,a][a,a^x,a^x,a]^{-1}[a,a^x,a,a^x]^{-1}\\&\qquad\quad\cdot[a,a^x,a,a^y]^{-1}[a,a^y,a,a^x]^{-1}[a,a^x,a^y,a]^{-1}[a,a^y,a^x,a]^{-1}\\&\qquad\quad\cdot[a,a^x,a^x,a^y][a,a^x,a^y,a^x][a,a^y,a^x,a^x]\\&\qquad=1.\end{aligned}$$

So

$$\begin{aligned}&[a,t,t,t]\\&\qquad=[a,[a,x],[a,y],[a,z]]\,[a,[a,x],[a,z],[a,y]]\,[a,[a,y],[a,x],[a,z]]\\&\qquad\quad\cdot[a,[a,y],[a,z],[a,x]]\,[a,[a,z],[a,x],[a,y]]\,[a,[a,z],[a,y],[a,x]]\\&\qquad=[a,b,c,d][a,b,d,c][a,c,b,d][a,c,d,b][a,d,b,c][a,d,c,b].\end{aligned}$$

Next, notice that

$$\begin{aligned}[a,xyz]&=[a,yz][a,x][a,x,yz]\\&=[a,z][a,y][a,y,z][a,x][a,x,yz],\end{aligned}$$

so that

$$\begin{aligned}t&=[a,x][a,y][a,z]\\&=[a,xyz][a,y,z]^{-1}[a,x,yz]^{-1}g\end{aligned}$$

for some $g$ in the derived group of the normal closure of $a$. It follows that $[a,t,t,t]$ is a product of terms of the form

$$[a,u,v,w]\qquad\text{where } u,v,w\in\{[a,xyz],[a,y,z]^{-1},[a,x,yz]^{-1}\}.$$

One final time we use the fact that elements of form 1 to 5 above are trivial to see that in this product the terms of the form $[a,u,v,w]$ where two or more of $u,v,w$ equal to $[a,xyz]$ all cancel with each other. For example, if we pick out the terms where two of $u,v,w$ equal $[a,xyz]$ and where the third equals $[a,y,z]^{-1}$, then we obtain

$$\left[a,[a,xyz],[a,xyz],[a,y,z]^{-1}\right]\left[a,[a,xyz],[a,y,z]^{-1},[a,xyz]\right]$$
$$\cdot\left[a,[a,y,z]^{-1},[a,xyz],[a,xyz]\right].$$

We can express $[a,y,z]^{-1}$ as a product $c_1c_2c_3c_4$, where $c_1,c_2,c_3,c_4$ are conjugates of $a$ or inverses of conjugates of $a$, and then expanding we obtain

$$\prod_{1\leqslant i\leqslant 4}\left[a,a^{-1}a^{xyz},a^{-1}a^{xyz},c_i\right]\left[a,a^{-1}a^{xyz},c_i,a^{-1}a^{xyz}\right]\left[a,c_i,a^{-1}a^{xyz},a^{-1}a^{xyz}\right],$$

and this equals 1 by an argument similar to those above.

So we see that $[a,t,t,t]$ is a product of terms of the form $[a,u,v,w]$ where $u,v,w\in\{[a,xyz],[a,y,z]^{-1},[a,x,yz]^{-1}\}$ and where at least two of $u,v,w\in\gamma_3(G)$. It follows that $[a,t,t,t]$ is a product of terms $[a,u,v,w]$ all of which lie in $\gamma_9(\langle a,x,y,z\rangle)$. But $\langle a,x,y,z\rangle$ is a 4-Engel 5-group, and by Theorem 3.3 of [5] we know that four generator 4-Engel 5-groups are nilpotent of class at most 8. So $[a,t,t,t]=1$, which implies that

$$[a,b,c,d][a,b,d,c][a,c,b,d][a,c,d,b][a,d,b,c][a,d,c,b]=1,$$

as claimed.

## 5. 4-*Engel* 2-*groups*

We finally show that the relation

$$[a,b,c,d][a,b,d,c][a,c,b,d][a,c,d,b][a,d,b,c][a,d,c,b]=1$$

also holds in 4-Engel 2-groups. As above, it is sufficient to show that if $G=\langle a,x,y,z\rangle$ is a 4-Engel 2-group and if $g$ is a commutator of weight 9 or more in $G$ with four of the entries in $G$ equal to $a$, then $g=1$. Actually, we prove a more general result.

PROPOSITION 2. *Let $G$ be a 4-Engel 2-group and let $g$ be a commutator of weight $k+4$ with entries $a,a,a,a,x_1,x_2,\ldots,x_k$ (in some order), where $k\geqslant 2$ and $a,x_1,x_2,\ldots,x_k\in G$. Then either $g=1$, or $g=[a,x_1,a,a,x_2,x_3,\ldots,x_k,a]$. Furthermore*

$$[a,x_1,a,a,x_2,x_3,\ldots,x_k,a]^2=1,$$
$$[a,x_1,a,a,x_2,x_3,\ldots,x_k,a]=[a,x_{1\sigma},a,a,x_{2\sigma},x_{3\sigma},\ldots,x_{k\sigma},a]$$

*for all permutations $\sigma$ of $\{1,2,\ldots,k\}$, and*

$$[a,x_1,a,a,x_2,x_3,\ldots,x_k,a]=1$$

*if $x_1,x_2,\ldots,x_k$ are not all distinct.*

(There are 4-Engel 2-groups with $[a,x_1,a,a,x_2,x_3,\ldots,x_k,a]\neq 1$ for $k=2,3,4,5$.) We then immediately obtain the following corollary, which implies that the normal closure of an element in a 4-Engel 2-group is 3-Engel.

COROLLARY 3. *Let $G=\langle a,x,y,z\rangle$ be a 4-Engel 2-group, and let $g$ be a commutator of weight 8 or more, with four of the entries in $g$ equal to $a$ and with the remaining entries from the set $\{x,y,z\}$. Then $g=1$.*

Proposition 2 has a similar flavour to some of the results established by Traustason [11] in his proof that the normal closure of an element in a 4-Engel 2-group is nilpotent of class at most 4, and we use several of Traustason's arguments in establishing Proposition 2.

To prove Proposition 2 we first use the GAP [2] implementation of Werner Nickel's nilpotent quotient algorithm [10] to find some identities which hold in three generator 4-Engel 2-groups. We compute the free 4-Engel group of rank three. If we let the free generators of this group be $a, x, y$ then we quickly establish that $[a, x, a, a, y, a]$ has order 10, that

$$[a, x, y, a, a, a] = 1,$$

and that the elements

$$[a, x, a, y, a, a], \quad [a, x, x, a, a, y, a], \quad [a, x, a, a, x, y, a]$$

all have order 5. It follows that if $a, x, y$ are elements of a 4-Engel 2-group then

$$[a, x, a, a, y, a]^2 = 1,$$
$$[a, x, y, a, a, a] = [a, x, a, y, a, a] = [a, x, x, a, a, y, a] = [a, x, a, a, x, y, a] = 1.$$

We use these identities to obtain some more general identities which hold in 4-Engel 2-groups.

First, consider the identity $[a, x, y, a, a, a] = 1$. Substituting $yz$ for $y$ we obtain

$$[[a, x, z][a, x, y][a, x, y, z], a, a, a] = 1.$$

Expanding, and using the fact that the normal closure of $a$ is nilpotent of class at most 4 we obtain

$$[a, x, z, a, a, a][a, x, y, a, a, a][a, x, y, z, a, a, a] = 1,$$

and hence

$$[a, x, y, z, a, a, a] = 1.$$

Repeating the argument we obtain

$$[a, x_1, x_2, \ldots, x_k, a, a, a] = 1 \tag{3}$$

for all $k \geqslant 1$.

Similarly, the identities

$$[a, x, a, y, a, a] = [a, x, x, a, a, y, a] = [a, x, a, a, x, y, a] = [a, x, a, a, y, a]^2 = 1$$

give

$$[a, x_1, x_2, \ldots, x_r, a, y_1, y_2, \ldots, y_s, a, a] = 1, \tag{4}$$
$$[a, x, x, a, a, y_1, y_2, \ldots, y_s, a] = 1, \tag{5}$$
$$[a, x, a, a, x, y_1, y_2, \ldots, y_s, a] = 1, \tag{6}$$
$$[a, x_1, x_2, \ldots, x_r, a, a, y_1, y_2, \ldots, y_s, a]^2 = 1, \tag{7}$$

for all $r, s \geqslant 1$. In fact, these identities also hold for $s = 0$. This follows from equation (3) for identities (4) and (7), and from calculations in the free 4-Engel group of rank two for identities (5) and (6).

Next, we substitute $xz$ for $x$ in equation (5). Expanding, we obtain

$$[a, x, z, a, a, y_1, y_2, \ldots, y_s, a][a, z, x, a, a, y_1, y_2, \ldots, y_s, a]u = 1,$$

where $u$ is a product of commutators of weight at least $s + 7$, with four entries $a$ and at least one entry $x$, at least one entry $z$, and one entry $y_i$ for each of $i = 1, 2, \ldots, s$. Denote the multiweight of $[a, x, z, a, a, y_1, y_2, \ldots, y_s, a]$ by $(4, 1, 1, \ldots, 1)$ (with $s + 2$ 1's), indicating that the commutator has four entries $a$, and one entry from each of $x, z, y_1, y_2, \ldots, y_s$. Then $u$ is a product of commutators with multiweights $(4, 2, 1, 1, \ldots, 1)$, $(4, 1, 2, 1, \ldots, 1)$ and $(4, 2, 2, 1, \ldots, 1)$, where multiweight $(4, 2, 1, 1, \ldots, 1)$ (for example) indicates that there are four entries $a$, two entries $x$, one entry $z$ and one entry $y_i$ for each of $i = 1, 2, \ldots, s$. We say that a multiweight $(i_1, i_2, \ldots, i_r)$ is higher than a multiweight $(j_1, j_2, \ldots, j_r)$ if $i_n \geqslant j_n$ for $n = 1, 2, \ldots, r$ and if $\sum_{n=1}^{r} i_n > \sum_{n=1}^{r} j_n$. Thus

$$[a, x, z, a, a, y_1, y_2, \ldots, y_s, a][a, z, x, a, a, y_1, y_2, \ldots, y_s, a]$$

is equal to a product of commutators each of which have higher multiweight. We write

$$[a, x, z, a, a, y_1, y_2, \ldots, y_s, a][a, z, x, a, a, y_1, y_2, \ldots, y_s, a] \sim 1$$

to denote this. Combining this with equation (7) we obtain

$$[a, [x, z], a, a, y_1, y_2, \ldots, y_s, a] \sim 1. \tag{8}$$

Following Traustason [11], we now perform some calculations in a certain Lie ring satisfying multilinear Lie relations which hold in the associated Lie rings of 4-Engel groups.

Consider the 4-Engel identity $[x, y, y, y, y] = 1$. If we substitute $y_1 y_2 y_3 y_4$ for $y$, expand, and pick out the terms which involve all of the variables $x$, $y_1$, $y_2$, $y_3$, $y_4$ then we obtain an identity of the form

$$\prod_{\sigma \in \mathrm{Sym}(4)} [x, y_{1\sigma}, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}]g = 1,$$

where $g$ is a product of commutators of weight at least 6, each of which involves all of the variables $x$, $y_1$, $y_2$, $y_3$, $y_4$. In other words, we obtain a relation of the form

$$\prod_{\sigma \in \mathrm{Sym}(4)} [x, y_{1\sigma}, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}] \sim 1$$

which holds whenever $x$, $y_1$, $y_2$, $y_3$, $y_4$ are elements of a 4-Engel group. This gives the multilinear Lie relation

$$\sum_{\sigma \in \mathrm{Sym}(4)} (x, y_{1\sigma}, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}) = 0 \tag{9}$$

which holds in the associated Lie rings of 4-Engel groups. (We will denote the Lie product of $x$ and $y$ by $(x, y)$, to distinguish Lie products from group commutators.) Similarly, substituting $y_1 y_2 y_3 y_4 y_5$ for $y$, expanding, and picking out the terms which involve all the variables $x$, $y_1$, $y_2$, $y_3$, $y_4$, $y_5$ we obtain a relation of the form

$$\prod_{\sigma \in \mathrm{Sym}(5)} [x, y_{1\sigma}, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}, y_{5\sigma}]^{\alpha_\sigma} \sim 1$$

for some integral coefficients $\alpha_\sigma$.

This yields the multilinear Lie relation

$$\sum_{\sigma\in \mathrm{Sym}(5)} \alpha_\sigma(x, y_{1\sigma}, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}, y_{5\sigma}) = 0. \tag{10}$$

Note that relation (9) has degree 5 and relation (10) has degree 6. We obtain another multilinear Lie relation of degree 6 as follows. We substitute $xz$ for $x$ in the identity $[x, y, y, y, y] = 1$ and obtain

$$[[x, y][x, y, z][z, y], y, y, y] = 1.$$

Expanding, and using the fact that any commutator with 5 or more entries $y$ is trivial in 4-Engel groups, we obtain

$$[x, y, y, y, y][x, y, z, y, y, y][z, y, y, y, y] = 1,$$

and hence

$$[x, y, z, y, y, y] = 1.$$

Substituting $y_1y_2y_3y_4$ for $y$ and expanding, we obtain the group relation

$$\prod_{\sigma\in \mathrm{Sym}(4)} [x, y_{1\sigma}, z, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}] \sim 1$$

and the multilinear Lie relation

$$\sum_{\sigma\in \mathrm{Sym}(4)} (x, y_{1\sigma}, z, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}) = 0. \tag{11}$$

In a similar way we obtain three multilinear Lie relations of degree 7. Substituting $zw$ for $z$ in the relation $[x, y, z, y, y, y] = 1$ we obtain the relation $[x, y, z, w, y, y, y] = 1$. This gives the group relation

$$\prod_{\sigma\in \mathrm{Sym}(4)} [x, y_{1\sigma}, z, w, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}] \sim 1$$

and the multilinear Lie relation

$$\sum_{\sigma\in \mathrm{Sym}(4)} (x, y_{1\sigma}, z, w, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}) = 0. \tag{12}$$

Similarly, substituting $y_1y_2y_3y_4y_5y_6$ for $y$ in the group relation $[x, y, y, y, y] = 1$, and substituting $y_1y_2y_3y_4y_5$ for $y$ in the group relation $[x, y, z, y, y, y] = 1$, we obtain two further multilinear Lie relations of degree 7. Thus we have one multilinear Lie relation of degree 5, two of degree 6 and three of degree 7. (These are the relations $t^{(1,4)} = 0$, $t^{(1,5)} = 0$, $t^{(2,4)} = 0$, $t^{(3,4)} = 0$, $t^{(1,6)} = 0$ and $t^{(2,5)} = 0$ from [**11**].)

We let $L$ be the (largest) Lie ring over the integers generated by $a, x_1, x_2, \ldots$ which satisfies these six multilinear Lie relations, and also satisfies the Lie relations $(x_i, x_j) = 0$ for all $i, j \geqslant 1$. Traustason [**11**] gives the following relations which hold

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in $L$ ($u$ and $w$ denote arbitrary elements of $L$):

$$(u, x_i, a, x_j, w) = (u, x_j, a, x_i, w) \text{ modulo } 2L, \tag{13}$$
$$(u, x_i, a, a, x_j, w) = (u, x_j, a, a, x_i, w) \text{ modulo } 2L, \tag{14}$$
$$(u, a, x_i, a, a, w) = (u, a, a, x_i, a, w) \text{ modulo } 2L, \tag{15}$$
$$(u, x_i, x_j, a, x_k, w) = (u, x_k, a, x_i, x_j, w) \text{ modulo } 2L, \tag{16}$$
$$(u, x_i, x_j, a, a, x_k, w) = (u, x_k, a, a, x_i, x_j, w) \text{ modulo } 2L. \tag{17}$$

Equations (13), (14) and (15) appear in [**11**, Lemma 5.4], and (16) and (17) appear in [**11**, Proposition 5.6]. Traustason's notation is different from that used here, and he works in the quotient $L/2L$, but otherwise the equations are the same. I also verified all these relations, using the nilpotent quotient algorithm for graded Lie rings [**4**] to compute in $L/2L$. (To do this, I needed to write special purpose subroutines to enforce the relations (9), (10), (11) and (12) as well as the remaining two relations of weight 7.)

We also need one further identity, which holds in any Lie ring:

$$\begin{aligned}
&(a, x, a, y, a)\\
&\quad = (a, x, y, a, a) + ((a, x), (a, y), a)\\
&\quad = (a, x, y, a, a) - ((a, y), a, (a, x)) - (a, (a, x), (a, y))\\
&\quad = (a, x, y, a, a) + ((a, x), (a, y, a)) - ((a, y), (a, x, a))\\
&\quad = -(a, x, a, a, y) + 2(a, x, a, y, a) + (a, y, x, a, a) + (a, y, a, a, x) - 2(a, y, a, x, a).
\end{aligned}$$

Combining this with equation (14) we see that in $L$

$$(a, x_i, a, x_j, a, w) = (a, x_j, x_i, a, a, w) \text{ modulo } 2L. \tag{18}$$

Using relations (13) – (18) we are able to prove the following lemma.

LEMMA 4. *Let $w$ be a Lie product in $L$ of multiweight $(4, 1, 1, \ldots, 1)$ in $a, x_1, x_2, \ldots, x_k$. Then, working modulo $2L$, $w$ is a linear combination of Lie products of the following forms:*

$$(a, x_1, a, a, x_2, x_3, \ldots, x_k, a),$$
$$(a, y_1, a, y_2, \ldots, y_r, a, a, y_{r+1}, \ldots, y_k),$$
$$(a, y_1, \ldots, y_r, a, a, a, y_{r+1}, \ldots, y_k),$$

*where $y_1, y_2, \ldots, y_k$ is some permutation of $x_1, x_2, \ldots, x_k$ and $1 \leqslant r \leqslant k$.*

*Proof.* Clearly, $w$ is a linear combination of Lie products of the form

$$(a, b_1, \ldots, b_r, a, c_1, \ldots, c_s, a, d_1, \ldots, d_t, a, e_1, \ldots, e_u)$$

where $r+s+t+u = k$, and where $b_1, \ldots, b_r, c_1, \ldots, c_s, d_1, \ldots, d_t, e_1, \ldots, e_u$ is some permutation of $x_1, x_2, \ldots, x_k$. Let us introduce the notation $f(r, s, t, u)$ to denote Lie products of this form. Clearly, we may assume that $r > 0$.

If $s > 0$ and $t > 0$, then working modulo $2L$ we can use equation (16) to express $f(r, s, t, u)$ in the form $f(1, 1, r+s+t-2, u)$, and then use equation (18) to express it in the form $f(2, 0, r+s+t-2, u)$.

So we need only consider elements of the form $f(r, 0, t, u)$ and $f(r, s, 0, u)$. If $t > 0$ then we can use equation (17) to express elements of the form $f(r, 0, t, u)$ in


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the form $f(1,0,r+t-1,u)$. Similarly, if $s>0$ then we can use equation (16) to express elements of the form $f(r,s,0,u)$ in the form $f(1,r+s-1,0,u)$.

So, working modulo $2L$, we see that $w$ is a linear combination of elements of the form $f(r,0,0,u)$, $f(1,0,t,u)$ and $f(1,s,0,u)$. Also if $u>0$ then we can use (17), (15) and (16) in succession, to express an element of the form $f(1,0,t,u)$ in the form $f(t,0,1,u)$, then $f(t,1,0,u)$ and then $f(1,t,0,u)$. So we are left with a linear combination of elements of the form $f(1,0,k-1,0)$, $f(1,s,0,u)$ and $f(r,0,0,u)$. To complete the proof of Lemma 4 we need to show that

$$(a,x_1,a,a,x_2,x_3,\dots,x_k,a)=(a,x_{1\sigma},a,a,x_{2\sigma},x_{3\sigma},\dots,x_{k\sigma},a) \text{ modulo } 2L$$

for all permutations $\sigma$ of $\{1,2,\dots,k\}$, and this follows immediately from equation (14) together with the fact that $(x_i,x_j)=0$ for all $i,j$. □

Corollary 5. *Let $F$ be the free Lie ring with free generators $a,x_1,x_2,\dots$ in the variety of Lie rings determined by the six multilinear Lie identities defined above, and let $w$ be a Lie product in $F$ of multiweight $(4,1,1,\dots,1)$ in $a,x_1,x_2,\dots,x_k$. If $q$ is any power of 2, then, working modulo $qF$, $w$ is a linear combination of Lie products of multiweight $(4,1,1,\dots,1)$ of the following forms:*

$$(a,x_1,a,a,x_2,x_3,\dots,x_k,a),$$
$$(a,z_1,a,a,z_2,z_3,\dots,z_m,a),$$
$$(a,y_1,a,y_2,\dots,y_r,a,a,y_{r+1},\dots,y_n),$$
$$(a,y_1,\dots,y_r,a,a,a,y_{r+1},\dots,y_n),$$

*where $m<k$, $n\leqslant k$, where $z_1,z_2,\dots,z_m,y_1,y_2,\dots,y_n$ are Lie products of elements from the set $\{x_1,x_2,\dots\}$, and where $z_1$ is a Lie product of weight at least 2.*

*Proof.* The proof is by induction on $k$, the result being trivial for $k=1$. First consider the case when $q=2$. Note that $L\cong F/I$, where $I$ is the ideal of $F$ generated by the Lie products $(x_i,x_j)$. So, working modulo $2F$, $w=u+v$ where $u$ is a linear combination of Lie products of the required form, and $v\in I$. All the relations of $F$ are multihomogenous, and so $v$ is a linear combination of elements of $I$ all of which have multiweight $(4,1,1,\dots,1)$. This means that $v$ is a linear combination of Lie products of multiweight $(4,1,1,\dots,1)$ of form

$$(a,b_1,\dots,b_r,a,c_1,\dots,c_s,a,d_1,\dots,d_t,a,e_1,\dots,e_u)$$

where $b_1,\dots,b_r,c_1,\dots,c_s,d_1,\dots,d_t,e_1,\dots,e_u$ are Lie products of elements from the set $\{x_1,x_2,\dots,x_k\}$, and where $r+s+t+u<k$ and at least one of the Lie products has weight at least 2. The result for $q=2$ now follows by induction on $k$.

Now consider the case when $q$ is some arbitrary (but fixed) power of 2. We have shown that $w=u+2v$ where $u$ is a linear combination of elements of the required form. But this relation must be multihomogeneous, and so $v$ is a linear combination of Lie products with multiweight $(4,1,1,\dots,1)$. And so $v=u'+2v'$ where $u'$ is a linear combination of elements of the required form. Thus $w=u+2u'+4v'$, where $v'$ is a linear combination of Lie products of multiweight $(4,1,1,\dots,1)$. So $w$ is a linear combination of elements of the required form modulo $4F$, and repeating this argument we see that $w$ is a linear combination of elements of the required form modulo $qF$. □

We are at last in a position to complete the proof of Proposition 2. Let $G$ be a finite 4-Engel 2-group, let $a, x_1, x_2, \ldots, x_k \in G$ for some $k \geqslant 2$, and let $g$ be a commutator of weight $k+4$ with entries $a, a, a, a, x_1, x_2, \ldots, x_k$ (in some order). Let the exponent of $G$ be $q$. Then by Corollary 5, $g \sim hh'$ where $h$ is a product of group commutators of the form

$$[a, x_1, a, a, x_2, x_3, \ldots, x_k, a],$$
$$[a, z_1, a, a, z_2, z_3, \ldots, z_m, a],$$
$$[a, y_1, a, y_2, \ldots, y_r, a, a, y_{r+1}, \ldots, y_n],$$
$$[a, y_1, \ldots, y_r, a, a, a, y_{r+1}, \ldots, y_n],$$

where $m < k$, $n \leqslant k$, where $z_1, z_2, \ldots, z_m, y_1, y_2, \ldots, y_n$ are commutators of elements from the set $\{x_1, x_2, \ldots\}$, where $z_1$ is a commutator of weight at least 2, and where $h'$ is a product of $q$th powers. Since $G$ has exponent $q$, $h' = 1$. Also, commutators in $G$ of the form

$$[a, y_1, a, y_2, \ldots, y_r, a, a, y_{r+1}, \ldots, y_n] \quad \text{or} \quad [a, y_1, \ldots, y_r, a, a, a, y_{r+1}, \ldots, y_n]$$

are trivial by equations (3) and (4). In addition, equation (8) implies that if $z_1$ is a commutator of weight at least 2 then

$$[a, z_1, a, a, z_2, z_3, \ldots, z_m, a] \sim 1.$$

Finally, $[a, x_1, a, a, x_2, x_3, \ldots, x_k, a]^2 = 1$ by equation (7), and so $g \sim 1$ or

$$g \sim [a, x_1, a, a, x_2, x_3, \ldots, x_k, a].$$

Note that if $\sigma$ is any permutation of $\{1, 2, \ldots, k\}$, then $g$ is also a commutator of weight $k+4$ with entries $a, a, a, a, x_{1\sigma}, x_{2\sigma}, \ldots, x_{k\sigma}$. It follows from the same argument that $g \sim 1$ or $g \sim [a, x_{1\sigma}, a, a, x_{2\sigma}, x_{3\sigma}, \ldots, x_{k\sigma}, a]$. This implies that

$$[a, x_1, a, a, x_2, x_3, \ldots, x_k, a] \sim [a, x_{1\sigma}, a, a, x_{2\sigma}, x_{3\sigma}, \ldots, x_{k\sigma}, a]$$

for all $\sigma$.

Now consider any commutator $h$ in $a, x_1, x_2, \ldots, x_k$ which has higher multiweight than $g$. Then this commutator must have a repeated entry $x_i$ for some $i$, and so $h \sim 1$ or

$$h \sim [a, x_i, a, a, x_i, x_j, \ldots, x_r, a] = 1 \quad \text{by equation (6).}$$

So if $h$ is a commutator in $a, x_1, x_2, \ldots, x_k$ which has higher multiweight than $g$, then $h \sim 1$. But this implies that $h$ is a product of commutators with higher multiweight than $h$, and each of these commutators is a product of commutators with yet higher multiweight. And so on. Since $G$ is nilpotent, this implies that $h = 1$. All this implies that $g = 1$ or $g = [a, x_1, a, a, x_2, x_3, \ldots, x_k, a]$, and the rest of Proposition 2 follows immediately.

## *References*


1. W. Bosma, J. Cannon and C. Playoust, 'The Magma algebra system I: The user language', *J. Symb. Comp.* 24 3/4 (1997) 235–265. 343
2. The GAP Group, 'GAP – Groups, Algorithms, and Programming, Version 4.4.4', 2004, http://gap-system.org. 343, 347

**3.** N. D. Gupta and F. Levin, 'On soluble Engel groups and Lie algebras', *Arch. Math.* 34 (1980) 289–295. 342

**4.** G. Havas, M. F. Newman, and M. R. Vaughan-Lee, 'A nilpotent quotient algorithm for graded Lie rings', *J. Symbolic Computation* 9 (1990) 653–664. 350

**5.** G. Havas and M. R. Vaughan-Lee, 'Computing with 4-Engel groups', *Groups St Andrews 2005*, London Mathematical Society Lecture Notes Series 340 (Cambridge University Press, 2007) 475–485. 346

**6.** G. Havas and M. R. Vaughan-Lee, '4-Engel groups are locally nilpotent', *Internat. J. Algebra and Computation* 15 (2005) 649–682. 341

**7.** H. Heineken, 'Engelsche Elemente der Länge drei', *Illinois J. Math.* 5 (1961) 681–707. 341

**8.** L. C. Kappe and W. P. Kappe, 'On three Engel groups', *Bull. Austral. Math. Soc.* 7 (1972) 391–405. 341

**9.** F. W. Levi, 'Groups in which the commutator operation satisfies certain algebraic conditions', *J. Indian Math. Soc.* 6 (1942) 87–97. 341

**10.** W. Nickel, 'Computation of nilpotent Engel groups', *J. Austral. Math. Soc. Ser. A* 67 (1999) 214–222. 341, 343, 347

**11.** G. Traustason, 'Locally nilpotent 4-Engel groups are Fitting groups', *J. Algebra* 270 (2003) 7–27. 341, 342, 347, 348, 349, 350

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