Abstract almost periodicity for group actions on uniform topological spaces

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Abstract. We present a unified theory for the almost periodicity of functions with values in an
arbitrary Banach space, measures and distributions via almost periodic elements for the action of a
locally compact abelian group on a uniform topological space. We discuss the relation between Bohr-
and Bochner-type almost periodicity, and similar conditions, and how the equivalence among such
conditions relates to properties of the group action and the uniformity. We complete the paper by
demonstrating how various examples considered earlier all fit in our framework.

1 Introduction

Almost periodicity plays an important role in many areas of mathematics, from
differential equations to aperiodic order. Let us recall that t is a period for a system
if translating by t the entire system we obtain an identical copy of the original system.
We say that a system is fully periodic if the set of periods is relatively dense.

As introduced by Bohr [5], the idea behind almost periodicity is to replace the
periods by almost periods. These are elements t such that after translation the system
"almost" agrees (with respect to a topology) with the original system. An element is
called *almost periodic* if the sets of almost periods are relatively dense. Bohr's original
definition was for the uniform convergence topology on the space of uniformly
continuous bounded functions on the real line, and in this case, the definition is
equivalent to the Bochner condition that the closed hull of the function is a compact
space in this topology. Many of these ideas have been extended to other topologies
on spaces of functions on various (or all) locally compact abelian groups by Stepanov
[28], Weyl [32], Besicovitch [4], to measures [3, 8, 10, 14] and even to distributions [30].

Almost periodicity plays a fundamental role in the area of Aperiodic Order due its
connection to pure point diffraction. This connection already appears (more or less
explicit) in the work of Meyer [18], Lagarias [11], and Solomyak [24, 25]. A first study
giving an explicit equivalence statement between pure point spectrum and almost
periodicity appears (in a specific situation) in [3] (see [8] as well for related work).
This was then extended and further studied in the framework of dynamical systems
in [9, 10, 15, 19, 29]. On a fundamental level, the equivalence was then established and

studied in an framework free of dynamical systems in [14]. It seems fair to say that by now the connection between pure point spectrum and almost periodicity, as well as the importance of almost periodicity to Aperiodic Order is well established and thoroughly understood.

In many situations, the Bohr and Bochner definition of almost periodicity are equivalent (see, for example, [8, 10, 16, 19] just to name a few). Moreover, the hull of an almost periodic function/measure is often a compact abelian group [12, 13, 15, 19]. Since the proofs in these related but different situations are similar, it is natural to ask if there may be any unified theory of almost periodicity which shows the equivalence between Bohr and Bochner definition in a very general situation. It is our goal here to answer this question.

Let us describe here our general approach. Both the Bohr and Bochner definition for almost periodicity are in terms of properties of the orbit of an element under the translation action of the group, and can be defined for an arbitrary group action on a nice topological space. While in most situations studied in literature, the topology is metrizable, this is neither the case in all situations nor necessary. Here, we deal with the more general case of group actions on uniform spaces (see Definition 2.1 for the definition of uniformity and entourages), that is, where the topology is defined by a uniformity. We should emphasize here that many of the results in the paper can probably be extended to group actions on arbitrary topological spaces. The uniformity structure seems to play an important role when studying Bochner-type almost periodicity, as in this case, pre-compactness is equivalent to total boundedness, and this is the reason why the setup is of uniform spaces.

Given the action α of a locally compact abelian group (LCAG) G on a uniform space X, we study the connection between the following three properties of an element $x \in X$:

- (Bohr-type almost periodicity) For each entourage $U \in \mathcal{U}$, the set $P_U(x) := \{t \in G : (x, \alpha(t, x)) \in U\}$ of almost periods is relatively dense.
- (Bochner-type almost periodicity) The orbit closure $\overline{\{\alpha(t, x) : t \in G\}}$ is compact in X.
- (Pseudo-Bochner-type almost periodicity) For each entourage $U \in \mathcal{U}$, the set $P_U(x)$ is finitely relatively dense.

Under the extra natural assumption that the uniformity is G-invariant (see Definition 3.1), we show in Lemma 4.3 that pseudo-Bochner-type almost periodicity is simply the total boundedness of the orbit $\{\alpha(t, x) : t \in G\}$. Therefore, for a G-invariant uniformity, Bochner-type almost periodicity implies pseudo-Bochner-type almost periodicity, and the two concepts are equivalent if the uniformity is also complete (Proposition 4.4). It is immediate from the definition that pseudo-Bochner-type almost periodicity implies Bohr-type almost periodicity. If the group action is equicontinuous (see Definition 3.4), we show in Proposition 4.6 that the converse also holds and hence Bohr- and pseudo-Bochner-type almost periodicity are equivalent. Combining the results, and using the fact that for G-invariant actions, equicontinuity on orbit closure is equivalent to continuity at 0 (see Lemma 3.8), we get in Theorem 4.7 that for continuous, G-invariant actions on uniform spaces, Bohr-, Bochner-, and

pseudo-Bochner-type almost periodicity are equivalent. Moreover, in this case, the orbit closure becomes a compact abelian group (Theorem 4.12).

We complete the paper by looking at some particular examples in Section 6.

Let us complete this section by discussing the (lack of) metrizability in the examples in Section 6. The examples in Sections 6.1, 6.3, and 6.4 and the first example in Section 6.9, respectively, are examples of group actions on Banach spaces, and hence on metrizable spaces. We show in Section 6.2 that the topology we consider on the spaces $C(G : H), C_b(G : H)$, and $C_u(G : H)$ is metrizable if and only if H is a metrizable group. The weak topology of Section 6.4 is not usually metrizable. Indeed, it is well known that the weak topology on a Banach space X is metrizable if and only if X is finite dimensional. This immediately implies that the weak topology on $C_u(G)$ is metrizable if and only if G is finite group. The autocorrelation topology of Section 6.6 is defined by mixing the uniformity defined by semi-metric d_A with the metrizable topology of G (as G is assumed to be second countable) and hence is metrizable. We show that the mixed norm uniformity from Section 6.9 is metrizable if and only if G is metrizable. It is well known that the weak* topology on a Banach space X is metrizable if and only if X is finite dimensional, one can easily show that the vague topology of Section 6.7 is in general not metrizable on $\mathcal{M}(G)$. We suspect that product topology from Section 6.8 is in general not metrizable on $\mathcal{M}^\infty(G)$.

2 Uniform topologies

In this section, we discuss the necessary background on uniform topologies needed for our considerations. The material is well known (see, for example, [6, Chapter 2] or [17]). Let us start with the definition of uniformity.

Definition 2.1 (Uniformity) Let X be a set. A nonempty set $\mathcal{U}$ consisting of subsets of $X \times X$ is called a *uniformity on X* if it satisfies the following conditions.

- The diagonal

$$\Delta := \{(x,x) \, : \, x \in X\} \subseteq U$$

  of $X \times X$ is contained in any $U \in \mathcal{U}$.
- The set $\mathcal{U}$ is closed upward in the sense that if U belongs to $\mathcal{U}$ and $U \subseteq V$ hold, then V belongs to $\mathcal{U}$ as well.
- If U, V belong to $\mathcal{U}$, then $U \cap V$ belongs to $\mathcal{U}$.
- For all $U \in \mathcal{U}$, there exists a $V \in \mathcal{U}$ such that

$$V \circ V := \{(x,y) \in X \times X \, : \, \text{ there exists } z \in X \text{ with } (x,y), (y,z) \in V\}$$

  is contained in U.
- For all $U \in \mathcal{U}$, the set

$$U^{-1} := \{(x,y) \, : \, (y,x) \in U\}$$

  belongs to $\mathcal{U}$.

The elements $U \in \mathcal{U}$ are called *entourages*.

Any uniformity $\mathcal{U}$ on X induces a topology $\tau_{\mathcal{U}}$ on X as follows: For $x \in X$ and $U \in \mathcal{U}$, we define

$$U[x] := \{y \in X : (x, y) \in U\}.$$

Then, the sets $\{U[x] : U \in \mathcal{U}\}$ define a basis at x for the topology $\tau_{\mathcal{U}}$. Specifically, a subset O belongs to $\tau_{\mathcal{U}}$ if and only if, for any $x \in O$, there exists $U \in \mathcal{U}$ with $U[x] \subseteq O$. We say that a topology is *uniform* if it is the topology induced by a uniformity.

Any metrizable topology is uniform. Indeed, if (X, d) is a metric space, for each $r > 0$, we can define

$$U_r := \{(x, y) \in X \times X : d(x, y) < r\}.$$

Then,

$$\mathcal{U}_d := \{U \subseteq X \times X : \text{ there exists } r > 0 \text{ such that } U_r \subseteq U\}$$

is a uniformity on X, which induces the topology defined by the metric d. As it is instructive and may help the reader to get some perspective on the definition of entourages, we briefly sketch the proof that $\mathcal{U}_d$ is a uniformity: As $d(x, x) = 0$, for all x, the system $\mathcal{U}_d$ contains the diagonal. The system $\mathcal{U}_d$ is upward closed by definition. The third point holds trivially. The fourth point is a consequence of the triangle inequality and the last point is a consequence of the symmetry of the metric.

Remark 2.2   In the preceding considerations, we do not need d to be a metric. It suffices that d is a pseudometric, i.e., a symmetric map $d : X \times X \longrightarrow [0, \infty]$ with $d(x, x) = 0$ for all $x \in X$ and

$$d(x, y) \le d(x, z) + d(z, y),$$

where the value ∞ is allowed.

Let $\mathcal{U}$ be a uniformity. Then, $\tau_{\mathcal{U}}$ is Hausdorff if and only if, for all $x, y \in X$ with $x \ne y$, there exists some $U \in \mathcal{U}$ such that $(x, y) \notin U$. Equivalently, $\tau_{\mathcal{U}}$ is Hausdorff if and only if

$$\bigcap_{U \in \mathcal{U}} U = \Delta.$$

All the uniform topologies we consider are assumed to be Hausdorff.

We next discuss the completion of the topology induced by a uniformity. Let us start by recalling the definition of a filter.

Definition 2.3   A *filter* on a set X is a family $\mathcal{F} \subseteq \mathcal{P}(X)$ of subsets of X with the following properties:

- If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.
- If $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \mathcal{F}$, then $A_1 \cap \cdots \cap A_n \in \mathcal{F}$.
- $\varnothing \notin \mathcal{F}$.

If $\mathcal{U}$ is a uniformity on X, a filter $\mathcal{F}$ is called *Cauchy* if, for all $U \in \mathcal{U}$, there exists some $A \in \mathcal{F}$ such that $A \times A \subseteq U$. We say that $\mathcal{F}$ is *convergent to* $x \in X$ if, for all $U \in \mathcal{U}$, we have $U[x] \in \mathcal{F}$.

A uniform space $(X, \mathcal{U})$ is called *complete* if every Cauchy filter is convergent.

Let us briefly discuss how to characterize uniform Hausdorff topologies which are given by a metric. First, we need the following definition.

Definition 2.4 (Basis of an entourage)   Let $\mathcal{U}$ be a uniformity on X. A set $\mathcal{B} \subseteq \mathcal{U}$ is called a *basis of entourages* if, for all $U \in \mathcal{U}$, there exists some $W \in \mathcal{B}$ such that $W \subseteq U$.

The following well-known result (see, e.g., [31]) characterizes the metrizability of a uniformity.

Theorem 2.5 (Characterization of metrizability of entourage)   *Let $\mathcal{U}$ be a Hausdorff uniformity on X. Then, there exists a metric d on X such that $\mathcal{U} = \mathcal{U}_d$ if and only if there exists a countable basis of entourages $\mathcal{B} \subseteq \mathcal{U}$.*

At the end of this section, let us review the notion of total boundedness and its connection to compactness.

Definition 2.6 (Total boundedness)   Let $(X, \mathcal{U})$ be a uniform Hausdorff space. A set $A \subseteq X$ is called *totally bounded* if, for all $U \in \mathcal{U}$, there exist $U_1, U_2, \ldots, U_n \subseteq X$ such that

$$A \subseteq \bigcup_{j=1}^{n} U_j \qquad \text{and} \qquad \bigcup_{j=1}^{n} (U_j \times U_j) \subseteq U.$$

The importance of totally boundedness is given by the following result.

Theorem 2.7 [6, Theorem II.4.3]   *Let $(X, \mathcal{U})$ be a uniform Hausdorff space.*

(a) *If $A \subseteq X$ is compact in $\tau_{\mathcal{U}}$, then A is totally bounded.*
(b) *If $A \subseteq X$ is totally bounded and $(X, \mathcal{U})$ is complete, then the closure $\overline{A}$ of A is compact.*

Remark 2.8 (Characterization of compactness)   From the theorem, we may easily derive the following characterization of compactness in a uniform space. A subset $A \subseteq X$ is compact if and only if A is totally bounded and complete (in the topology inherited from $\mathcal{U}$).

The topology of a compact set is always uniform.

Theorem 2.9 [6, Theorem II.4.1]   *Let (X, τ) be a compact space. Then, there exists a unique uniformity $\mathcal{U}$ on X such that $\tau = \tau_{\mathcal{U}}$.*

3 Group actions on uniform topological spaces

In this section, we consider groups acting on uniform spaces. The material is certainly known.

Let $(X, \mathcal{U})$ be a uniform Hausdorff space, and let $\tau_{\mathcal{U}}$ be the topology induced by $\mathcal{U}$ on X. Let G be a group. Then, a map $\alpha : G \times X \to X$ is called a *group action* if it satisfies:

- $\alpha(e, x) = x$ for all $x \in X$ (where e is the neutral element of G).
- $\alpha(s, \alpha(t, x)) = \alpha(st, x)$ for all $x \in X$ and $t, s \in G$.

If $\alpha : G \times X \longrightarrow X$ is a group action, the group G is said to act on X (via α).

Next, we review and introduce various definitions for the action α, which will play a central role in this paper. Note that we do not at first assume any continuity property of α. Indeed, we have not assumed that G carries a topology so far.

We start by introducing the notion of G-invariance for uniformities and (pseudo) G-invariance for metrics on X.

Definition 3.1   Let α be an action of G on X.

(a) A uniformity $\mathcal{U}$ on X is called *G-invariant* if, for each $U \in \mathcal{U}$, there exists a $U' \in \mathcal{U}$ such that, for all $t \in G$, we have

$$\alpha(t, U') := \{(\alpha(t, x), \alpha(t, y)) : (x, y) \in U'\} \subseteq U.$$

(b) A metric d on X is called *G-invariant* if, for all $x, y \in X$ and $t \in G$, we have

$$d(\alpha(t, x), \alpha(t, y)) = d(x, y).$$

(c) A metric d on X is called *pseudo G-invariant* if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all $x, y \in X$ with $d(x, y) < \delta$ and all $t \in G$, we have

$$d(\alpha(t, x), \alpha(t, y)) < \varepsilon.$$

It is obvious that any G-invariant metric is pseudo G-invariant. It is also clear from the definition in (a) that U' itself is contained in U (as we can take t to be the neutral element of G). This will be used tacitly subsequently.

Note that another way to phrase (c) of the preceding definition is that the family α_t, $t \in G$, (with $\alpha_t(x) = \alpha(t, x)$) is equicontinuous. By changing the metric to an equivalent metric, we can then make α_t into a families of isometries. This is discussed next.

Lemma 3.2 (Invariance and pseudo invariance)   *Let α be an action of G on a metric space (X, d), and let $\mathcal{U}_d$ be the uniformity on X defined by d. Then, the following statements are equivalent.*

(i) *The uniformity $\mathcal{U}_d$ is G-invariant.*
(ii) *The metric d is pseudo G-invariant.*
(iii) *There exists a G-invariant metric d' on X with $\mathcal{U}_d = \mathcal{U}_{d'}$.*

Proof   (i) $\Longrightarrow$ (ii): Let $\varepsilon > 0$. Since $U_\varepsilon := \{(x, y) : d(x, y) < \varepsilon\} \in \mathcal{U}$ and $\mathcal{U}$ is G-invariant, there exists some $V \in \mathcal{U}$ such that, for all $t \in G$ and $(x, y) \in V$, we have $(\alpha(t, x), \alpha(t, y)) \in U_\varepsilon$.

Now, since $V \in \mathcal{U}$, there exists some $\delta > 0$ such that $d(x, y) < \delta$ implies $(x, y) \in V$. Therefore, we have $(x, y) \in V$ for all $x, y \in X$ with $d(x, y) < \delta$ and all $t \in G$, and hence $(\alpha(t, x), \alpha(t, y)) \in U_\varepsilon$. This implies

$$d(\alpha(t, x), \alpha(t, y)) < \varepsilon.$$

(ii) $\Longrightarrow$ (iii): Define

$$\bar{d}(x, y) := \sup\{d(\alpha(t, x), \alpha(t, y)) : t \in G\}.$$

It follows immediately from the definition that we have

$$(1) \qquad\qquad \bar{d}(x, y) = \bar{d}(\alpha(t, x), \alpha(t, y))$$

for all $x, y \in X$ and $t \in G$. Now, $\bar{d}$ may not be a metric. Indeed, since X is not necessarily compact, d could be unbounded, i.e., there could exist some $x, y \in X$ such that $\bar{d}(x, y) = \infty$. To fix this issue, define

$$d'(x, y) := \begin{cases} \bar{d}(x, y), & \text{if } \bar{d}(x, y) < 1, \\ 1, & \text{otherwise.} \end{cases}$$

We claim that this is a metric which has the desired properties.

- We show that d' is a metric.

    First, note that $d'(x, y) \geq 0$ for all $x, y \in G$ by construction. Moreover, $d'(x, y) = 0 \iff \bar{d}(x, y) = 0 \iff d(\alpha(t, x), \alpha(t, y)) = 0$ for all $t \in G \iff x = y$.

    Next, the definition of $\bar{d}$ gives that $\bar{d}(x, y) = \bar{d}(y, x)$ for all x, y, and hence $d'(x, y) = d'(y, x)$.

    Finally, let us prove the triangle inequality. Let $x, y, z \in X$. We split the problem into three cases:

    Case 1: Assume $d'(x, z) \geq 1$. Then, one finds

    $$d'(x, y) \leq 1 \leq d'(x, z) \leq d'(x, z) + d'(y, z).$$

    Case 2: Assume $d'(y, z) \geq 1$. Then, we get

    $$d'(x, y) \leq 1 \leq d'(y, z) \leq d'(x, z) + d'(y, z).$$

    Case 3: Assume $d'(x, z) < 1$ and $d'(y, z) < 1$. Then, $d'(x, z) = \bar{d}(x, z)$ and $d'(y, z) = \bar{d}(y, z)$ imply

    $$\begin{aligned} d'(x, y) \leq \bar{d}(x, y) &= \sup\{d(\alpha(t, x), \alpha(t, y)) : t \in G\} \\ &\leq \sup\{d(\alpha(t, x), \alpha(t, z)) + d(\alpha(t, y), \alpha(t, z)) : t \in G\} \\ &\leq \sup\{d(\alpha(t, x), \alpha(t, z)) : t \in G\} + \sup\{d(\alpha(t, y), \alpha(t, z)) : t \in G\} \\ &= \bar{d}(x, z) + \bar{d}(y, z) = d'(x, z) + d'(y, z). \end{aligned}$$

    This proves that d' is a metric.
- d' is G-invariant.

    This follows immediately from (1) and the definition of d'.
- d and d' induce the same uniformity.

    First, note that we have

    $$d'(x, y) < r \implies \bar{d}(x, y) < r \implies d(x, y) < r$$

    for all $0 < r < 1$. Hence,

    $$\{(x, y) \in X \times X : d'(x, y) < r\} \subseteq \{(x, y) \in X \times X : d(x, y) < r\}$$

    for all $0 < r <$. This gives, $\mathcal{U}_d \subseteq \mathcal{U}_{d'}$.

    Next, let $\varepsilon > 0$ be given. Assume without loss of generality that $\varepsilon < 1$. Since d is pseudo-invariant, there exists some $\delta > 0$ such that $d(\alpha(t, y), \alpha(t, z)) < \varepsilon$ for all $y, z \in X$ with $d(y, z) < \delta$ and all $t \in G$. In particular, we have

    $$d'(x, y) \leq \varepsilon,$$

whenever $d(x, y) < \delta$. This gives

$$\{(x, y) \in X \times X : d(x, y) < \delta\} \subseteq \{(x, y) \in X \times X : d'(x, y) \le \varepsilon\}.$$

This gives $\mathcal{U}_{d'} \subseteq \mathcal{U}_d$, and hence $\mathcal{U}_d = \mathcal{U}_{d'}$.

(iii) $\Longrightarrow$ (i): Since $\mathcal{U}_d$ agrees with $\mathcal{U}_{d'}$ and d' is G-invariant, it follows immediately that $\mathcal{U}_d$ is G-invariant. ∎

Next, consider a uniform Hausdorff space $(X, \mathcal{U})$, and let $\tau_{\mathcal{U}}$ be the topology defined by $\mathcal{U}$. Let G be an LCAG. We then write the group operation as addition with $+$, and denote the neutral element of G as 0 and the inverse of $t \in G$ by $-t$.

Let α be an action of G on X. Recall that α is called *continuous* if

$$\alpha : G \times X \to X$$

is a continuous mapping with respect to the topologies of G and $(X, \tau_{\mathcal{U}})$.

We want to look at a stronger version of continuity for α. To do so, we start with the following lemma.

Lemma 3.3 *Let $(X, \mathcal{U})$ be a Hausdorff uniform space and $\alpha : G \times X \longrightarrow X$ the continuous action of an LCAG. The following assertions are equivalent.*

(i) *For all $U \in \mathcal{U}$, there exist an open set $O \subseteq G$ containing 0 and a set $V \in \mathcal{U}$ such that*

$$\{(\alpha(s, x), \alpha(t, y)) : s, t \in O, (x, y) \in V\} \subseteq U.$$

(ii) *For all $U \in \mathcal{U}$, there exist an open set $O \subseteq G$ containing 0 and a set $V \in \mathcal{U}$ such that*

$$\{(\alpha(s, x), y) : s \in O, (x, y) \in V\} \subseteq U.$$

(iii) *For all $U \in \mathcal{U}$, there exists an open set $O \subseteq G$ containing 0 such that*

$$\{(\alpha(s, x), x) : s \in O, x \in X\} \subseteq U.$$

(iv) *The family of function $\{\alpha_x\}_{x \in X}$ defined by $\alpha_x : G \to X$*

$$\alpha_x(t) = \alpha(t, x)$$

is equicontinuous at $t = 0$ (i.e., for every $U \in \mathcal{U}$, there exists an open set $O \subseteq G$ containing 0 with $(\alpha_x(s), x) = (\alpha_x(s), \alpha_x(0)) \in U$ for all $s \in O$).

(v) *The family of function $\{\alpha_x\}_{x \in X}$ is uniformly equicontinuous (i.e., for every $U \in \mathcal{U}$, there exists an open set $O \subseteq G$ containing 0 with $(\alpha_x(s), \alpha_x(t)) \in U$ for all $s, t \in G$ with $s - t \in O$).*

Proof (i) $\Longrightarrow$ (ii): This is obvious, since

$$\{(\alpha(s, x), y) : s \in O, (x, y) \in V\} \subseteq \{(\alpha(s, x), \alpha(t, y)) : s, t \in O, (x, y) \in V\}.$$

(ii) $\Longrightarrow$ (iii): This is obvious, since $\Delta = \{(x, x) : x \in X\} \subseteq V$.

(iii) $\Longrightarrow$ (i): Let $U \in \mathcal{U}$, and let W' be so that $W' \circ W' \circ W' \subseteq U$. Let

$$W = W' \cap (W')^{-1}.$$

Then, by (iii), there exists an open set $O \subseteq G$ containing 0 such that

$$\{(\alpha(s, x), x) : s \in O, x \in X\} \subseteq W.$$

Now, we have $(\alpha(s,x),x) \in W$ for all $s,t \in O$ and $(x,y) \in W'$ and $(\alpha(t,y),y) \in W$. Since $W = W' \cap (W')^{-1}$, we obtain

$$(\alpha(s,x),x) \in W', \qquad (x,y) \in W', \qquad \text{and} \qquad (y,\alpha(t,y)) \in W'.$$

This gives $(\alpha(s,x),\alpha(t,y)) \in W' \circ W' \circ W' \subseteq U$. This shows that

$$\{(\alpha(s,x),\alpha(t,y)) \,:\, s,t \in O, \, (x,y) \in W'\} \subseteq U.$$

(iii) $\implies$ (v): Let $U \in \mathcal{U}$. By (iii), there exists an open set $O \subseteq G$ containing 0 such that

$$(2) \qquad\qquad \{(\alpha(s,y),y) \,:\, s \in O, \, y \in X\} \subseteq U.$$

Now, let $s - t \in O$. Then, for all $x \in X$, setting $y = \alpha(t,x)$ in (2), we get

$$(\alpha(s-t,\alpha(t,x)),\alpha(t,x)) \in U, \quad (\alpha(s,x)),\alpha(t,x)) \in U, \quad \text{and} \quad (\alpha_x(s),\alpha_x(t)) \in U.$$

This shows that $(\alpha_x(s),\alpha_x(t)) \in U$ holds for all $x \in X$ and all $s - t \in O$, proving the equicontinuity of this family.

(v) $\implies$ (iv): This is obvious.

(iv) $\implies$ (iii): This is obvious.   ∎

We can now define the notion of equicontinuous group actions.

Definition 3.4 (Equicontinuous group action)   Let α be an action of G on the uniform Hausdorff space $(X,\mathcal{U})$. We say that α is *equicontinuous* if it satisfies one (and thus all) the equivalent conditions of Lemma 3.3.

Remark 3.5   We will discuss examples in Section 6, where the action is not continuous, or continuous but not equicontinuous.

Remark 3.6   The action α is continuous if, for all $U \in \mathcal{U}$ and $x \in X$, the set

$$O_{x,U} = \{t \in G \,:\, (\alpha(t,x),x) \in \mathcal{U}\}$$

is an open neighborhood of 0 in G.

The action α is equicontinuous if, for all $U \in \mathcal{U}$, the set

$$O_U = \{t \in G \,:\, (\alpha(t,x),x) \in U \text{ for all } x \in X\} = \bigcap_{x \in X} O_{x,U}$$

is an open neighborhood of 0 in G.

Definition 3.7 (Orbit and hull)   Let α be an action of G on the uniform Hausdorff space $(X,\mathcal{U})$. For $x \in X$, the *orbit of x* $O(x)$ is defined as

$$O(x) := \{\alpha(t,x) \,:\, t \in G\}.$$

The *hull of x* (also called the *orbit closure of x*), denoted by $H_x^{\mathcal{U}}$, is defined as the closure of the orbit in $(X,\tau_{\mathcal{U}})$, i.e.,

$$H_x^{\mathcal{U}} := \overline{\{\alpha(t,x) \,:\, t \in G\}}^{\tau_{\mathcal{U}}}.$$

Next, let us show that for G-invariant uniformities, the continuity of the action implies the equicontinuity on orbit closures.

Lemma 3.8 (Equicontinuity on orbit closures)   *Let α be an action of G on the uniform Hausdorff space $(X, \mathcal{U})$. Assume that α is continuous and $\mathcal{U}$ is G-invariant. Then, for each $x \in X$, α is equicontinuous on $H_x^{\mathcal{U}}$.*

Proof   Let $U \in \mathcal{U}$, and let $V \in \mathcal{U}$ be such that $V \circ V \circ V \circ V \subseteq U$. Since α is G-invariant, there exists some $V' \in \mathcal{U}$ such that

$$(3) \qquad \{(\alpha(t, x), \alpha(t, y)) : t \in G, (x, y) \in V'\} \subseteq V.$$

Clearly, $V' \subseteq V$ holds (as we can take $t = 0$ on the left-hand side). Since α is continuous, there exists an open set $O \subseteq G$ containing 0 such that

$$(4) \qquad (\alpha(t, x), x) \in V'$$

for all $t \in O$. Now, let $t \in O$ and $y, z \in H_x$ be arbitrary such that $(y, z) \in V'$. Since $y, z \in H_z$, there exist $s, r \in G$ such that $(y, \alpha(s, x)) \in V'$ and $(z, \alpha(r, x)) \in V'$. Then, since $(y, \alpha(s, x)) \in V'$, we have, by (3),

$$(\alpha(t, y), \alpha(s + t, x)) \in V.$$

Moreover, since $t \in O$, by (4),

$$(\alpha(t, x), x) \in V'$$

and hence, again by (3),

$$(\alpha(t + s, x), \alpha(s, x)) \in V.$$

Therefore,

$$(\alpha(t, y), \alpha(s, x)) \in V \circ V.$$

Finally, since $(y, \alpha(s, x)) \in V' \subseteq V$, we get

$$(\alpha(t, y), y) \in V \circ V \circ V.$$

Using $(y, z) \in V' \subseteq V$, we obtain

$$(\alpha(t, y), z) \in V \circ V \circ V \circ V \subseteq U.$$

Therefore, we have

$$(\alpha(t, y), z) \in U$$

for all $t \in O$ and $y, z \in H_x$ with $(y, z) \in V'$. By Lemma 3.3, the action α is equicontinuous on H_x. ∎

4 Bohr- and Bochner-type almost periodicity

For this entire section, we let an LCAG G be given and α is an action of G on a uniform Hausdorff space $(X, \mathcal{U})$. We discuss the standard definitions of almost periodicity in this context.

Let $A \subseteq G$ be any set. Then, A is *relatively dense* if there exists a compact set K such that

$$A + K = G.$$

We say that A is *finitely relatively dense* if there exists a set finite $F \subseteq G$ such that $A + F = G$. Clearly, a finitely relatively dense set is relatively dense.

We can now introduce the following definitions.

Definition 4.1 For $x \in X$ and $U \in \mathcal{U}$, define

$$P_U(x) := \{t \in G : (\alpha(t, x), x) \in U\}.$$

(a) The element $x \in X$ is called *Bohr-type almost periodic* if, for all $U \in \mathcal{U}$, the set $P_U(x)$ is relatively dense.
(b) The element $x \in X$ is called *Bochner-type almost periodic* if $H_x^{\mathcal{U}}$ is compact in $(X, \tau_{\mathcal{U}})$.
(c) The element $x \in X$ is called *pseudo-Bochner-type almost periodic*[1] if, for all $U \in \mathcal{U}$, the set $P_U(x)$ is finitely relatively dense.

Remark 4.2 (a) Let α be an action of G on a uniform Hausdorff space $(X, \mathcal{U})$ such that the uniformity is G-invariant. Then, for all $U \in \mathcal{U}$, there exists some $V \in \mathcal{U}$ such that, for all $x \in X$ and $t \in G$, we have

$$P_V(x) \subseteq P_U(\alpha(t, x)) \qquad \text{and} \qquad P_V(\alpha(t, x)) \subseteq P_U(x).$$

(b) Consider the space $\mathcal{B}\mathrm{ap}_{2,\mathcal{A}}(G)$ of Besicovitch almost periodic functions on an LCAG G (see [14] for details) together with the uniformity $\mathcal{U}$ given by the metric $d(f, g) := \|f - g\|_{b,2,\mathcal{A}}$, and the action $\alpha(t, f) = \tau_t f$ (see [14]). Then, an element $f \in \mathcal{B}\mathrm{ap}_{2,\mathcal{A}}(G)$ is Bohr-type almost periodic if and only if f is a mean almost periodic function.

   We will see below more examples, where Bohr-/Bochner-type almost periodicity is equivalent with some other standard notion of almost periodicity. For this reason, to avoid confusion, we used the names Bohr-type and Bochner-type almost periodicity in Definition 4.1, instead of the simpler Bohr or Bochner almost periodicity.

Next, we will see that there is a connection between the total boundedness of the orbit of an element x and pseudo-Bochner-type almost periodicity.

Lemma 4.3 *Let α be an action of G on $(X, \mathcal{U})$, and let $\mathcal{U}$ be G-invariant. Then, an element $x \in X$ is pseudo-Bochner-type almost periodic if and only if the orbit $O(x)$ is totally bounded.*

Proof We start with a preliminary observation used repeatedly in the proof. Let $U \in \mathcal{U}$. Due to the G-invariance, there exists a $V \in \mathcal{U}$ with $(\alpha(s, x), \alpha(s, y)) \in U$ for all $(x, y) \in V$ and $s \in G$. Hence, $(\alpha(t, x), \alpha(r, x)) \in V$ for some $r, s \in G$ implies

$$(\alpha(t - r, x), x) = (\alpha(-r, \alpha(t, x)), \alpha(-r, \alpha(r, x))) \in U,$$

which in turn gives

$$t \in r + P_U(x).$$

---

[1]We will see below that this definition is closely related to the total boundedness of the orbit O_x, and hence to Bochner-type almost periodicity.

$\Longleftarrow$: Let $U \in \mathcal{U}$ be given. Choose $V \in \mathcal{U}$ with $(\alpha(s, x), \alpha(s, y)) \in U$ for all $(x, y) \in V$ and $s \in G$. Since $O(X)$ is totally bounded, there exists some $V_1, \ldots, V_n \subseteq X$ such that

$$O(X) \subseteq V_1 \cup \cdots \cup V_n \qquad \text{and} \qquad \bigcup_{j=1}^{n}(V_j \times V_j) \subseteq V.$$

Without loss of generality, we can assume $O(X) \cap V_j \neq \varnothing$, as otherwise, we can erase U_j from our list. Therefore, for each $1 \leq j \leq n$, there exists some $t_j \in G$ such that $\alpha(t_j, x) \in V_j$. We claim

$$G = \bigcup_{j=1}^{n}(t_j + P_U(x))$$

(which gives finitely relative denseness of $P_U(x)$ and, hence, Bochner-type almost periodicity of x). Indeed, let $t \in G$. Then,

$$\alpha(t, x) \in O(X) \subseteq V_1 \cup \cdots \cup V_n$$

holds and, hence, there exists some $1 \leq j \leq n$ such that $\alpha(t, x) \in V_j$. Therefore,

$$(\alpha(t, x), \alpha(t_j, x)) \in V_j \times V_j \subseteq V$$

holds and

$$t \in P_U(\alpha(t_j, x))$$

follows from the observation at the beginning of the proof.

$\Longrightarrow$: Let $U \in \mathcal{U}$ be given. Let $W \in \mathcal{U}$ be given with $W \subseteq U$ and $W \circ W^{-1} \subseteq U$. Choose $V \in \mathcal{U}$ with $(\alpha(s, x), \alpha(s, y)) \in W$ for all $s \in G$ and $(x, y) \in V$. Since x is pseudo-Bochner-type almost periodic, there exist $t_1, \ldots, t_n$ with

$$G = \bigcup_{j=1}^{n}(t_j + P_V(x)).$$

For each $1 \leq j \leq n$, define

$$U_j = \{y \in X : (y, \alpha(t_j, x)) \in W\}.$$

Now, let $t \in G$ be arbitrary. Then, there exists some $1 \leq j \leq n$ with $t \in t_j + P_V(x)$ and hence

$$(\alpha(t - t_j, x), x) \in V.$$

This gives

$$(\alpha(t, x), \alpha(t_j, x)) \in W$$

and, hence, $\alpha(t, x) \in U_j$. As $t \in G$ was arbitrary, we infer

$$O(X) \subseteq \bigcup_{j=1}^{n} U_j.$$

It remains to show $U_j \times U_j \subseteq U$ for $j = 1, \ldots, n$. So, let $1 \le j \le n$ and $(y, z) \in U_j \times U_j$ be given. Then $(y, \alpha(t_j, x)), (z, \alpha(t_j, x)) \in W$ giving $(y, \alpha(t_j, x)) \in W$ and $(\alpha(t_j, x), z) \in W^{-1}$ and, hence,

$$(y, z) \in W \circ W^{-1} \subseteq U.$$

This shows that $U_j \times U_j \subseteq U$ for all j and the proof is finished. ∎

By combining this result with Theorem 2.7, we obtain the following connection between Bochner-type almost periodicity and pseudo-Bochner-type almost periodicity.

Proposition 4.4 *Let G act on a uniform topological Hausdorff space, and let $\mathcal{U}$ be G-invariant. Then, the following holds for $x \in X$:*

(a) *If x is Bochner-type almost periodic, then x is pseudo-Bochner-type almost periodic.*
(b) *If (X, τ_U) is complete and x is pseudo-Bochner-type almost periodic, then x is Bochner-type almost periodic.*

In fact, it is easy to see that the equivalence between Bochner-type and pseudo-Bochner-type almost periodicity is a matter of the completeness of $\mathcal{U}$.

Corollary 4.5 *Let G act on a uniform topological Hausdorff space and assume that the uniformity is G-invariant. Let $x \in X$ be pseudo-Bochner almost periodic. Then x is Bochner almost periodic if and only if $H_x^{\mathcal{U}}$ is complete.*

By definition, any pseudo-Bochner-type almost periodic point is Bohr-type almost periodic (as a finitely relatively dense set is clearly relatively dense). Next, we want to show that for equicontinuous group actions, Bohr-type almost periodicity implies pseudo-Bochner-type almost periodicity. This will allow us show, in the case of equicontinuous group actions of G on a complete G-invariant uniform Hausdorff space $(X, \mathcal{U})$, the equivalence between Bohr-type, Bochner-type, and pseudo-Bochner-type almost periodicity.

Proposition 4.6 *Let G act on a uniform topological Hausdorff space $(X, \mathcal{U})$ such that the action is equicontinuous, and let $x \in X$. Then, x is Bohr-type almost periodic (i.e., $P_U(x)$ is relatively dense for any $U \in \mathcal{U}$) if and only if x is pseudo-Bochner-type almost periodic (i.e., for all $U \in \mathcal{U}$, the set $P_U(x)$ is finitely relatively dense).*

Proof $\Longleftarrow$: This follows immediately from the fact that every finite set in G is compact.

$\Longrightarrow$: Let $U \in \mathcal{U}$ be arbitrary. Since the action is equicontinuous, there exists an open set $O \subseteq G$ and $V \in \mathcal{U}$ such that

$$(5) \qquad \{(\alpha(s, x), \alpha(t, y)) : s, t \in O, (x, y) \in V\} \subseteq U.$$

Since x is Bohr-type almost periodic, there exists a compact set K such that

$$P_V(x) + K = G.$$

Next, since K is compact, and O is open, there exists some finite set F such that

$$K \subseteq F + O.$$

We will show that

$$G = P_U(x) + F.$$

Indeed, let $t \in G$. Then, there exists some $s \in P_V(x)$ and $k \in K$ such that $t = s + k$. Since $k \in K \subseteq F + O$, there exists some $f \in F$ and $u \in O$ such that

$$k = f + u.$$

Therefore,

$$t = s + f + u.$$

We claim that $s + u \in P_U(x)$, which will complete the proof. Indeed, $s \in P_V(x)$ implies that

$$(\alpha(s,x),x) \in V.$$

Since $u, 0 \in O$ and $(\alpha(s,x),x) \in V$, by (5), we have

$$(\alpha(u,\alpha(s,x)),\alpha(0,x)) = (\alpha(u+s,x),x) \in U$$

and, hence, $s + u \in P_U(x)$ as claimed.   ∎

The next result is a consequence of the previous proposition.

Theorem 4.7 (Main result - I) *Let G act on a complete uniform topological Hausdorff space $(X, \mathcal{U})$ such that the action is continuous and $\mathcal{U}$ is G-invariant. Let $x \in X$. Then, the following statements are equivalent.*

(i) *x is Bohr-type almost periodic.*

(ii) *x is Bochner-type almost periodic.*

(iii) *x is pseudo-Bochner-type almost periodic.*

Proof Note that $\mathcal{U}$ induces a uniformity $\mathcal{U}_x$ on $H_x^{\mathcal{U}}$, which is complete and G-invariant. By Lemma 3.8, the action α is equicontinuous on $(H_x^{\mathcal{U}}, \mathcal{U}_x)$. Moreover, it is easy to see that x is Bohr-type, Bochner-type, or pseudo-Bochner-type almost periodic in $(X, \mathcal{U})$, respectively, if and only if x is Bohr-type, Bochner-type, or pseudo-Bochner-type almost periodic in $(H_x^{\mathcal{U}}, \mathcal{U}_x)$.

The equivalence of (i) and (iii) now follows from Proposition 4.6, and the equivalence of (ii) and (iii) follows from Proposition 4.4 (applied to $(H_x^{\mathcal{U}}, \mathcal{U}_x)$).   ∎

We complete the section by discussing how—if the uniformity is G-invariant and complete—$H_x^{\mathcal{U}}$ has a natural abelian group structure.

Proposition 4.8 *Let G act on a uniform topological Hausdorff space $(X, \mathcal{U})$ such that the uniformity $\mathcal{U}$ is G-invariant and α is continuous. Let $x \in X$ be given such that $H_x^{\mathcal{U}}$ is complete. Then, the following statements hold.*

(a) *$H_x^{\mathcal{U}}$ is a topological abelian group with the addition $\oplus$ induced by*

$$\alpha(t,x) \oplus \alpha(s,x) := \alpha(s+t,x).$$

(b) *$F : G \to H_x^{\mathcal{U}}$ via*

$$F(t) = \alpha(t,x)$$

is a uniformly continuous group homomorphism, with dense range.

Proof Note first that α is equicontinuous on $H_x^{\mathcal{U}}$ by Lemma 3.8.

(a) We first show that $\oplus$ defined by

$$\alpha(t, x) \oplus \alpha(s, x) := \alpha(s + t, x)$$

is well defined on $O(x)$, defines an abelian group structure, and addition and inversion are continuous with respect to the topology induced by $\mathcal{U}$.

To show that addition is well defined consider $t, t', s, s' \in G$ with $\alpha(t, x) = \alpha(t', x)$ and $\alpha(s, x) = \alpha(s', x)$. Then a direct computation gives

$$\alpha(t + s, x) = \alpha(t, \alpha(s, x)) = \alpha(t, \alpha(s', x)) = \alpha(s', \alpha(t, x))$$
$$= \alpha(s', \alpha(t', x)) = \alpha(t' + s', x).$$

This shows that $\oplus$ is well defined. By definition, $O(x)$ is closed under $\oplus$.

Associativity of the addition on G yields that $\oplus$ is associative. Indeed, a direct computation gives

$$(\alpha(t, x) \oplus \alpha(s, x)) \oplus \alpha(r, x) = \alpha((t + s) + r, x) = \alpha(t + (s + r), x)$$
$$= \alpha(t, x) \oplus (\alpha(s, x) \oplus \alpha(r, x)).$$

Since G is abelian, it is obvious that $\oplus$ is commutative. Moreover, for all $t \in G$, we have

$$\alpha(t, x) \oplus \alpha(0, x) = \alpha(t, x).$$

This shows that $x = \alpha(0, x)$ is the identity in $O(x)$. Finally, for all $t \in G$, we have

$$\alpha(t, x) \oplus \alpha(-t, x) = \alpha(0, x).$$

This shows that $(O(x), \oplus)$ is an abelian group.

We now show that $(O(x), \oplus)$ becomes a topological group when equipped with the topology induced by $\mathcal{U}$.

Indeed, let $U \in \mathcal{U}$ be any entourage. Let W be such that $W \circ W \subseteq U$. Since α is G-invariant, there exists some $V \in \mathcal{U}$ such that, for all $(y, z) \in V$ and $t \in G$, we have

$$(6) \qquad\qquad\qquad (\alpha(t, y), \alpha(t, z)) \in W.$$

Now, for all $\alpha(t_1, x), \alpha(t_2, x), \alpha(s_1, x), \alpha(s_2, x) \in O_x$ with

$$(\alpha(s_1, x), \alpha(s_2, x)) \in V \qquad \text{and} \qquad (\alpha(t_1, x), \alpha(t_2, x)) \in V,$$

(6) implies

$$(\alpha(s_1 + t_1, x), \alpha(s_2 + t_1, x)) \in W \qquad \text{and} \qquad (\alpha(t_1 + s_2, x), \alpha(t_2 + s_2, x)) \in W$$

and, hence,

$$(\alpha(s_1 + t_1, x), \alpha(s_2 + t_2, x)) \in U.$$

It follows that

$$\big(\alpha(s_1, x) \oplus \alpha(t_1, x), \alpha(s_2, x) \oplus \alpha(t_2, x)\big) \in U.$$

This proves that $\oplus : O(x) \times O(x) \to O(x)$ is uniformly continuous with respect to $\mathcal{U}$.

We now turn to inversion. As already discussed, any $\alpha(t,x) \in O(x)$ has the inverse

$$\ominus\alpha(t,x) = \alpha(-t,x).$$

Let $U \in \mathcal{U}$ be arbitrary, and let $V = U^{-1}$. Since α is G-invariant, there exists some $W \in \mathcal{U}$ such that, for all $(y,z) \in V$ and $t \in G$, we have

$$(7) \qquad\qquad\qquad (\alpha(t,y),\alpha(t,z)) \in V.$$

Now, if $(\alpha(t,x),\alpha(s,x)) \in V$, then we have by (7)

$$(\alpha(-s-t,\alpha(t,x)),\alpha(-s-t,\alpha(s,x))) \in V = U^{-1}$$

and, hence,

$$(\alpha(-t,x),\alpha(-s,x)) \in U.$$

This show $\ominus : G \to G$ is uniformly continuous with respect to $\mathcal{U}$.

Now, $O(x)$ is a dense subset of the complete set $H_x^{\mathcal{U}}$. Therefore, by [6, Chapter II and Proposition 13], $H_x^{\mathcal{U}}$ is—what is known as—the Hausdorff completion of $O(x)$. Since $(O(x),\oplus,\mathcal{U})$ is a topological abelian group, we then infer by [6, Chapter III and Theorems I and II] that $H_x^{\mathcal{U}}$ is an abelian group and the inclusion $i : O(x) \to H_x^{\mathcal{U}}$ is a group homomorphism.

(b) It is obvious that $F' : G \longrightarrow O(x), t \mapsto \alpha(t,x)$ is a group homomorphism as is the inclusion

$$i : O(x) \longrightarrow H_x^{\mathcal{U}}, \qquad y \mapsto y.$$

Thus, we have the following group homomorphisms:

$$G \xrightarrow{F'} O(x) \overset{i}{\hookrightarrow} H_x^{\mathcal{U}}.$$

Since the inclusion i is uniformly continuous, to complete the proof, we need to show that F' is uniformly continuous.

Let $U \in \mathcal{U}$ be arbitrary. Since $\mathcal{U}$ is G-invariant, there exists some $U' \in \mathcal{U}$ such that

$$(8) \qquad\qquad \{(\alpha(t,x),\alpha(t,y)) : t \in G, (x,y) \in U'\} \subseteq U.$$

Next, since α is continuous, it is equicontinuous on $H_x^{\mathcal{U}}$ by Lemma 3.8. Thus, there exists an open set $O \subseteq G$ containing 0 and $V \in \mathcal{U}$ such that

$$(9) \qquad \{(\alpha(s,x),\alpha(t,y)) : s,t \in O, (x,y) \in V \cap H_x^{\mathcal{U}} \times H_x^{\mathcal{U}}\} \subseteq U'.$$

Now, let $s,t \in G$ be so that $s-t \in U$. Then, by (9), we have $(\alpha(s-t,x),\alpha(0,x)) \in U'$ and hence, by (8), we have

$$(F'(t),F'(s)) = (\alpha(t,\alpha(s-t,x)),\alpha(t,\alpha(0,x))) \in U.$$

This shows that F' is uniformly continuous, and completes the proof. ∎

Remark 4.9 (a) Let us emphasize that we do not need compactness (i.e., almost periodicity of any form) of $H_x^{\mathcal{U}}$ to obtain the group structure on $H_x^{\mathcal{U}}$. To illustrate this, we include the following example. Consider the translation action of $\mathbb{R}$ on the vector space $C_u(\mathbb{R})$ of uniformly continuous and bounded functions equipped with the supremum norm $\|\cdot\|_\infty$. For any nontrivial $f \in C_u(\mathbb{R})$ with $\lim_{x\to\pm\infty} f(x) = 0$ is

easy to see that H_f is isomorphic to $\mathbb{R}$, under the canonical isomorphism $t \in G \leftrightarrow T_t f \in H_f$.

(b) Since $F' : G \longrightarrow O(x)$ is onto, by the fundamental theorem of group isomorphism, we have

$$O(x) \simeq G/\mathrm{Per(x)}.$$

In the case that the hull is compact we can even say more.

Corollary 4.10 *Let G act on a uniform topological Hausdorff space X such that the uniformity $\mathcal{U}$ is G-invariant and α is continuous. If $x \in X$ is Bochner-type almost periodic, then $H_x^{\mathcal{U}}$ is a compact abelian group with the addition $\oplus$ induced by*

$$\alpha(t, x) \oplus \alpha(s, x) := \alpha(s + t, x)$$

and $F : G \longrightarrow H_x^{\mathcal{U}}, t \mapsto \alpha(t, x)$, is a continuous group homomorphism with dense range.

Proof Since x is Bochner-type almost periodic, $H_x^{\mathcal{U}}$ is compact. Hence, it is also complete. Now, the statement follows from Proposition 4.8. ∎

The previous corollary can be understood in terms of the *Bohr-compactification G_b* of G. This is the (unique) compact group with the universal property that there exists a continuous group homomorphism $\iota_b : G \longrightarrow G_b$ such that, for any continuous group homomorphism $\psi : G \longrightarrow H$ into a compact group H, there exists a unique mapping $\Psi : G_b \longrightarrow H$ with $\psi = \Psi \circ \iota$ (see, for example, [20, Proposition 4.2.6] for details).

Corollary 4.11 *Let G act on a uniform topological Hausdorff space X such that the uniformity $\mathcal{U}$ is G-invariant and α is continuous. If $H_x^{\mathcal{U}}$ is a compact abelian group with the addition $\oplus$ induced by*

$$\alpha(t, x) \oplus \alpha(s, x) := \alpha(s + t, x),$$

then there exists a surjective continuous group homomorphism $\Psi : G_b \longrightarrow H_x^{\mathcal{U}}$ such that

$$\alpha(t, x) = \Psi(\iota_b(t))$$

holds for all $t \in G$.

Proof Since $H_x^{\mathcal{U}}$ is a compact abelian group, by the universal property, there exists a continuous group homomorphism $\Psi : G_b \to H_x^{\mathcal{U}}$ such that

$$\alpha(t, x) = F(t) = \Psi(i_b(t)) \quad \text{for all } t \in G.$$

Since F has dense range, so does Ψ. Furthermore, since G_b is compact, so is $F(G_b)$. It follows that the range of F is compact and hence closed. In particular, F is surjective. ∎

Combination of the previous results gives our next main result.

Theorem 4.12 (Main result - II) *Let G act on a complete uniform Hausdorff topological X space such that the action is continuous and the uniformity $\mathcal{U}$ is G-invariant. Then, for $x \in X$, the following assertions are equivalent.*

(i) *x is Bohr-type almost periodic.*
(ii) *x is Bochner-type almost periodic.*

(iii) x is pseudo-Bochner-type almost periodic.

(iv) $H_x^{\mathcal{U}}$ is a compact abelian group with the group operation $\oplus$ induced by

$$\alpha(t, x) \oplus \alpha(s, x) := \alpha(s + t, x).$$

(v) There exists a continuous function $\Psi : G_{\mathrm{b}} \to H_x^{\mathcal{U}}$ such that

$$\alpha(t, x) = \Psi(i_{\mathrm{b}}(t)) \quad \text{for all } t \in G.$$

Moreover, in this case, $F(t) = \alpha(t, x)$ define a group homomorphism with dense range $F : G \to H_x^{\mathcal{U}}$, and Ψ is an onto group homomorphism.

Proof   The equivalences between (i), (ii), and (iii) follow from Theorem 4.7. The implication (ii) $\Longrightarrow$ (iv) follows from Corollary 4.10. This corollary gives also that $F : G \longrightarrow H_x^{\mathcal{U}}, t \mapsto \alpha(t, x)$, is a continuous group homomorphism with dense range. The implication (iv) $\Longrightarrow$ (v) follows from Corollary 4.11. Next, we will show (v) $\Longrightarrow$ (ii): Since Ψ is continuous, and G_{b} is compact, so is $\Psi(G_{\mathrm{b}}) =: K$. Moreover, by (v) we also have $\alpha(t, x) = \Psi(i_{\mathrm{b}}(t))$, for all $t \in G$, which implies $O(x) \subset K$. As the compact K is closed, we infer

$$H_x^{\mathcal{U}} = \overline{O(x)} \subseteq K.$$

Thus, $H_x^{\mathcal{U}}$ must be compact as it is a closed subset of a compact set. This shows (ii). Moreover, as K is contained in $H_x^{\mathcal{U}}$ (by definition of Ψ), we have even

$$H_x^{\mathcal{U}} \subseteq K = \Psi(G_{\mathrm{b}}) \subseteq H_x^{\mathcal{U}}$$

giving $H_x^{\mathcal{U}} = K$ and hence Ψ is onto. This completes the proof.   ∎

5   The mixed uniformity

In the preceding discussion and most notably in Lemma 3.8, we have seen that the continuity of α is equivalent to the equicontinuity of α on orbit closures (provided $\mathcal{U}$ is G-invariant). This—so to say—booster of continuity has been a main tool in our considerations. Now, it may happen that α is not continuous. However, even if we lack continuity, by a simple mixing process, we can make α equicontinuous, without changing the Bohr-type almost periodicity, or the other basic properties of the uniformity. This is discussed in this section.

Note, however, that since the equivalence between Bohr-type and pseudo-Bochner-type almost periodicity relies on equicontinuity, this mixing process can actually change pseudo-Bochner- and Bochner-type almost periodicity, and we will see such an example in the next section.

The mixed uniformity we study in this section has appeared before in some particular cases (see, for example, [3, p. 15] and [19, Definition 4.2]).

Let G be an LCAG, and let $\mathcal{U}$ be a G-invariant uniformity on some set X and $\alpha : G \times X \to X$ any group action. Denote, by $\mathcal{O}$, the set of all open sets in G containing 0.

Now, for $U \in \mathcal{U}$ and $O \in \mathcal{O}$, define

$$V[O, U] := \{(\alpha(t, x), y) : t \in O, (x, y) \in U\},$$

and set

$$\mathcal{U}_{\mathrm{mix}} := \{ V \subseteq X \times X : \text{ there exists } U \in \mathcal{U}, O \in \mathcal{O} \text{ such that } V[O, U] \subseteq V \}.$$

Let us first show that $\mathcal{U}_{\mathrm{mix}}$ is indeed a uniformity on X.

Proposition 5.1 ($\mathcal{U}_{\mathrm{mix}}$ as uniformity)   *If $\mathcal{U}$ is G-invariant, then $\mathcal{U}_{\mathrm{mix}}$ is a uniformity on X.*

Proof   We have to show that the five points defining a uniformity are satisfied. It is rather straightforward to show this. For the convenience of the reader, we include a proof.

- Let $V \in \mathcal{U}_{\mathrm{mix}}$. Then, there exists some $O \in \mathcal{O}$ and $U \in \mathcal{U}$ so that $V[O, U] \subseteq V$. Then, for all $x \in X$, we have $0 \in O$ and $(x, x) \in U$ and, hence,

$$(x, x) = (\alpha(0, x), x) \in V[O, U] \subseteq V.$$

  This shows that $\Delta \subseteq V$.
- Let $V \in \mathcal{U}_{\mathrm{mix}}$ and $V \subseteq W$. Since $V \in \mathcal{U}_{\mathrm{mix}}$, there exists some $O \in \mathcal{O}$ and $U \in \mathcal{U}$ such that $V[O, U] \subseteq V \subseteq W$. This shows that $W \in \mathcal{U}_{\mathrm{mix}}$.
- Let $V_1, V_2 \in \mathcal{U}$. Then, there exist some $O_1, O_2 \in \mathcal{O}$ and $U_1, U_2 \in \mathcal{U}$ such that $V[O_j, U_j] \subseteq V_j$. We have $O := O_1 \cap O_2 \in \mathcal{O}$, $U = U_1 \cap U_2 \in \mathcal{U}$. Moreover, it follows immediately from the definition that, for $1 \le j \le 2$, we have

$$V[O, U] \subseteq V[O_j, U_j].$$

  Therefore,

$$V[O, U] \subseteq V[O_1, U_1] \cap V[O_2, U_2] \subseteq V_1 \cap V_2,$$

  which shows that $V_1 \cap V_2 \in \mathcal{U}_{\mathrm{mix}}$.
- Let $V \in \mathcal{U}_{\mathrm{mix}}$. Then, there exists some $O \in \mathcal{O}$ and $U \in \mathcal{U}$ so that $V[O, U] \subseteq V$. Then, since $O \in \mathcal{O}$, we can find some $O' \in \mathcal{O}$ so that $O' + O' \subseteq O$. Since $U \in \mathcal{U}$, there exists some $U_1 \in \mathcal{U}$ such that $U_1 \circ U_1 \subseteq U$.

  By the G-invariance of $\mathcal{U}$, there exists some $U' \in \mathcal{U}$ such that

$$(10) \qquad \{ (\alpha(t, x), \alpha(t, y)) : t \in G, (x, y) \in U' \} \subseteq U_1.$$

Note that U' is contained in U (as we can set $t = 0$) and set $V' := V[O', U']$.

Let $(x, y) \in V' \circ V'$. Then, there exists some $z \in X$ such that $(x, z), (z, y) \in V' = V[O', U']$. Therefore, there exists some $x', z' \in X$ and $t, s \in O$ such that

$$(x', z), (z', y) \in U', \qquad x = \alpha(t, x'), \qquad \text{and} \qquad z = \alpha(s, z').$$

Note here that $z' = \alpha(-s, \alpha(s, z')) = \alpha(-s, z)$ and $x' = \alpha(-t, x)$ hold. Therefore, by (10), we have $(\alpha(-t - s, x), \alpha(-s, z)) \in U' \subseteq U_1$ and $(\alpha(-s, z), y) \in U' \subseteq U_1$ and, hence, $(\alpha(-t - s, x), y) \in U_1 \circ U_1 \subseteq U$. This gives

$$(x, y) = (\alpha(t + s, \alpha(-t - s, x)), y) \in V[O, U] \subseteq V.$$

Hence, we obtain $V' \circ V' \subseteq V$.

- Let $V \in \mathcal{U}_{\text{mix}}$. Then, there exists some $O \in \mathcal{O}$ and $U \in \mathcal{U}$ so that $V[O, U] \subseteq V$. Next, since $U \in \mathcal{U}$ and the $\mathcal{U}$ is G-invariant, there exists some $U' \in \mathcal{U}$ such that

$$\{(\alpha(t, x), \alpha(t, y)) : t \in G, (x, y) \in U'\} \subseteq U.$$

Now, let $(x, y) \in V[-O, (U')^{-1}]$. Then, there exists some $t \in -O$ and $x' \in X$ such that $(x', y) \in (U')^{-1}$ and $x = \alpha(t, x')$. Next, $(x', y) \in (U')^{-1}$ gives $(y, x') \in U'$ and, hence, $(\alpha(t, y), x) \in U$. Therefore, since $-t \in O$, we have

$$(y, x) = (\alpha(-t, \alpha(t, y)), x) \in V[O, U] \subseteq V$$

and, hence, $(x, y) \in V^{-1}$. This proves that

$$V[-O, (U')^{-1}] \subseteq V^{-1}$$

and, hence, $V^{-1} \in \mathcal{U}_{\text{mix}}$.

∎

Definition 5.2 (Mixed uniformity)   Let the LCAG G act on the uniform space $(X, \mathcal{U})$ such that $\mathcal{U}$ is a G-invariant. Then, $\mathcal{U}_{\text{mix}}$ is called the *mixed uniformity* induced from the action of G.

Proposition 5.3 (Characterization of $\mathcal{U}_{\text{mix}}$)   *Let the LCAG G act on the uniform space $(X, \mathcal{U})$ such that $\mathcal{U}$ is G-invariant. Then, the following statements hold.*

(a) *The action α is equicontinuous on $(X, \mathcal{U}_{\text{mix}})$ and $\mathcal{U}$ is finer than $\mathcal{U}_{\text{mix}}$, that is, $\mathcal{U}_{\text{mix}} \subseteq \mathcal{U}$ holds.*
(b) *$\mathcal{U}$ is the finest uniformity satisfying (a), i.e., whenever $\mathcal{U}'$ is a uniformity such that $\mathcal{U}$ is finer than $\mathcal{U}'$ and α is equicontinuous on $(X, \mathcal{U}')$, then $\mathcal{U}' \subseteq \mathcal{U}_{\text{mix}}$ holds.*

In particular, α is equicontinuous on $(X, \mathcal{U})$ if and only if $\mathcal{U} = \mathcal{U}_{\text{mix}}$.

Proof   The last statement is immediate from (a) and (b).

(a) From Lemma 3.3(iii), it follows easily that α is equicontinuous with respect to $(X, \mathcal{U}_{\text{mix}})$. To show that $\mathcal{U}$ is finer that $\mathcal{U}_{\text{mix}}$ we consider an arbitrary $V \in \mathcal{U}_{\text{mix}}$. Then, there exists some $U \in \mathcal{U}$ and $O \in \mathcal{O}$ such that

$$V[O, U] \subseteq V.$$

Now, clearly $U \subseteq V[O, U]$ holds and we get $U \subseteq V$. Since $\mathcal{U}$ is a uniformity, we get $V \in \mathcal{U}$ and, therefore, $\mathcal{U}_{\text{mix}} \subseteq \mathcal{U}$ follows as $V \in \mathcal{U}_{\text{mix}}$ was arbitrary.

(b) Let $\mathcal{U}'$ with $\mathcal{U}' \subseteq \mathcal{U}$ be given such that α is equicontinuous on $(X, \mathcal{U}')$. Let $V \in \mathcal{U}'$ be arbitrary. Since α is equicontinuous on $(X, \mathcal{U}')$, by Lemma 3.3(ii), there exists some $O \in \mathcal{O}$ and $U \in \mathcal{U}$ such that

$$V[O, U] \subseteq V.$$

As $\mathcal{U}_{\text{mix}}$ is a uniformity it is upward closed and $V \in \mathcal{U}_{\text{mix}}$ follows. As $V \in \mathcal{U}'$ was arbitrary, the desired inclusion $\mathcal{U}' \subseteq \mathcal{U}_{\text{mix}}$ follows.   ∎

We now turn to the question of metrizability of $\mathcal{U}_{\text{mix}}$.

Proposition 5.4 (Metrizability of $\mathcal{U}_{\text{mix}}$)   *If $\mathcal{U}$ and the topology on G are metrizable, then $\mathcal{U}_{\text{mix}}$ is metrizable.*

Proof Since $\mathcal{U}$ is metrizable, we can find a countable basis of entourages $\mathcal{B} \subseteq \mathcal{U}$. Moreover, since the topology on G is metrizable, we can find some countable $\mathcal{C} \subseteq \mathcal{O}$ which is a basis of open sets at $t = 0$ for the topology of G. It follows immediately that $\{V[O, U] : O \in \mathcal{C}, U \in \mathcal{B}\}$ is a countable bases of entourages for $\mathcal{U}_{\text{mix}}$. The desired statement now follows from Theorem 2.5. ∎

Remark 5.5 The converse is not true. Indeed, consider the action of any group G on an arbitrary set X. Let $\mathcal{U}$ be the uniformity on X given by the discrete metric d.

It is easy to see that

$$\mathcal{U} = \{U \subseteq X \times X : \Delta \subseteq X\}.$$

Then, it follows immediately that $\mathcal{U}_{\text{mix}} = \mathcal{U}$ is metrizable, no matter if the topology on G is metrizable or not.

Now, we show that if $\mathcal{U}$ is G-invariant, complete and metrizable, $\mathcal{U}_{\text{mix}}$ is complete.

Proposition 5.6 *Let $\mathcal{U}$ be a G-invariant uniformity. If $\mathcal{U}$ and the topology on G are metrizable and if $\mathcal{U}$ is complete, then $\mathcal{U}_{\text{mix}}$ is complete.*

Proof Since $\mathcal{U}$ is G-invariant and metrizable, by Lemma 3.2, there exists a G-invariant metric $d_{\mathcal{U}}$ which defines the uniformity $\mathcal{U}$. Next, let d_G be any G-invariant metric on G which gives the topology of G. Since $\mathcal{U}_{\text{mix}}$ is metrizable (by the preceding proposition), to prove completeness, it suffices to show that every sequence $(x_n) \in X$ which is Cauchy with respect to $\mathcal{U}_{\text{mix}}$ has a subsequence which is convergent in $\mathcal{U}_{\text{mix}}$ to some $x \in X$.

Let (x_n) be a Cauchy sequence with respect to $\mathcal{U}_{\text{mix}}$. Let

$$O_n = \{t \in G : d_G(t, 0) < 2^{-n}\} = B_{2^{-n}}(0),$$
$$U_n = \{(x, y) \in X \times X : d_{\mathcal{U}}(x, y) < 2^{-n}\} \in \mathcal{U},$$
$$V_n = V[O_n, U_n] \in \mathcal{U}_{\text{mix}}.$$

We note

$$V_n = \{(\alpha(t, x), y) : d_U(x, y) < 2^{-n}, d_G(t, 0) < 2^{-n}\}.$$

Now, since (x_n) is Cauchy with respect to $\mathcal{U}_{\text{mix}}$, for each $m \in \mathbb{N}$, there exists some $M(m) \in \mathbb{N}$ such that, for all $n_1, n_2 > M(m)$, we have

$$(x_{n_1}, x_{n_2}) \in V_m.$$

Hence, we can construct inductively a sequence (n_k) such that

$$n_1 > M(1) \quad \text{and} \quad n_{k+1} > \max\{n_k, M(1), M(2), \ldots, M(k), M(k+1)\}.$$

Note here that $n_{k+1} > n_k$ by construction and hence (x_{n_k}) is a subsequence of (x_n). Next, since $n_k, n_{k+1} > M_k$, the definition of M_k gives

$$(x_{n_{k+1}}, x_{n_k}) \in V_k.$$

Therefore, there exist some $t_k \in O_k$ and y_k such that

$$(x_{n_{k+1}}, x_{n_k}) = (\alpha(r_k, y_k), x_{n_k}) \quad \text{and} \quad d_U(y_k, x_k) < 2^{-k}.$$

Let $t_k := -r_k$. Then $d(t_k, 0) = d(0, r_k) < 2^{-k}$, and $x_{n_{k+1}} = \alpha(r_k, y_k)$ gives

$$y_k = \alpha(-r_k, x_{n_{k+1}}) = \alpha(t_n, x_{n_{k+1}}).$$

In particular,

$$d_G(t_k, 0) < 2^{-k} \qquad \text{and} \qquad d_U(\alpha(t_k, x_{n_{k+1}}), x_{n_k}) = d_U(y_k, x_k) < 2^{-k}.$$

Define $s_k = t_1 + t_2 + \cdots + t_{k-1} \in G$. Then, since d_G is G-invariant, we have

$$d(s_{k+1}, s_k) = d(s_{k+1} - s_k, 0) = d(t_k, 0) < 2^{-k}$$

and a standard telescopic argument shows that (s_k) is a Cauchy sequence in G, and hence, convergent to some $s \in G$.

Next, since the metric d_U is G-invariant, we have

$$\begin{aligned}
d_U(\alpha(s_{k+1}, x_{n_{k+1}}), \alpha(s_k, x_{n_k})) &= d_U(\alpha(s_{k+1} - s_k, x_{n_{k+1}}), x_{n_k}) \\
&= d_U(\alpha(t_k, x_{n_{k+1}}), x_{n_k} < 2^{-k}.
\end{aligned}$$

Again, by a standard telescopic argument, the sequence (y_k) with $y_k = \alpha(s_k, x_{n_k})$ is a Cauchy sequence in (X, d_U). Since $\mathcal{U}$ is complete, it follows that (y_k) converges to some $y \in X$.

Now, since (x_{n_k}) with $x_{n_k} = \alpha(-s_k, y_k), y_k$ converges in $\mathcal{U}$ to $y \in X$ and (s_k) converges in G to s, the definition of $\mathcal{U}_{\mathrm{mix}}$ implies immediately that (x_{n_k}) converges to $\alpha(-s, y)$. This completes the proof. ∎

Next, we show that mixing the uniformity does not change Bohr-type almost periodicity for an element.

Theorem 5.7 *Let α be an action of G on a uniform space $(X, \mathcal{U})$ such that $\mathcal{U}$ is G-invariant. Then, $x \in X$ is Bohr-type almost periodic with respect to $\mathcal{U}$ if and only if x is Bohr-type almost periodic with respect to $\mathcal{U}_{\mathrm{mix}}$.*

Proof $\Longrightarrow$: This follows immediately from the definition of Bohr-type almost periodicity and $\mathcal{U}_{\mathrm{mix}} \subseteq \mathcal{U}$.

$\Longleftarrow$: Let an arbitrary $U \in \mathcal{U}$ be given. Fix $O \in \mathcal{O}$ with compact closure and consider

$$V = V[O, U] \in \mathcal{U}_{\mathrm{mix}}.$$

Since x is Bohr-type almost periodic with respect to $\mathcal{U}_{\mathrm{mix}}$, the set

$$P_V(x) := \{t \in G : (\alpha(t, x), x) \in V\}$$

is relatively dense. Hence, there exists some compact set $K \subseteq G$ such that

$$P_V(x) + K = G.$$

Now, let $t \in P_V(x)$ be arbitrary. This means

$$(\alpha(t, x), x) \in V[O, U]$$

and, therefore, there exist some $s \in O$ and $y \in X$ such that

$$\alpha(t, x) = \alpha(s, y) \text{ and } (y, x) \in U$$

holds. The first relation gives $y = \alpha(t - s, x)$ and, therefore,

$$(\alpha(t - s, x), x) \in U.$$

This shows

$$t - s \in P_U(x) = \{t \in G : (\alpha(t, x), x) \in U\}.$$

Thus, we obtain

$$t \in P_U(x) + s \subseteq P_U(x) + O \subseteq P_U(x) + \overline{O}.$$

Since $t \in P_V(x)$ was arbitrary, we get

$$P_V(x) \subseteq P_U(x) + \overline{O},$$

and hence

$$G = P_V(x) + K \subseteq P_U(x) + (\overline{O} + K).$$

Since $\overline{O} + K$ is compact, we get that $P_U(x)$ is relatively dense. As $U \in \mathcal{U}$ was arbitrary, the claim follows. ∎

When we now collect all results from this section and combine them with Theorem 4.12, the following result is an immediate consequence.

Corollary 5.8 *Let G act on a complete metric space (X, d) space such that d is pseudo G-invariant, and let $\mathcal{U} = \mathcal{U}_d$. Then, for $x \in X$, the following assertions are equivalent.*

(i) *x is Bohr-type almost periodic in $(X, \mathcal{U}, \alpha)$.*
(ii) *x is Bohr-type almost periodic in $(X, \mathcal{U}_{\mathrm{mix}}, \alpha)$.*
(iii) *x is Bochner-type almost periodic in $(X, \mathcal{U}_{\mathrm{mix}}, \alpha)$.*
(iv) *x is pseudo-Bochner-type almost periodic in $(X, \mathcal{U}_{\mathrm{mix}}, \alpha)$.*
(v) *$H_x^{\mathcal{U}_{\mathrm{mix}}}$ is a compact abelian group with the group operation $\oplus$ induced by*

$$\alpha(t, x) \oplus \alpha(s, x) := \alpha(s + t, x).$$

(vi) *There exists a continuous function $\Psi : G_{\mathrm{b}} \to H_x^{\mathcal{U}_m}$ such that*

$$\alpha(t, x) = \Psi(i_{\mathrm{b}}(t)) \quad \text{for all } t \in G.$$

Moreover, in this case, $F(t) = \alpha(t, x)$ define a group homomorphism with dense range $F : G \to H_x^{\mathcal{U}}$, and Ψ is an onto group homomorphism.

6 Examples

In this section, we present a wealth of examples for our results. This will, in particular, show how all earlier corresponding results on almost periodicity that we are aware of fall within our framework.

6.1 Bohr and Bochner almost periodicity for functions

Consider X to be the vector space $C_{\mathrm{u}}(G)$ of uniformly continuous bounded functions on an LCAG G with the topology given by $\| \cdot \|_\infty$ and the translation action

$$\alpha(t, f)(x) := f(x - t).$$

It is easy to see that α is G-invariant, equicontinuous and that the uniformity is complete. Therefore, Theorem 4.12 gives the following well-known result.

Theorem 6.1   *Let $f \in C_u(G)$. Then, the following are equivalent.*

 (i) *f is Bohr almost periodic.*
 (ii) *f is Bochner almost periodic.*
(iii) *$H_f := \overline{\{T_t f \, : \, t \in G\}}$ is a compact abelian group with the addition operation induced by*

$$(T_t f) \oplus (T_s f) = T_{s+t} f.$$

(iv) *There exists a continuous function $\Psi : G_b \to H_f$ such that*

$$T_t f = \Psi(i_b(t)) \quad \text{for all } t \in G.$$

Moreover, in this case, $F(t) = \alpha(t, x)$ define a group homomorphism with dense range $F : G \to H_f$, and Ψ is an onto group homomorphism.

Note here that the mapping $\delta_0 : C_u(G) \to \mathbb{C}$ with

$$\delta_0(f) = f(0)$$

is uniformly continuous, and hence so is $\delta_0 \circ \Psi \in C(G_b)$. Then, (iv) in Theorem 6.1 implies that there exists some $g := \delta_0 \circ \Psi \in C(G_b)$ such that

$$f(-t) = g(i_b(t)).$$

It follows easily from here that (iv) in Theorem 6.1 can be replaced by

(iv') *There exists some $g \in C(G_b)$ such that $f = g \circ i_b$.*

6.2   Group-valued almost periodic functions

In this section, we review the concept of group-valued almost periodic functions, as discussed recently in [12].

Let G, H be two LCAG, with H complete. Let $\mathcal{H}$ be the set of all functions $f : G \to H$. For each neighborhood, $W \subseteq H$ of 0 define

$$U_W := \{(f, g) \in \mathcal{H} \times \mathcal{H} \, : \, f(x) - g(x) \in W \text{ for all } x \in G\},$$

and let

$$\mathcal{U} = \{U \subseteq \mathcal{H} \times \mathcal{H} \, : \, \text{there exists } W \text{ such that } U_W \subseteq U\}.$$

As usual, let $C(G : H)$, $C_b(G : H)$, and $C_u(G : H)$ denote the subspaces of $\mathcal{H}$ consisting of continuous, continuous bounded, and uniformly continuous and bounded functions, respectively.

It is easy to see that the topology induced by $\mathcal{U}$ on $C(G : H)$, $C_b(G : H)$, and $C_u(G : H)$, respectively, is metrizable, if and only if H is metrizable. Indeed, if the topology induced by $\mathcal{U}$ on any of these spaces is metrizable, then its restriction to the subspace of constant functions is metrizable, from which we immediately get that H is metrizable. On another hand, if $\{W_n \, : \, n \in \mathbb{N}\}$ is a countable basis of open sets at $0 \in H$, then $\{U_{W_n}[0] \, : \, n \in \mathbb{N}\}$ becomes an open basis at the constant function 0 for the topology induced by $\mathcal{U}$.

From [12, Lemma 6], we find the following.

Lemma 6.2 (a) $\mathcal{U}$ *is a uniformity on* $\mathcal{H}$.
(b) $\mathcal{H}$ *is complete with respect to* $\mathcal{U}$.
(c) $C(G : H), C_b(G : H),$ *and* $C_u(G : H)$ *are closed in* $\mathcal{H}$.

Note here that G acts naturally on $\mathcal{H}$ via the translation

$$(T_t g)(x) = g(x - t),$$

and that $\mathcal{U}$ is G-invariant. While the translation action of G is not continuous on $\mathcal{H}$, it is equicontinuous on $C_u(G : H)$. Therefore, Theorem 4.12 gives the following statement (compare [12, Proposition 9]).

Theorem 6.3 *Let* $f \in C_u(G : H)$. *Then, the following are assertions equivalent.*

 (i) f *is Bohr-type almost periodic.*
 (ii) f *is Bochner-type almost periodic.*
 (iii) $H_f := \overline{\{T_t f : t \in G\}}$ *is a compact abelian group with the addition operation induced by*

$$(T_t f) \oplus (T_s f) = T_{s+t} f.$$

 (iv) *There exists a continuous function* $\Psi : G_b \to H_f$ *such that*

$$T_t f = \Psi(i_b(t)) \quad \text{for all } t \in G.$$

Moreover, in this case, $F(t) = \alpha(t, x)$ *define a group homomorphism with dense range* $F : G \to H_f$, *and* Ψ *is an onto group homomorphism.*

Exactly as before, we can define a continuous mapping $F : C(G : H) \to H$ via $F(h) = h(0)$. Then, for each f satisfying the conditions of Theorem 6.3, the function $f_b(x) := (F \circ \Psi)(-x)$ belongs to $C(G_b : H)$ and satisfies

$$f_b(i_b(t)) = f(t) \quad \text{for all } t \in G.$$

6.3 Almost periodic functions with values in a Banach space

Bohr almost periodic functions with values in a Banach space have attracted particular attention (see, e.g., [7, 23]). Note also that this class includes functions of the form

$$G \longrightarrow \mathcal{H}, \qquad t \mapsto T_t f,$$

whenever $\mathcal{H}$ is a Hilbert space, f belongs to $\mathcal{H}$ and T is a unitary representation of G on $\mathcal{H}$. This class of functions is of prime importance in the study of diffraction (see, e.g., [10, 15]).

Now, the class of Bohr almost periodic functions on Banach spaces is actually a particular case of group-valued Bohr almost periodic functions discussed in the preceding section. Indeed, if $(B, \|\cdot\|)$ is Banach space over $\mathbb{R}$ or $\mathbb{C}$, Bohr almost periodicity in

$$BC^0(\mathbb{R}, B) := \{ f : \mathbb{R} \to B : f \text{ bounded and continuous } \}$$

is simply Bohr almost periodicity in $C_u(G : H)$, where $G = (\mathbb{R}, +)$ and $H = (B, +)$. Here, we can work in $C_u(G : H)$ because every Bohr almost periodic function in $BC(\mathbb{R}, B)$ is uniformly continuous [7]. So, Theorem 6.3 applies.

6.4 Stepanov almost periodicity

Here, we follow closely [26]. Let G be an LCAG and $K \subseteq G$ be a fixed compact set with non-empty interior, and let $1 \leq p < \infty$.

Let $BS_K^p(G)$ denote the space of all $f \in L_{loc}^p(G)$ such that

$$\|f\|_{S_K^p} := \sup_{y \in G} \left(\frac{1}{|K|} \int_{y+K} |f(t)|^p \mathrm{d}t \right)^{\frac{1}{p}} < \infty.$$

Then, $(BS_K^p(G), \|\cdot\|_{S_K^p})$ is a Banach space [26, Proposition 2.2]. Now, let $\mathcal{U}_{step}$ be the uniformity defined by this norm on $BS_K^p(G)$. Since the Stepanov norm is G-invariant by definition, so is $\mathcal{U}_{step}$. Moreover, the translation action is continuous [26, Lemma 2.5] and hence uniformly continuous on every $H_f^{\mathcal{U}_{step}}$ by Lemma 3.8.

Now, Bohr-type almost periodicity with respect to this uniformity is called *Stepanov almost periodicity*. Therefore, Stepanov almost periodicity is equivalent to Bohr-type almost periodicity with respect to this uniformity, giving [26, Proposition 2.7].

6.5 Weak almost periodicity

Consider $X = C_u(G)$ with the weak topology of the Banach space $(C_u(G), \|\cdot\|_\infty)$. Note here that the weak topology is not complete for infinite dimensional vector spaces, and hence is not complete on $C_u(G)$ unless G is a finite group. Moreover, the translation action $T_t f(x) = f(x - t)$ is not equicontinuous.

Because of this, in this case, we only get the implications.

Bochner-type almost periodicity $\implies$ pseudo-Bochner-type almost periodicity.

$\implies$ Bohr-type almost periodicity.

This is the reason, why weak almost periodicity for functions is defined via Bochner-type almost periodicity.

6.6 Autocorrelation topology

Let G be a second countable LCAG, and let $\mathcal{A} = (A_n)$ be a van Hove sequence (which exists due to the second countability of G, see [27, Proposition B.6]). Fix an open set $U = -U \subseteq G$ and define

$$\mathcal{D}_U(G) := \{\Lambda \subseteq G : \Lambda \text{ is } U\text{-uniformly discrete}\}.$$

Next, let $\mathcal{A}$ be an exhaustive nested van Hove sequence (see [19] for definitions and properties). Then, $\mathcal{A}$ induces [19, Proposition 2.1] a semi-metric $d_{\mathcal{A}}$ on $\mathcal{D}_U$ via

$$d_{\mathcal{A}}(\Lambda, \Gamma) := \limsup_{n \to \infty} \frac{\mathrm{card}((\Lambda \Delta \Gamma) \cap A_n)}{|A_n|},$$

which becomes a metric on the set $\mathcal{D} := \mathcal{D}_U / \equiv$ of equivalence classes under the equivalence relation

$$\Lambda \equiv \Gamma \Leftrightarrow d_A(\Lambda, \Gamma) = 0.$$

By [19, Proposition 2.1 and Corollary 3.10], $\mathcal{D}$ is complete and d is G-invariant.

Let $\mathcal{U}$ denote the uniformity defined by d. Note here that the uniform discreteness of elements in $\mathcal{D}_U$ imply that the translation action is never continuous. Since $(\mathcal{D}, \mathcal{U})$ is a complete uniform space, $\mathcal{U}$ is G-invariant and both the topology of G and $\mathcal{U}$ are metrizable, we can mix the uniformity as in Section 5, and then [19, Lemma 4.5 and Proposition 4.6] are simply consequences of Corollary 5.8.

6.7 Vague topology

Consider now $\mathbb{X} = \mathcal{M}(G)$ the space of unbounded measures on G. Recall that the vague topology on $\mathcal{M}(G)$ is defined as pointwise convergence on test functions $\varphi \in C_c(G)$. The group G acts via translations on $\mathcal{M}(G)$. Since the vague topology is complete, Bochner-type and pseudo-Bochner-type almost periodicity are equivalent. Moreover, a measure $\mu \in \mathbb{X}$ is Bochner-type almost periodic if and only if it is translation bounded [2, 27].

Now, the translation action of G on $\mathbb{X}$ is not equicontinuous, so Bochner- and Bohr-type almost periodicity may not be equivalent. Since every translation bounded measure is Bochner-type almost periodic, it is automatically Bohr-type almost periodic.

Next, consider the measure

$$\mu := \sum_{n \in \mathbb{N}} n \, \delta_{5^{n+1}\mathbb{Z} + 2 \cdot 5^n}.$$

Now, for each N, define

$$\mu_N := \sum_{n=1}^{N-1} n \, \delta_{5^{n+1}\mathbb{Z} + 2 \cdot 5^n} \qquad \text{and} \qquad \nu_N := \sum_{n=N}^{\infty} n \, \delta_{5^{n+1}\mathbb{Z} + 2 \cdot 5^n}.$$

Then, by construction, μ_N is 5^N periodic and $\mathrm{supp}(\nu_N) \subseteq 2 \cdot 5^N + 5^{N+1}\mathbb{Z}$. In particular, for all $m \in 5^N + 5^{N+1}\mathbb{Z}$, we have

(11)
$$T_m \mu_N = \mu_N, \quad \mathrm{supp}(\nu_N) \cap (-5^N, 5^N) = \varnothing, \quad \text{and} \quad \mathrm{supp}(T_m \nu_N) \cap (-5^N, 5^N) = \varnothing.$$

This immediately implies that μ is Bohr-type almost periodic for the vague topology. Indeed, let U be an entourage for the vague topology. Then, there exist some $\varphi_1, \dots, \varphi_n \in C_c(G)$ and $\varepsilon > 0$ such that

$$U[\varphi_1, \dots, \varphi_n; \varepsilon] := \{(\omega, \varpi) : |\omega(\varphi_j) - \varpi(\varphi_j)| < \varepsilon \text{ for all } 1 \leq j \leq n\} \subseteq U.$$

Let N be so that, for all $1 \leq j \leq N$, we have

$$\mathrm{supp}(\varphi_j) \subseteq (-5^N, 5^N).$$

Let $m \in 5^N + 5^{N+1}\mathbb{Z}$ be arbitrary. By (11), we have

$$T_m \mu_N(\varphi_j) = \mu_N(\varphi_j) \qquad \text{for all } 1 \leq j \leq n,$$

as well as

$$T_m v_N(\varphi_j) = v_N(\varphi_j) \qquad \text{for all } 1 \le j \le n.$$

As $\mu = \mu_N + v_N$, we get

$$(\mu, T_m\mu) \in U \quad \text{for all } m 5^N + 5^{N+1}\mathbb{Z}.$$

This shows that μ is Bohr-type almost periodic, but, since it is not translation bounded, it is not Bochner-type almost periodic.

6.8 Product topology

Let G act via translation of functions on a uniform space of $(X, \mathcal{U})$ of functions on G, with the property that $C_u(G) \subseteq X$. We can then *push* the uniformity $\mathcal{U}$ to $\mathcal{M}^\infty(G)$ the following way.

For each $U \in \mathcal{U}$, $n \in \mathbb{N}$, and all $\varphi_1, \ldots, \varphi_n \in C_c(G)$, define

$$V[U : \varphi_1, \ldots, \varphi_n] := \{(\mu, v) \in \mathcal{M}^\infty(G) \times \mathcal{M}^\infty(G) : (\mu * \varphi_j, v * \varphi) \in U \text{ for all } 1 \le j \le n\}.$$

Next, define

$$\mathcal{U}_{pu} := \{V \subseteq \mathcal{M}^\infty(G) \times \mathcal{M}^\infty(G) : \text{there exist } U \in \mathcal{U}, \, n \in \mathbb{N}, \, \varphi_1, \ldots, \varphi_n \in C_c(G)$$
$$\text{such that } V[U : \varphi_1, \ldots, \varphi_n] \subseteq V\}.$$

Then, $\mathcal{U}_{pu}$ is a uniformity on $\mathcal{M}^\infty(G)$. If $\mathcal{U}$ is G-invariant, so is $\mathcal{U}_{pu}$. Moreover, if the translation action is equicontinuous on $\mathcal{U}$, it is also equicontinuous on $\mathcal{U}_{pu}$. Completeness is in general more subtle.

This process allows us carry many results about almost periodicity from functions to measures. We look next at one such example.

Now, consider the case when $X = C_u(G)$ with the uniformity given by the norm $\|\cdot\|_\infty$. The uniformity induced by the push of this uniformity to $\mathcal{M}^\infty(G)$ is called the *product uniformity*, and is denoted by $\mathcal{U}_p$. The corresponding topology is the *product topology for measures* (see [8] for properties). It is easy to see that $\mathcal{U}_p$ is G-invariant, and the translation action is equicontinuous.

Now, it is not known if this space, or if $\mathcal{M}^\infty(G)$ is complete, but this space is quasi-complete (meaning any equi-translation bounded closed subset is complete [8]). This turns out to be enough in this situation. Indeed, if $\mu \in \mathcal{M}^\infty(G)$, then all measures in $H_\mu^{\mathcal{U}_p}$ are equi-translation bounded, and hence this orbit is complete. Therefore, we get the standard characterization of $\mathcal{SAP}(G)$, the first three equivalences appearing in [1, 8, 20], whereas the equivalence to condition (iv) appearing in [12, 13, 16].

Theorem 6.4 *Let $\mu \in \mathcal{M}^\infty(G)$. Then, the following assertions are equivalent.*

(i) *For all $\varphi_1, \ldots, \varphi_n \in C_c(G)$ and all $\varepsilon > 0$, the set*

$$P_\varepsilon = \{t \in G : \|T_t(\varphi_j * \mu) - (\varphi_j * \mu)\|_\infty < \varepsilon \text{ for all } 1 \le j \le n\}$$

is relatively dense.

(ii) *For all $\varphi_1, \ldots, \varphi_n \in C_c(G)$ and all $\varepsilon > 0$, the set*

$$P_\varepsilon = \{t \in G \ : \ \|T_t(\varphi_j * \mu) - (\varphi_j * \mu)\|_\infty < \varepsilon \text{ for all } 1 \leq j \leq n\}$$

is finitely relatively dense.

(iii) $H_\mu^{\mathcal{U}_p}$ *is compact in the product topology.*

(iv) $H_\mu^{\mathcal{U}_p}$ *is a compact abelian group, with the group operation induced by*

$$(T_t\mu) \oplus (T_s\mu) := T_{t+s}\mu.$$

(v) *There exists a continuous function $\Psi : G_b \to H_\mu^{\mathcal{U}_p}$ such that*

$$T_t f = \Psi(i_b(t)) \quad \text{for all } t \in G.$$

Moreover, in this case, $F(t) = \alpha(t, x)$ define a group homomorphism with dense range $F : G \to H_\mu^{\mathcal{U}_p}$, and Ψ is an onto group homomorphism.

6.9 Norm and mixed norm topology

Consider $X = \mathcal{M}^\infty(G)$, the space of translation bounded measures. Let $K \subseteq G$ be a fixed compact set with non-empty interior. Then, K defines a norm on X [3] via

$$\|\mu\|_K := \sup_{t \in G} |\mu| (t + K).$$

The space $(X, \|\cdot\|_K)$ is a Banach space [22]. Let $\mathcal{U}_{\text{norm}}$ be the uniformity defined on $\mathcal{M}^\infty(G)$ by this norm.

Let us first recall the following definition [3].

Definition 6.5 A measure μ in $\mathcal{M}^\infty(G)$ is called *norm almost periodic* if μ is Bohr-type almost periodic in this topology.

Norm almost periodic pure point measures are appear naturally in the CPS [3, 13, 21, 29] and have been fully characterized in [29].

Note that the translation action is not continuous on X, and while it is continuous on some orbits (for example, it is continuous on the orbit of measures of the form $\mu = f\theta_G$ for $f \in C_u(G)$) it is never continuous on the orbit of nontrivial pure point measures (see Lemma 6.6). In particular, in this case, we have

Bochner-type almost periodicity $\iff$ pseudo-Bochner-type almost periodicity

$\implies$ Bohr-type almost periodicity.

It turns out that for pure point measures, Bochner-type almost periodicity is not a relevant concept, as we have the following lemma.

Lemma 6.6 Let $\mu \in \mathcal{M}^\infty(G)$ be a pure point measure.

(a) *If the translation action is continuous at a point $s \in G$ on the set $\{T_t\mu \ : \ t \in G\}$, then $\mu = 0$.*

(b) *If G is σ-compact and uncountable and μ is pseudo-Bochner-type almost periodic, then $\mu = 0$.*

Proof (a) Assume by contradiction that $\mu \neq 0$. Then, since μ is pure point, there exists some $x \in G$ such that

$$|\mu(\{x\})| = a > 0.$$

Next, fix some open precompact set U with $s \in U$. Since μ is a measure, we have

$$|\mu|(x - s + U) < \infty,$$

and hence, the set

$$F := \left\{ y \in x - s + U \ : \ |\mu(\{y\})| \geq \frac{a}{2} \right\}$$

is a finite set. Let $F' = F \backslash \{x\}$. Then, the set

$$V := U \backslash (s - x + F')$$

is an open neighborhood of s, and for all $t \in V$, we have

$$|T_s \mu(\{-s + x\}) - T_t \mu\{-s + x\}| \geq |T_s \mu(\{-s + x\})| - |T_t \mu\{-s + x\}|$$

$$= |\mu(\{x\})| - |\mu\{t - s + x\}| \geq a - \frac{a}{2} = \frac{a}{2},$$

and hence,

$$\|T_s \mu - T_t \mu\|_K \geq \frac{a}{2}.$$

This contradicts the fact that T is continuous at s.

(b) Assume by contradiction that $\mu \neq 0$. The argument is similar to the one in (a). Since μ is pure point, there exists some $x \in G$ such that

$$|\mu(\{x\})| = a > 0.$$

Next, fix some open precompact set U with $0 \in U$ and set $\varepsilon = \frac{a}{2}$. We show that

$$P_\varepsilon := \{t \in G \ : \ \|T_t \mu - \mu\|_K < \varepsilon\}$$

is locally finite and hence countable, by the σ-compactness of G. Since P_ε is finitely relatively dense, this implies that G is countable, a contradiction.

Indeed, let $K \subseteq G$ be any compact set, and let

$$F := P_\varepsilon \cap K.$$

We want to show that F is finite. First note that, for all $t \in F$, we have $\|T_t \mu - \mu\|_K < \varepsilon$ and, hence,

$$|\mu(\{t + x\}) - \mu(\{x\})| < \frac{a}{2}.$$

Since $\mu(\{x\}) = a$, this gives $|\mu(\{t + x\})| > \frac{a}{2}$.

It follows from here that

$$\sum_{t \in F} \frac{a}{2} \leq \sum_{t \in F} |\mu(\{t + x\})| \leq |\mu|(x + K) < \infty,$$

giving that F is finite, and completing the proof.   ∎

Since $\mathcal{U}_{\mathrm{norm}}$ is given by a metric, and G-invariant, we can define the mixed uniformity from Section 5. We will denote the mixed uniformity by $\mathcal{U}_{m-n}$ and refer to the topology induced by this uniformity as the *mixed-norm topology* for measures.

Since the norm topology is given by a norm, if G is metrizable, it follows immediately that the mixed norm topology is metrizable. On another hand, if the mixed norm topology is metrizable, it is metrizable on the set $\{T_s\delta_0 \,:\, s \in G\}$, which immediately implies that the topology on G is metrizable. Therefore, the mixed-norm topology is metrizable if and only if G is metrizable.

Now, Corollary 5.8 has the following consequence.

Corollary 6.7 *Let G be metrizable, and let $\mu \in \mathcal{M}^\infty(G)$. Then, the following assertions are equivalent.*

(i) *μ is norm almost periodic.*

(ii) *$H_\mu^{\mathcal{U}_{m-n}}$ is compact.*

(iv) *$H_\mu^{\mathcal{U}_{m-n}}$ is a compact abelian group, with the group operation induced by*

$$\left(T_t\mu\right) \oplus \left(T_s\mu\right) \coloneqq T_{t+s}\mu.$$

(v) *There exists a continuous function $\Psi : G_{\mathsf{b}} \to H_\mu^{\mathcal{U}_{m-n}}$ such that*

$$T_t f = \Psi\left(i_{\mathsf{b}}(t)\right) \text{ for all } t \in G.$$

Moreover, in this case, $F(t) = \alpha(t, x)$ define a group homomorphism with dense range $F : G \to H_\mu^{\mathcal{U}_{m-n}}$, and Ψ is an onto group homomorphism.

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