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Nonlinear Electrostatic Equations for Collisionless Plasmas

This monograph presents a perturbative nonlinear kinetic theory of plasma turbulence, known as the *weak turbulence theory*. At the outset, it should be pointed out that this book does not include the effects of ambient magnetic field. Plasmas in real situations are usually magnetized, so that applications of the method discussed in this book will be somewhat limited, but the purpose is to lay out the fundamental methodology and conceptual foundations so that more general applications for magnetized plasmas may be developed on the basis of this book. This book also limits the discussions to spatially homogeneous plasma.

Plasma kinetic theory has a long history, and many early papers can be found in the literature that discuss the perturbative nonlinear kinetic theory of plasma turbulence – see, for example, papers by Vedenov and Velikhov (1962); Kovrizhnykh and Tsytovich (1964, 1965); Kovrizhnykh (1965); Gorbunov and Silin (1965); Gorbunov et al. (1965); Tsytovich (1967); Rogister and Oberman (1968, 1969), to name just several. These are merely sample papers, among those that personally influenced the author of this book.

If one is interested in the general background on plasma kinetic theory, there are some excellent early monographs, among which may be, for instance, those by Montgomery and Tidman (1964); Kadomtsev (1965); Klimontovich (1967, 1982); Pitaevskii and Lifshitz (1981); Sagdeev and Galeev (1969); Tsytovich (1970, 1977a,b); Davidson (1972); Ichimaru (1973); Krall and Trivelpiece (1973); Akhiezer et al. (1975); Hasegawa (1975); Kaplan and Tsytovich (1973); Sitenko (1967, 1982); Melrose (1980a, 1986); Nicholson (1983); Alexandrov et al. (1984); Chen (1987), etc. This list is incomplete, but they represent some standard works that treat the foundations of plasma kinetic theory and/or weak plasma turbulence theory. More recent books are also available. See, for example, those by Musher et al. (1995); Sitenko and Malnev (1995); Treumann and Baumjohann (1997); Tsytovich (1995); Kono and Škorić (2010); Diamond et al. (2010), etc., which deal with the subject of plasma kinetic theory and nonlinear phenomena.

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So, as the readers may appreciate, there is an abundance of resources on the topic of plasma kinetic theory, and one may ask why another book? The rationale for this book is as follows: Discussions of nonlinear plasma theories, particularly those concerning the weak turbulence theory found in many of the above-cited works, are sometimes not so easy to follow, especially for young researchers. Moreover, many of the monographs cover wide-ranging topics with generally brief descriptions for each subject area without going much into in-depth discussions. It is the purpose of this book to focus only on the kinetic theory of weak plasma turbulence, but to present the detailed fundamental discussions and derivations as clearly as possible, without sacrificing the intermediate mathematical steps. Talking of the latter, many authors omit too many intermediate steps, which can be a source of much frustrations for young scientists. This book does not spare the readers the mathematical details. This strategy means that some materials in the book can be a bit lengthy, and casual readers may get lost in the maths. However, if one approaches the material with enough patience, he or she will be rewarded with the intimate knowledge on how the weak turbulence theory actually works, what are the essential assumptions behind the theory, and so forth. Owing to the space devoted to mathematical details, some standard topics often included in the textbooks and monographs on nonlinear plasma theory are left out. For instance, parametric instabilities, solitary wave theory, coherent nonlinear structure formation in plasma, etc., are not covered in this book.

This book is intended for advanced undergraduate, graduate students, or young researchers who are already familiar with the introductory level of plasma kinetic theory, but wishing to familiarize themselves with a more in-depth understanding on nonlinear theory of weak plasma turbulence. In spite of this, this book expounds on foundational principles at the conceptual level as much as possible without assuming too much prior knowledge on the part of the readers.

1.1 Preamble: Fundamental Concepts

We are interested in physical phenomena that are described as turbulent, which loosely means physical quantities that are fluctuating in space and time. In order to characterize such fluctuations, we employ statistical methods and concepts. That is, we deal with averages in time, space, or over hypothetical collection of different possible states called the *ensemble*. One is particularly interested in how fluctuating quantities measured in two or more different times or in two or more different spatial locations are correlated. We begin by considering many-body correlations associated with fluctuating physical quantities, and the spectral transformation of such quantities in space and time.

The statistical correlation is an important concept that characterizes the nature of turbulence. Suppose that one measures a particular physical quantity, say velocity or electromagnetic field, in a turbulent medium at a given time. Suppose also that one measures the same quantity at another time separated by an interval. If one repeats such series of measurements over and over again, then if the physical quantities are uncorrelated, that is, if there is no cause and effect relationship between the two measurements, then on average, the product of two measurements made at two different time intervals may be zero, since by the very nature of turbulence, velocity or field may have random directions. On the other hand, if the first measurement affects the second measurement because there exists an underlying cause-and-effect relationship, then the average of the products may be finite. A systematic way to characterize how the statistical average of the products of physical quantities, or equivalently, their correlation function, behaves in space and time can thus be useful for understanding and characterizing the turbulence. Consequently, in this book we will be concerned with the description of how the statistical average of the correlation of fluctuating (i.e., turbulent) quantity, $\langle \delta a^2 \rangle$, dynamically evolves. Here, δa represents any dynamical quantity, and the symbol $\langle \cdots \rangle$ denotes the statistical average.

The convention adopted in this book for the definition of spatial Fourier transformation and its inverse is

$$f_{\mathbf{k}} = (2\pi)^{-3} \int d\mathbf{r}\; f(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}, \qquad f(\mathbf{r}) = \int d\mathbf{k}\; f_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{1.1}$$

Here $f(\mathbf{r})$ is any physical quantity, which is a function of spatial coordinate $\mathbf{r}$, and which is bounded in space. The Fourier transformation of a product of two functions is represented by the convolution

$$(2\pi)^{-3} \int d\mathbf{r}\; f(\mathbf{r})\, g(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}} = \int d\mathbf{k}'\; f_{\mathbf{k}'}\, g_{\mathbf{k}-\mathbf{k}'} = \int d\mathbf{k}'\; f_{\mathbf{k}-\mathbf{k}'}\, g_{\mathbf{k}'}. \tag{1.2}$$

The proof of this "convolution theorem" is straightforward. All one has to do is to insert for $f(\mathbf{r})$ and $g(\mathbf{r})$, their respective Fourier transformations, and make use of the well-known delta function identity

$$\int d\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} = \delta(\mathbf{k}). \tag{1.3}$$

Fourier transformation of a function $f(\mathbf{r},t)$ in both space and time can be defined by

$$f_{\mathbf{k},\omega} = (2\pi)^{-4} \int d\mathbf{r} \int dt\; f(\mathbf{r},t)\, e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t},$$

$$f(\mathbf{r},t) = \int d\mathbf{k} \int d\omega\; f_{\mathbf{k},\omega}\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}. \tag{1.4}$$

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Convolution theorem for the spatio-temporal Fourier transformation is

$$(2\pi)^{-4}\int d\mathbf{r}\int dt\; f(\mathbf{r},t)\,g(\mathbf{r},t)\,e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}$$
$$=\int d\mathbf{k}'\int d\omega'\; f_{\mathbf{k}',\omega'}\,g_{\mathbf{k}-\mathbf{k}',\omega-\omega'}$$
$$=\int d\mathbf{k}'\int d\omega'\; f_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\,g_{\mathbf{k}',\omega'}. \tag{1.5}$$

When the angular frequency ω satisfies the dispersion relation $\omega=\omega_{\mathbf{k}}+i\gamma_{\mathbf{k}}$, that is, when (generally complex) ω is a function of $\mathbf{k}$, then the Fourier representation of function $f(\mathbf{r},t)$ can be re-expressed by virtue of the fact that we may write the spectral amplitude as

$$f_{\mathbf{k},\omega}=f_{\mathbf{k}}\delta(\omega-\omega_{\mathbf{k}}-i\gamma_{\mathbf{k}}), \tag{1.6}$$

or

$$f(\mathbf{r},t)=\int d\mathbf{k}\;f_{\mathbf{k}}\;\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t+\gamma_{\mathbf{k}}t). \tag{1.7}$$

If $f(\mathbf{r},t)$ is real then obviously $f^*(\mathbf{r},t)=f(\mathbf{r},t)$, where the asterisk $*$ represents the complex conjugate. From this it follows that

$$\int d\mathbf{k}\;f^*_{\mathbf{k}}\;\exp(-i\mathbf{k}\cdot\mathbf{r}+i\omega_{\mathbf{k}}t)=\int d\mathbf{k}\;f_{\mathbf{k}}\;\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t), \tag{1.8}$$

which leads to the following symmetry relations:

$$f^*_{\mathbf{k}}=f_{-\mathbf{k}},\qquad \omega_{-\mathbf{k}}=-\omega_{\mathbf{k}},\qquad \gamma_{-\mathbf{k}}=\gamma_{\mathbf{k}}. \tag{1.9}$$

Let $\delta f(\mathbf{r},t)$ represent a fluctuating quantity whose ensemble average is zero:

$$\langle\delta f(\mathbf{r},t)\rangle=0. \tag{1.10}$$

In our notation, any quantity preceded by δ indicates that this quantity is fluctuating in space and time, that is, turbulent. By “ensemble average” we may mean an average over phase, space, or time. Or it could mean an average over all possible configurations. Turbulence is called “homogeneous” if the spatial dependence of the two-body correlation is only upon the relative distance,

$$\langle\delta f(\mathbf{r},t)\,\delta f(\mathbf{r}',t)\rangle=\langle\delta f^2\rangle_{\mathbf{r}-\mathbf{r}',t,t'}=\langle\delta f^2\rangle_{\mathbf{r}'-\mathbf{r},t,t'}, \tag{1.11}$$

and “stationary” if the temporal two-body correlation is a function of relative time difference,

$$\langle\delta f(\mathbf{r},t)\,\delta f(\mathbf{r},t')\rangle=\langle\delta f^2\rangle_{\mathbf{r},\mathbf{r}',t-t'}=\langle\delta f^2\rangle_{\mathbf{r},\mathbf{r}',t'-t}. \tag{1.12}$$

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Thus, for homogeneous and stationary turbulence the two-body correlation function is given by

$$\langle \delta f(\mathbf{r},t)\, \delta f(\mathbf{r}',t') \rangle = \langle \delta f^2 \rangle_{\mathbf{r}-\mathbf{r}',t-t'}. \tag{1.13}$$

It should be noted that not all fluctuating quantities in nature satisfy the zero ensemble average property (1.10). Physical processes whose fluctuations satisfy (1.10) are called "incoherent" phenomena, while "coherent" processes may be associated with a nonvanishing ensemble average. For incoherent processes different phases are uncorrelated such that when averaged over them, the result vanishes; hence, such processes are characterized by the zero ensemble average property specified by (1.10).

In a similar way, the three-body correlation function for homogeneous and stationary turbulence is a function of distances between any two points, say $(\mathbf{r},t)$ and $(\mathbf{r}',t')$, among three points $(\mathbf{r},t)$, $(\mathbf{r}',t')$, $(\mathbf{r}'',t'')$, in coordinate-time space:

$$\langle \delta f(\mathbf{r},t)\, \delta f(\mathbf{r}',t')\, \delta f(\mathbf{r}'',t'') \rangle = \langle \delta f^3 \rangle_{\mathbf{r}-\mathbf{r}',\mathbf{r}-\mathbf{r}'';t-t',t-t''}. \tag{1.14}$$

The four-body correlation function for homogeneous and stationary turbulence can be defined likewise:

$$\begin{aligned} &\langle \delta f(\mathbf{r},t)\, \delta f(\mathbf{r}',t')\, \delta f(\mathbf{r}'',t'')\, \delta f(\mathbf{r}''',t''') \rangle \\ &\quad = \langle \delta f^2 \rangle_{\mathbf{r}-\mathbf{r}';t-t'} \langle \delta f^2 \rangle_{\mathbf{r}''-\mathbf{r}''';t''-t'''} + \langle \delta f^2 \rangle_{\mathbf{r}-\mathbf{r}'';t-t''} \langle \delta f^2 \rangle_{\mathbf{r}'-\mathbf{r}''';t'-t'''} \\ &\qquad + \langle \delta f^2 \rangle_{\mathbf{r}-\mathbf{r}''';t-t'''} \langle \delta f^2 \rangle_{\mathbf{r}'-\mathbf{r}'';t'-t''} \\ &\qquad + \langle \delta f^4 \rangle_{\mathbf{r}-\mathbf{r}',\mathbf{r}'-\mathbf{r}'',\mathbf{r}''-\mathbf{r}''';t-t',t'-t''',t''-t'''}. \end{aligned} \tag{1.15}$$

Let us represent the two-body correlation function in spectral form:

$$\begin{aligned} \langle \delta f(\mathbf{r},t)\, \delta f(\mathbf{r}',t') \rangle &= \int d\mathbf{k} \int d\omega\, \langle \delta f^2 \rangle_{\mathbf{k},\omega}\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-i\omega(t-t')} \\ &= \int d\mathbf{k} \int d\omega \int d\mathbf{k}' \int d\omega'\, \langle \delta f_{\mathbf{k},\omega}\, \delta f_{\mathbf{k}',\omega'} \rangle\, e^{i\mathbf{k}\cdot\mathbf{r}+i\mathbf{k}'\cdot\mathbf{r}'-i\omega t-i\omega' t'}, \end{aligned} \tag{1.16}$$

where in the second line we have made use of the spectral representations for individual functions $\delta f(\mathbf{r},t)$ and $\delta f(\mathbf{r}',t')$. From this, it is seen that the equality can be obtained if the following condition is satisfied:

$$\langle \delta f_{\mathbf{k},\omega}\, \delta f_{\mathbf{k}',\omega'} \rangle = \delta(\mathbf{k}+\mathbf{k}')\, \delta(\omega+\omega')\, \langle \delta f^2 \rangle_{\mathbf{k},\omega}. \tag{1.17}$$

If we write the spectral component $\delta f_{\mathbf{k},\omega}$ with an explicit phase factor,

$$\delta f_{\mathbf{k},\omega} = \hat{f}_{\mathbf{k},\omega} e^{i\phi_{\mathbf{k},\omega}}, \tag{1.18}$$

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where $\phi_{\mathbf{k},\omega}$ represents the phase, then we have

$$\langle \delta f_{\mathbf{k},\omega}\, \delta f_{\mathbf{k}',\omega'} \rangle = \langle \hat{f}_{\mathbf{k},\omega}\, \hat{f}_{\mathbf{k}',\omega'} e^{i\phi_{\mathbf{k},\omega}+i\phi_{\mathbf{k}',\omega'}} \rangle. \tag{1.19}$$

For homogeneous and stationary turbulence the phase is assumed to be random (or uncorrelated). As such, the ensemble average over random phases becomes nonzero only if

$$\phi_{\mathbf{k},\omega} + \phi_{\mathbf{k}',\omega'} = 0, \tag{1.20}$$

which can be satisfied under the assumption that, if for $\mathbf{k} = -\mathbf{k}'$ and $\omega = -\omega'$, the following is also satisfied:

$$\phi_{-\mathbf{k},-\omega} = -\phi_{\mathbf{k},\omega}. \tag{1.21}$$

This is but the rephrasing of condition (1.17). The assumption of homogeneous and stationary turbulence is thus equivalent to the "random phase approximation." In short, the property

$$\langle \delta f^2 \rangle_{\mathbf{k},\omega} = \langle \delta f_{\mathbf{k},\omega}\, \delta f_{-\mathbf{k},-\omega} \rangle \tag{1.22}$$

is a useful spectral characteristic for homogeneous and stationary turbulence, or equivalently, fluctuations with random phases.

Next, consider the three-body correlation, which we may write as

$$\begin{aligned}\langle \delta f(\mathbf{r},t)\, \delta f(\mathbf{r}',t')\, \delta f(\mathbf{r}'',t'') \rangle &= \int d\mathbf{k} \int d\omega \int d\mathbf{k}' \int d\omega'\, \langle \delta f^3 \rangle_{\mathbf{k},\omega;\mathbf{k}',\omega'} \\ &\quad \times e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')+i\mathbf{k}'\cdot(\mathbf{r}'-\mathbf{r}'')-i\omega(t-t')-i\omega'(t'-t'')} \\ &= \int d\mathbf{k} \int d\omega \int d\mathbf{k}' \int d\omega' \int d\mathbf{k}'' \int d\omega'' \\ &\quad \times \langle \delta f_{\mathbf{k},\omega}\, \delta f_{\mathbf{k}',\omega'}\, \delta f_{\mathbf{k}'',\omega''} \rangle \\ &\quad \times e^{i\mathbf{k}\cdot\mathbf{r}+i\mathbf{k}'\cdot\mathbf{r}'+i\mathbf{k}''\cdot\mathbf{r}''-i\omega t-i\omega' t'-i\omega'' t''}. \end{aligned} \tag{1.23}$$

From this we obtain the identity

$$\langle \delta f_{\mathbf{k},\omega}\, \delta f_{\mathbf{k}',\omega'}\, \delta f_{\mathbf{k}'',\omega''} \rangle = \delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')\, \delta(\omega+\omega'+\omega'')\, \langle \delta f^3 \rangle_{\mathbf{k},\omega;\mathbf{k}+\mathbf{k}',\omega+\omega'}. \tag{1.24}$$

A similar analysis can be carried out for the four-body correlation. The derivation is tedious but straightforward, and is thus omitted.

We summarize the general properties of the many-body correlations, or many-body cumulants for homogeneous and stationary turbulence:

$$\begin{aligned}\langle \delta f_{\mathbf{k},\omega}\, \delta f_{\mathbf{k}',\omega'} \rangle &= \delta(\mathbf{k}+\mathbf{k}')\, \delta(\omega+\omega') \langle \delta f^2 \rangle_{\mathbf{k},\omega}, \\ \langle \delta f_{\mathbf{k},\omega}\, \delta f_{\mathbf{k}',\omega'}\, \delta f_{\mathbf{k}'',\omega''} \rangle &= \delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}'') \delta(\omega+\omega'+\omega'') \langle \delta f^3 \rangle_{\mathbf{k},\omega;\mathbf{k}+\mathbf{k}',\omega+\omega'},\end{aligned}$$

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$$
\begin{aligned}
\langle \delta f_{\mathbf{k},\omega}\, \delta f_{\mathbf{k}',\omega'}\, \delta f_{\mathbf{k}'',\omega''}\, \delta f_{\mathbf{k}'''\omega'''} \rangle = {} & \delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}''')\, \delta(\omega+\omega'+\omega''+\omega''') \\
& \times [\delta(\mathbf{k}+\mathbf{k}')\, \delta(\omega+\omega')\, \langle \delta f^2 \rangle_{\mathbf{k},\omega}\, \langle \delta f^2 \rangle_{\mathbf{k}'',\omega''} \\
& + \delta(\mathbf{k}+\mathbf{k}'')\, \delta(\omega+\omega'')\, \langle \delta f^2 \rangle_{\mathbf{k},\omega}\, \langle \delta f^2 \rangle_{\mathbf{k}',\omega'} \\
& + \delta(\mathbf{k}'+\mathbf{k}'')\, \delta(\omega'+\omega'')\, \langle \delta f^2 \rangle_{\mathbf{k},\omega}\, \langle \delta f^2 \rangle_{\mathbf{k}',\omega'} \\
& + \langle \delta f^4 \rangle_{\mathbf{k},\omega;\,\mathbf{k}+\mathbf{k}',\omega+\omega';\,\mathbf{k}+\mathbf{k}'+\mathbf{k}'',\omega+\omega'+\omega''}]. \qquad (1.25)
\end{aligned}
$$

An important consequence of this result is that an ensemble average of two fluctuating quantities δf and δg, where they are related to each other, can be expressed in terms of their spectral counterparts as follows:

$$
\langle \delta f(\mathbf{r},t)\, \delta g(\mathbf{r},t) \rangle = \int d\mathbf{k} \int d\omega\, \langle \delta f_{\mathbf{k},\omega}\, \delta g_{-\mathbf{k},-\omega} \rangle. \qquad (1.26)
$$

1.2 Electrostatic Vlasov Equation

A simple and intuitive definition of *plasma* is that it is an *ionized gas*. Individual electrons and ions that make up the plasma interact through collective electromagnetic force. Collective behavior of a plasma is described by a statistical means. In this book we are concerned with a fully ionized plasma. For partially ionized plasma, atomic processes such as the recombination and collisions between charged particles and neutrals cannot be ignored, which complicate the matter. Vlasov equation (Vlasov, 1938) describes the statistical property of a plasma governed by collective processes. The system under consideration is a spatially uniform plasma made of single-species ions (protons) and electrons, and there is no net electric or magnetic field. We also assume zero average charge or current in the system. If we make the simplifying approximation that the plasma particles interact primarily through electrostatic field, then the dynamics can be described by the Vlasov–Poisson system of equations

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{e_a}{m_a}\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{v}} \right) f_a &= 0, \\
\nabla\cdot\mathbf{E} &= 4\pi \sum_a e_a \int d\mathbf{v}\, f_a, \qquad (1.27)
\end{aligned}
$$

where e_a and m_a are charge and mass of species a $(= e, i)$ for electrons and ions ($e_a = e$ for protons and $e_a = -e$ for electrons). The one-particle distribution function $f_a(\mathbf{r},\mathbf{v},t)$ is the probability density of finding a collection of plasma particles of species a, at a particular state in phase space $(\mathbf{r},\mathbf{v})$ at a given time t. Consequently, if we integrate $f_a(\mathbf{r},\mathbf{v},t)$ over $\mathbf{v}$, or equivalently, if we collect all possible configuration in velocity space, then the result becomes the density of charged particle species labeled a,

$$\rho_a(\mathbf{r},t) = \int d\mathbf{v}\, f_a(\mathbf{r},\mathbf{v},t). \tag{1.28}$$

Multiplying the charge e_a and summing over all charged particle species leads to the total charge density

$$\rho(\mathbf{r},t) = \sum_a e_a \rho_a(\mathbf{r},t). \tag{1.29}$$

Since $f_a(\mathbf{r},\mathbf{v},t)$ is the probability density, it is normalized to the ambient charged particle number density n_a,

$$\frac{1}{\mathscr{V}} \int d\mathbf{r} \int d\mathbf{v}\, f_a(\mathbf{r},\mathbf{v},t) = n_a, \tag{1.30}$$

where $\mathscr{V}$ is the volume of the system. That is, if we collect all possible configurations in velocity space at a given time, and integrate over the entire volume under consideration and divide by $\mathscr{V}$, that is, take the spatial average, then the result should be the total number of particles per volume, $n_a = N_a/\mathscr{V}$, or equivalently, the ambient density. Since in the absence of source or sink, plasma particles cannot be created or annihilated (that is, no recombination into neutrals or reionization), the one-particle distribution function must be conserved. Hence,

$$\frac{df_a}{dt} = \left(\frac{\partial}{\partial t} + \dot{\mathbf{r}}\cdot\nabla + \dot{\mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{v}}\right) f_a = 0. \tag{1.31}$$

By virtue of the equation of motion,

$$\dot{\mathbf{r}} = \mathbf{v} \text{ and } \dot{\mathbf{v}} = \frac{e_a}{m_a}\mathbf{E}, \tag{1.32}$$

we obtain the Vlasov equation in (1.27). Because of the charge neutrality condition, the ambient density is the same for both ions and electrons,

$$n_e = n_i = n. \tag{1.33}$$

Let us separate the physical quantities into average and fluctuating parts. The average particle distribution function is independent of the spatial coordinate $\mathbf{r}$ since we assume uniform plasma, and there is no average electric field, so that we may write

$$\begin{aligned} f_a(\mathbf{r},\mathbf{v},t) &= n_a F_a(\mathbf{v},t) + \delta f_a(\mathbf{r},\mathbf{v},t), \\ \mathbf{E}(\mathbf{r},t) &= \delta\mathbf{E}(\mathbf{r},t), \end{aligned} \tag{1.34}$$

where δ represents fluctuating quantities whose phases are supposed to be random. When averaged over their phases, these quantities vanish. In (1.34) we have introduced a normalized one-particle distribution function $F_a(\mathbf{v},t)$ $[\int d\mathbf{v}\, F_a(\mathbf{v},t) = 1]$. Inserting (1.34) back into the coupled Vlasov–Poisson equation, we obtain

$$\left(\frac{\partial}{\partial t} + \frac{e_a}{m_a}\,\delta\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{v}}\right) n_a F_a + \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{e_a}{m_a}\,\delta\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\delta f_a = 0,$$
$$\nabla\cdot\delta\mathbf{E} = 4\pi\sum_a e_a\int d\mathbf{v}\,\delta f_a. \tag{1.35}$$

Upon averaging (1.35) over random phases of the fluctuations, we obtain the formal particle kinetic equation:

$$\frac{\partial n_a F_a}{\partial t} = -\frac{e_a}{m_a}\frac{\partial}{\partial\mathbf{v}}\cdot\langle\delta f_a\,\delta\mathbf{E}\rangle. \tag{1.36}$$

Let us subtract the formal particle kinetic equation from the original equation in order to obtain the equation for perturbed distribution function:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\delta f_a = -\frac{e_a}{m_a}\,\delta\mathbf{E}\cdot\frac{\partial n_a F_a}{\partial\mathbf{v}}$$
$$-\frac{e_a}{m_a}\frac{\partial}{\partial\mathbf{v}}\cdot\left(\delta f_a\,\delta\mathbf{E} - \langle\delta f_a\,\delta\mathbf{E}\rangle\right). \tag{1.37}$$

Note that (1.37) is nonlinear since it contains terms of order $\mathscr{O}(\delta^2)$.

We assume that the fluctuations can be decomposed in the sense of Fourier–Laplace transformation over the fast-time scales of fluctuations while the spectral amplitudes may vary slowly in time:

$$\delta f_a(\mathbf{r},\mathbf{v},t) = \int d\mathbf{k}\int_L d\omega\;\delta f^a_{\mathbf{k},\omega}(\mathbf{v},t)\,e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t},$$
$$\delta f^a_{\mathbf{k},\omega}(\mathbf{v},t) = \frac{1}{(2\pi)^4}\int d\mathbf{r}\int_0^\infty dt\;\delta f_a(\mathbf{r},\mathbf{v},t)\,e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t},$$
$$\delta\phi(\mathbf{r},t) = \int d\mathbf{k}\int_L d\omega\;\delta\phi_{\mathbf{k},\omega}(t)\,e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t},$$
$$\delta\phi_{\mathbf{k},\omega}(t) = \frac{1}{(2\pi)^4}\int d\mathbf{r}\int_0^\infty dt\;\delta\phi(\mathbf{r},t)\,e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}, \tag{1.38}$$

where we have assumed

$$\delta\mathbf{E}(\mathbf{r},t) = -\nabla\delta\phi(\mathbf{r},t), \qquad \text{or} \qquad \delta\mathbf{E}_{\mathbf{k},\omega} = -i\mathbf{k}\,\delta\phi_{\mathbf{k},\omega}, \tag{1.39}$$

since we work under the electrostatic approximation. In (1.38), the integration $\int_L d\omega$ is taken along the path L that stretches from $\omega = -\infty + i\sigma$ to $\omega = \infty + i\sigma$ ($\sigma > 0$ and $\sigma \to 0$). The infinitesimal positive imaginary part σ signifies that we are only interested in causal solutions. The causality requirement is related to the Laplace transformation being defined only over $0 < t < \infty$ rather than the entire integral domain, $-\infty < t < \infty$. The reason for the positive infinitesimal imaginary part, $\sigma > 0$, in the ω integration along the path L, is to ensure the temporal convergence for $t \to \infty$, hence the causality. By the choice of positive integral $\int_0^\infty dt$, one is effectively breaking the time reversal symmetry, and this forces physical processes to proceed in forward time, $t > 0$. That is, the Laplace transformation in place of the symmetric temporal Fourier transformation is equivalent to imposing the causal relationship to the otherwise time-reversible Vlasov equation. In Appendix A we review the treatment of time-irreversible small amplitude plasma perturbation, as discussed originally by Landau (1946). We also discuss the notion of Landau damping in Appendix A.

In the Fourier–Laplace transformation defined in (1.38), we have made an assumption that the spectral amplitudes $\delta f^a_{\mathbf{k},\omega}(\mathbf{v},t)$ and $\delta\phi_{\mathbf{k},\omega}(t)$ have slow and adiabatic time dependence. These quantities are assumed to vary slowly in time, while the temporal dependence dictated by $\exp(-i\omega t)$ is assumed to be fast. That is, the time dependence of amplitudes is much weaker than that associated with the wave time scale, $\delta f, \delta\phi \sim \mathscr{O}(t_{\rm slow})$, where $\mathscr{O}(t_{\rm slow}) \gg \mathscr{O}(\omega^{-1})$. These amplitudes are calculated as if they are independent of time on the fast wave time scale $(t \sim \omega^{-1})$.

Employing the transformation (1.38), the equation for fluctuating field, formal particle kinetic equation, and the equation for perturbed distribution function are expressed, respectively, as follows:

$$
\begin{aligned}
\delta\phi_{\mathbf{k},\omega}(t) &= \sum_a \frac{4\pi e_a}{k^2}\int d\mathbf{v}\,\delta f^a_{\mathbf{k},\omega}(\mathbf{v},t),\\
\frac{\partial n_a F_a(\mathbf{v},t)}{\partial t} &= \frac{i e_a}{m_a}\int d\mathbf{k}\int d\omega\int d\mathbf{k}'\int d\omega'\\
&\quad\times\left(\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{v}}\right)\langle\delta\phi_{\mathbf{k}',\omega'}(t)\delta f^a_{\mathbf{k},\omega}(\mathbf{v},t)\rangle\, e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}-i(\omega+\omega')t},\\
\left(\omega-\mathbf{k}\cdot\mathbf{v}+i\frac{\partial}{\partial t}\right)\delta f^a_{\mathbf{k},\omega}(\mathbf{v},t) &= -\frac{e_a}{m_a}\,\delta\phi_{\mathbf{k},\omega}(t)\,\mathbf{k}\cdot\frac{\partial n_a F_a(\mathbf{v},t)}{\partial\mathbf{v}}\\
&\quad-\frac{e_a}{m_a}\int d\mathbf{k}'\int d\omega'\,\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{v}}\\
&\quad\times\Big[\delta\phi_{\mathbf{k}',\omega'}(t)\,\delta f^a_{\mathbf{k}-\mathbf{k}',\omega-\omega'}(\mathbf{v},t)\\
&\quad-\langle\delta\phi_{\mathbf{k}',\omega'}(t)\,\delta f^a_{\mathbf{k}-\mathbf{k}',\omega-\omega'}(\mathbf{v},t)\rangle\Big].
\end{aligned}
\tag{1.40}
$$

The present treatment of slow and adiabatic time dependence of spectral amplitudes is not rigorous. Mathematically consistent treatment involves the multiple time-scale analysis as employed by Davidson (1972). The present discussion resorts to the shortcut approach, following the method employed in the standard literature, for example, Sitenko (1982), Akhiezer et al. (1975), etc.

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1.3 Fast-Time Scale Solution

Formal solutions for the fast-time scale quantities necessitate the inversion of differential operator, $\omega - \mathbf{k}\cdot\mathbf{v} + i\partial/\partial t$. To simplify the matter, we temporarily ignore the adiabatic time derivative by absorbing the time derivative as part of the "new" definition for ω,

$$\omega + i\,\partial/\partial t \to \omega, \tag{1.41}$$

within the definition for $\omega - \mathbf{k}\cdot\mathbf{v} + i0 + i\partial/\partial t$. We will reinstitute the explicit slow-time derivative when we discuss the wave kinetic equation later. Meanwhile, let us define

$$\mathbf{g}_{\mathbf{k},\omega} \equiv -\frac{e_a}{m_a}\frac{1}{\omega - \mathbf{k}\cdot\mathbf{v} + i0}\frac{\partial}{\partial\mathbf{v}}. \tag{1.42}$$

Note that $+i0$ is meant to indicate that the angular frequency ω is always to be interpreted as having an infinitesimally small but positive imaginary part, which is intimately related to the causality condition associated with the Laplace transformation, and discussed in Appendix A. The notion of ω having positive imaginary part arises from the asymptotic treatment associated with the analytic continuation, and it is further discussed in detail in Appendix B. Then the equation for perturbed particle distribution function in (1.40) is re-expressed as follows:

$$\begin{aligned}\delta f^a_{\mathbf{k},\omega} &= \mathbf{k}\cdot\mathbf{g}_{\mathbf{k},\omega}\, n_a F_a\, \delta\phi_{\mathbf{k},\omega} \\ &\quad + \int d\mathbf{k}' \int d\omega'\, \mathbf{k}'\cdot\mathbf{g}_{\mathbf{k},\omega}\left[\delta\phi_{\mathbf{k}',\omega'}\,\delta f^a_{\mathbf{k}-\mathbf{k}',\omega-\omega'} - \langle\delta\phi_{\mathbf{k}',\omega'}\,\delta f^a_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\rangle\right].\end{aligned} \tag{1.43}$$

To solve the equation for $\delta f^a_{\mathbf{k},\omega}$, we expand $\delta f^a_{\mathbf{k},\omega}$ in a formal perturbation series with successive terms proportional to increasing powers of $\delta\phi_{\mathbf{k},\omega}$:

$$\delta f^a_{\mathbf{k},\omega} = \delta f^{a(1)}_{\mathbf{k},\omega} + \delta f^{a(2)}_{\mathbf{k},\omega} + \delta f^{a(3)}_{\mathbf{k},\omega} + \cdots, \tag{1.44}$$

where $\delta f^{a(n)}_{\mathbf{k},\omega}$ scales as $\mathcal{O}(\delta\phi^n_{\mathbf{k},\omega})$. It is not so difficult to see why $\delta f^{a(n)}_{\mathbf{k},\omega}$ should scale as $\mathcal{O}(\delta\phi^n_{\mathbf{k},\omega})$. If we ignore the nonlinear coupling term in (1.43), then it is readily obvious that the perturbed distribution function $\delta f^a_{\mathbf{k},\omega}$ is proportional to the perturbed wave amplitude $\delta\phi_{\mathbf{k},\omega}$. From this it logically follows that the expansion parameter should scale as $\delta\phi_{\mathbf{k},\omega}$. This consideration also leads to the conceptually "necessary" condition for the validity of so-called *weak turbulence* expansion (1.44). If we consider that the characteristic energy associated with the average particle distribution $n_a F_a$ is the thermal energy,

$$\mathcal{E}_{\text{particle}} = \sum_a n_a \int d\mathbf{v}\,\frac{m_a v^2}{2}\,F_a(\mathbf{v}) = \sum_a \frac{n_a m_a v_{Ta}^2}{2} = \sum_a n_a T_a, \tag{1.45}$$

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and that the perturbed distribution function, which is defined by $\delta f_a = f_a - n_a F_a$, scales as the wave amplitude, $\delta f_a \propto \delta\phi$, then in order for perturbative expansion (1.44) to converge, the characteristic energy associated with the perturbed distribution function, which is proportional to the wave energy density,

$$\sum_a \int d\mathbf{v}\, \frac{m_a v^2}{2}\, \delta f_a(\mathbf{v}) \propto \mathscr{E}_{\rm wave} = \frac{\langle \delta E^2 \rangle}{8\pi}, \tag{1.46}$$

where $\delta\mathbf{E} = -\nabla\delta\phi$, must be sufficiently lower than the thermal energy. In short, the following condition must be satisfied for the weak turbulence expansion to be valid:

$$\mathscr{E}_{\rm particle} \gg \mathscr{E}_{\rm wave}. \tag{1.47}$$

This is the general requirement for the validity of weak turbulence theory.

Inserting the series expansion (1.44) to the original equation for $\delta f^a_{\mathbf{k},\omega}(\mathbf{v})$, we readily obtain, order by order,

$$\begin{aligned}
\delta f^{a(1)}_{\mathbf{k},\omega} &= \mathbf{k}\cdot\mathbf{g}_{\mathbf{k},\omega}\, n_a F_a\, \delta\phi_{\mathbf{k},\omega},\\
\delta f^{a(2)}_{\mathbf{k},\omega} &= \int d\mathbf{k}' \int d\omega'\, \mathbf{k}'\cdot\mathbf{g}_{\mathbf{k},\omega} \left[\delta\phi_{\mathbf{k}',\omega'}\, \delta f^{a(1)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} - \langle \delta\phi_{\mathbf{k}',\omega'}\, \delta f^{a(1)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} \rangle \right],\\
\delta f^{a(3)}_{\mathbf{k},\omega} &= \int d\mathbf{k}' \int d\omega'\, \mathbf{k}'\cdot\mathbf{g}_{\mathbf{k},\omega} \left[\delta\phi_{\mathbf{k}',\omega'}\, \delta f^{a(2)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} - \langle \delta\phi_{\mathbf{k}',\omega'}\, \delta f^{a(2)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} \rangle \right],\\
&\cdots
\end{aligned} \tag{1.48}$$

etc. Iterative solutions for $\delta f^{a(2)}_{\mathbf{k},\omega}$ and $\delta f^{a(3)}_{\mathbf{k},\omega}$ are given below:

$$\begin{aligned}
\delta f^{a(2)}_{\mathbf{k},\omega} &= \int d\mathbf{k}' \int d\omega'\, (\mathbf{k}'\cdot\mathbf{g}_{\mathbf{k},\omega}) \left[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{g}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} \right]\\
&\quad \times n_a F_a \left(\delta\phi_{\mathbf{k}',\omega'}\, \delta\phi_{\mathbf{k}-\mathbf{k}',\omega-\omega'} - \langle \delta\phi_{\mathbf{k}',\omega'}\, \delta\phi_{\mathbf{k}-\mathbf{k}',\omega-\omega'} \rangle \right),\\
\delta f^{a(3)}_{\mathbf{k},\omega} &= \int d\mathbf{k}' \int d\omega' \int d\mathbf{k}'' \int d\omega''\, (\mathbf{k}'\cdot\mathbf{g}_{\mathbf{k},\omega})\, (\mathbf{k}''\cdot\mathbf{g}_{\mathbf{k}-\mathbf{k}',\omega-\omega'})\\
&\quad \times \left[(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\cdot\mathbf{g}_{\mathbf{k}-\mathbf{k}'-\mathbf{k}'',\omega-\omega'-\omega''} \right] n_a F_a\\
&\quad \times \left(\delta\phi_{\mathbf{k}',\omega'}\, \delta\phi_{\mathbf{k}'',\omega''}\, \delta\phi_{\mathbf{k}-\mathbf{k}'-\mathbf{k}'',\omega-\omega'-\omega''} \right.\\
&\quad - \delta\phi_{\mathbf{k}',\omega'}\, \langle \delta\phi_{\mathbf{k}'',\omega''}\, \delta\phi_{\mathbf{k}-\mathbf{k}'-\mathbf{k}'',\omega-\omega'-\omega''} \rangle\\
&\quad \left. - \langle \delta\phi_{\mathbf{k}',\omega'}\, \delta\phi_{\mathbf{k}'',\omega''}\, \delta\phi_{\mathbf{k}-\mathbf{k}'-\mathbf{k}'',\omega-\omega'-\omega''} \rangle \right).
\end{aligned} \tag{1.49}$$

Let us introduce the following simplified notations:

$$\begin{aligned}
\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} &= \int d\mathbf{k}_1 \int d\mathbf{k}_2\, \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}),\\
\sum_{\omega_1+\omega_2=\omega} &= \int d\omega_1 \int d\omega_2\, \delta(\omega_1+\omega_2-\omega),
\end{aligned}$$

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$$\sum_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}} = \int d\mathbf{k}_1 \int d\mathbf{k}_2 \int d\mathbf{k}_3 \, \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}),$$
$$\sum_{\omega_1+\omega_2+\omega_3=\omega} = \int d\omega_1 \int d\omega_2 \int d\omega_3 \, \delta(\omega_1+\omega_2+\omega_3-\omega), \tag{1.50}$$

etc. Then, the series solution for $\delta f^a_{\mathbf{k},\omega}$ is given by

$$\begin{aligned}
\delta f^a_{\mathbf{k},\omega} &= \mathbf{k}\cdot\mathbf{g}_{\mathbf{k},\omega}\, n_a F_a\, \delta\phi_{\mathbf{k},\omega} \\
&\quad + \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} \sum_{\omega_1+\omega_2=\omega} (\mathbf{k}_1\cdot\mathbf{g}_{\mathbf{k},\omega})\,(\mathbf{k}_2\cdot\mathbf{g}_{\mathbf{k}_2,\omega_2})\, n_a F_a \\
&\quad \times (\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2} - \langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\rangle) \\
&\quad + \sum_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}} \sum_{\omega_1+\omega_2+\omega_3=\omega} (\mathbf{k}_1\cdot\mathbf{g}_{\mathbf{k},\omega})\,(\mathbf{k}_2\cdot\mathbf{g}_{\mathbf{k}_2+\mathbf{k}_3,\omega_2+\omega_3}) \\
&\quad \times (\mathbf{k}_3\cdot\mathbf{g}_{\mathbf{k}_3,\omega_3})\, n_a F_a\, (\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}_3,\omega_3} \\
&\quad - \delta\phi_{\mathbf{k}_1,\omega_1}\,\langle\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}_3,\omega_3}\rangle - \langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}_3,\omega_3}\rangle). \\
&\quad \vdots
\end{aligned} \tag{1.51}$$

The right-hand side of (1.51) is not symmetric with respect to the interchange of dummy variables, $(\mathbf{k}_1,\omega_1)$ and $(\mathbf{k}_2,\omega_2)$. The second-order nonlinear term should be symmetrized with respect to variables $(\mathbf{k}_1,\omega_1)$ and $(\mathbf{k}_2,\omega_2)$, while for the third-order term, the symmetrization should be implemented with respect to $(\mathbf{k}_2,\omega_2)$ and $(\mathbf{k}_3,\omega_3)$. The interchange of these dummy variables leaves (1.51) intact. When the above quantities are symmetrized with respect to these variables, then the correct symmetrized expression emerges,

$$\begin{aligned}
\delta f^a_{\mathbf{k},\omega} &= \alpha(\mathbf{k},\omega)\, n_a F_a\, \delta\phi_{\mathbf{k},\omega} \\
&\quad + \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} \sum_{\omega_1+\omega_2=\omega} \alpha^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)\, n_a F_a \\
&\quad \times (\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2} - \langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\rangle) \\
&\quad + \sum_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}} \sum_{\omega_1+\omega_2+\omega_3=\omega} \alpha^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3)\, n_a F_a \\
&\quad \times \Big(\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}_3,\omega_3} - \delta\phi_{\mathbf{k}_1,\omega_1}\,\langle\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}_3,\omega_3}\rangle \\
&\quad - \langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}_3,\omega_3}\rangle\Big),
\end{aligned} \tag{1.52}$$

where

$$\begin{aligned}
\alpha(\mathbf{k},\omega) &= \mathbf{k}\cdot\mathbf{g}_{\mathbf{k},\omega}, \\
\alpha^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) &= \frac{1}{2}\,[(\mathbf{k}_1\cdot\mathbf{g}_{\mathbf{k}_1+\mathbf{k}_2,\omega_1+\omega_2})\,(\mathbf{k}_2\cdot\mathbf{g}_{\mathbf{k}_2,\omega_2}) \\
&\quad + (\mathbf{k}_2\cdot\mathbf{g}_{\mathbf{k}_1+\mathbf{k}_2,\omega_1+\omega_2})\,(\mathbf{k}_1\cdot\mathbf{g}_{\mathbf{k}_1,\omega_1})],
\end{aligned}$$

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$$\alpha^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3) = \frac{1}{2}\,(\mathbf{k}_1\cdot\mathbf{g}_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3,\,\omega_1+\omega_2+\omega_3}) \times [\,(\mathbf{k}_2\cdot\mathbf{g}_{\mathbf{k}_2+\mathbf{k}_3,\,\omega_2+\omega_3})\,(\mathbf{k}_3\cdot\mathbf{g}_{\mathbf{k}_3,\,\omega_3}) + (\mathbf{k}_3\cdot\mathbf{g}_{\mathbf{k}_2+\mathbf{k}_3,\,\omega_2+\omega_3})\,(\mathbf{k}_2\cdot\mathbf{g}_{\mathbf{k}_2,\,\omega_2})\,], \tag{1.53}$$

1.4 Perturbed Wave Equation

We insert the iterative solution for $\delta f^a_{\mathbf{k},\omega}$ (1.52) to the right-hand side of perturbed Poisson equation in (1.40),

$$\begin{aligned}
0 = &\left(1 - \sum_a \frac{4\pi e_a n_a}{k^2}\int d\mathbf{v}\,\alpha(\mathbf{k},\omega)\,F_a\right)\delta\phi_{\mathbf{k},\omega} \\
&- \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}\ \sum_{\omega_1+\omega_2=\omega}\ \sum_a \frac{4\pi e_a n_a}{k^2}\int d\mathbf{v}\,\alpha^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)\,F_a \\
&\times\left(\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2} - \langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\rangle\right) \\
&- \sum_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}}\ \sum_{\omega_1+\omega_2+\omega_3=\omega}\ \sum_a \frac{4\pi e_a n_a}{k^2}\int d\mathbf{v}\,F_a\,\alpha^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3) \\
&\times\left(\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}_3,\omega_3} - \langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}_3,\omega_3}\rangle\right. \\
&\left.- \delta\phi_{\mathbf{k}_1,\omega_1}\,\langle\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}_3,\omega_3}\rangle\right).
\end{aligned} \tag{1.54}$$

Let us define the linear dielectric response function,

$$\begin{aligned}
\epsilon(\mathbf{k},\omega) &= 1 + \chi(\mathbf{k},\omega) = 1 + \sum_a \chi_a(\mathbf{k},\omega) \\
&= 1 + \frac{4\pi}{\omega}\,\sigma(\mathbf{k},\omega) = 1 + \frac{4\pi}{\omega}\sum_a \sigma_a(\mathbf{k},\omega), \\
\chi_a(\mathbf{k},\omega) &= \frac{4\pi}{\omega}\,\sigma_a(\mathbf{k},\omega) = -\frac{4\pi e_a n_a}{k^2}\int d\mathbf{v}\,\alpha(\mathbf{k},\omega)\,F_a,
\end{aligned} \tag{1.55}$$

where $\chi(\mathbf{k},\omega)$ is the linear dielectric susceptibility and $\sigma(\mathbf{k},\omega)$ is the linear dielectric conductivity. The second-order nonlinear response function, or equivalently, the second-order nonlinear susceptibility, is likewise defined

$$\begin{aligned}
\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) &= \sum_a \chi^{(2)}_a(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2), \\
\chi^{(2)}_a(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) &= -\frac{4\pi i e_a n_a}{k_1\,k_2\,|\mathbf{k}_1+\mathbf{k}_2|}\int d\mathbf{v}\,\alpha^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)\,F_a,
\end{aligned} \tag{1.56}$$

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and the third-order nonlinear response function, or third-order nonlinear susceptibility, may also be defined

$$\bar{\chi}^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3) = \sum_a \bar{\chi}_a^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3),$$

$$\bar{\chi}_a^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3) = \frac{4\pi e_a n_a}{k_1\,k_2\,k_3\,|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|} \times \int d\mathbf{v}\,\alpha^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3)\,F_a,$$

$$\chi^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3) = \frac{1}{3}\,[\,\bar{\chi}^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3) + \bar{\chi}^{(3)}(\mathbf{k}_2,\omega_2|\mathbf{k}_1,\omega_1|\mathbf{k}_3,\omega_3) + \bar{\chi}^{(3)}(\mathbf{k}_3,\omega_3|\mathbf{k}_2,\omega_2|\mathbf{k}_1,\omega_1)\,]. \tag{1.57}$$

To sum up, the linear and nonlinear susceptibilities can be expressed as

$$\chi_a(\mathbf{k},\omega) = -\frac{4\pi e_a n_a}{k^2}\int d\mathbf{v}\,\mathbf{k}\cdot\mathbf{g}_{\mathbf{k},\omega}\,F_a, \tag{1.58}$$

$$\chi_a^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) = -\frac{1}{2}\,\frac{4\pi i e_a n_a}{k_1\,k_2\,|\mathbf{k}_1+\mathbf{k}_2|}\int d\mathbf{v}\,\mathbf{g}_{\mathbf{k}_1+\mathbf{k}_2,\omega_1+\omega_2} \cdot [\,\mathbf{k}_1\,(\mathbf{k}_2\cdot\mathbf{g}_{\mathbf{k}_2,\omega_2}) + \mathbf{k}_2\,(\mathbf{k}_1\cdot\mathbf{g}_{\mathbf{k}_1,\omega_1})\,]\,F_a, \tag{1.59}$$

$$\bar{\chi}_a^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3) = \frac{1}{2}\,\frac{4\pi e_a n_a}{k_1\,k_2\,k_3\,|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|} \times \int d\mathbf{v}\,(\mathbf{g}_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3,\omega_1+\omega_2+\omega_3}\cdot\mathbf{k}_1)\mathbf{g}_{\mathbf{k}_2+\mathbf{k}_3,\omega_2+\omega_3} \cdot \left[\mathbf{k}_2\,(\mathbf{k}_3\cdot\mathbf{g}_{\mathbf{k}_3,\omega_3}) + \mathbf{k}_3\,(\mathbf{k}_2\cdot\mathbf{g}_{\mathbf{k}_2,\omega_2})\right]\,F_a. \tag{1.60}$$

After explicitly writing out the various objects, making use of their respective definitions, we may also rewrite the susceptibilities in concrete forms as follows:

$$\chi_a(\mathbf{k},\omega) = \frac{\omega_{pa}^2}{k^2}\int d\mathbf{v}\,\frac{\mathbf{k}\cdot\partial F_a/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}+i0}, \tag{1.61}$$

$$\chi_a^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) = \frac{-i}{2}\,\frac{e_a}{m_a}\,\frac{\omega_{pa}^2}{k_1\,k_2\,|\mathbf{k}_1+\mathbf{k}_2|} \times \int d\mathbf{v}\,\frac{1}{\omega_1+\omega_2-(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{v}+i0} \times \left[\mathbf{k}_1\cdot\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{k}_2\cdot\partial F_a/\partial\mathbf{v}}{\omega_2-\mathbf{k}_2\cdot\mathbf{v}+i0}\right) + \mathbf{k}_2\cdot\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{k}_1\cdot\partial F_a/\partial\mathbf{v}}{\omega_1-\mathbf{k}_1\cdot\mathbf{v}+i0}\right)\right], \tag{1.62}$$

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$$\bar{\chi}_a^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3) = \frac{(-i)^2}{2}\frac{e_a^2}{m_a^2}\frac{\omega_{pa}^2}{k_1\,k_2\,k_3\,|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|}$$
$$\times \int d\mathbf{v}\,\frac{1}{\omega_1+\omega_2+\omega_3-(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{v}+i0}$$
$$\times\,\mathbf{k}_1\cdot\frac{\partial}{\partial\mathbf{v}}\left\{\frac{1}{\omega_2+\omega_3-(\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{v}+i0}\right.$$
$$\times\left[\mathbf{k}_2\cdot\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{k}_3\cdot\partial F_a/\partial\mathbf{v}}{\omega_3-\mathbf{k}_3\cdot\mathbf{v}+i0}\right)\right.$$
$$\left.\left.+\,\mathbf{k}_3\cdot\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{k}_2\cdot\partial F_a/\partial\mathbf{v}}{\omega_2-\mathbf{k}_2\cdot\mathbf{v}+i0}\right)\right]\right\}, \tag{1.63}$$

where

$$\omega_{pa} = \left(\frac{4\pi n_a e_a^2}{m_a}\right)^{1/2} \tag{1.64}$$

is the plasma frequency for species a. In (1.64), we again iterate that the positive infinitesimal imaginary part associated with the angular frequency within the resonant denominators is the result of imposing the causality condition in the Laplace transformation (1.38) – see also Appendix B. Various linear and nonlinear susceptibilities in (1.55)–(1.63) are known by their respective names:

- $\epsilon(\mathbf{k},\omega)$; linear dielectric constant
- $\chi(\mathbf{k},\omega)$; linear dielectric susceptibility
- $\sigma(\mathbf{k},\omega)$; linear dielectric conductivity
- $\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)$; second-order nonlinear susceptibility
- $\bar{\chi}^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3)$; partial third-order nonlinear susceptibility
- $\chi^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3)$; fully symmetrized third-order nonlinear susceptibility.

These definitions and designation of conventions (and even the notations) follow Sitenko (1982). In terms of these susceptibilities, (1.54) can be written compactly as

$$0 = k\,\epsilon(\mathbf{k},\omega)\,\delta\phi_{\mathbf{k},\omega} - i\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}\sum_{\omega_1+\omega_2=\omega} k_1\,k_2\,\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)$$
$$\times\left(\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2} - \langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\rangle\right)$$
$$-\sum_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}}\sum_{\omega_1+\omega_2+\omega_3=\omega} k_1\,k_2\,k_3\,\bar{\chi}^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3)$$
$$\times\left(\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}_3,\omega_3} - \langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}_3,\omega_3}\rangle\right.$$
$$\left.-\,\delta\phi_{\mathbf{k}_1,\omega_1}\,\langle\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}_3,\omega_3}\rangle\right). \tag{1.65}$$

Recall that in the statistical theory of turbulence we are interested in the correlation function, $\langle\delta\phi^2\rangle$. We may construct the equation for correlation from (1.65).

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Let us therefore multiply $\delta\phi_{\mathbf{k}',\omega'}$ to (1.65), and take the ensemble average of the resulting equation:

$$
\begin{aligned}
0 = {} & k\,\epsilon(\mathbf{k},\omega)\,\langle\delta\phi_{\mathbf{k},\omega}\,\delta\phi_{\mathbf{k}',\omega'}\rangle \\
& -i\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}\;\sum_{\omega_1+\omega_2=\omega} k_1\,k_2\,\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)\,\langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}',\omega'}\rangle \\
& -\sum_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}}\;\sum_{\omega_1+\omega_2+\omega_3=\omega} k_1\,k_2\,k_3\,\bar{\chi}^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3) \\
& \times\Big(\langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}_3,\omega_3}\,\delta\phi_{\mathbf{k}',\omega'}\rangle \\
& -\langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}',\omega'}\rangle\,\langle\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}_3,\omega_3}\rangle\Big)\,.
\end{aligned}
\tag{1.66}
$$

This equation relates the two-body electric field amplitude correlation $\langle\delta\phi^2\rangle$ to three- and four-body correlations $\langle\delta\phi^3\rangle$ and $\langle\delta\phi^4\rangle$. As a consequence, (1.66) is not closed, but instead forms an infinite chain, or hierarchy, of correlations. To break the hierarchy one must introduce certain assumptions. This is the "closure problem." To break the hierarchy, we write the four-body cumulant $\langle\delta\phi^4\rangle$ as the sum of products of two body cumulants while ignoring the irreducible four-body correlation. This is equivalent to ignoring the last term on the right-hand side of (1.15). Let us thus consider the third-order nonlinear term associated with $\bar{\chi}^{(3)}$. Making use of the shorthand notations,

$$
\begin{aligned}
K &= (\mathbf{k},\omega), \quad 1\equiv K_1 = (\mathbf{k}_1,\omega_1), \\
2 &\equiv K_2 = (\mathbf{k}_2,\omega_2), \quad 3\equiv K_3 = (\mathbf{k}_3,\omega_3),
\end{aligned}
\tag{1.67}
$$

we may write

$$
\begin{aligned}
& \langle\delta\phi_1\,\delta\phi_2\,\delta\phi_3\,\delta\phi_{K'}\rangle - \langle\delta\phi_1\,\delta\phi_{K'}\rangle\,\langle\delta\phi_2\,\delta\phi_3\rangle \\
& = \delta(1+2+3+K')\left[\delta(1+2)\,\langle\delta\phi^2\rangle_1\,\langle\delta\phi^2\rangle_3\right. \\
& \quad \left. +\,\delta(1+3)\,\langle\delta\phi^2\rangle_1\,\langle\delta\phi^2\rangle_2\right],
\end{aligned}
\tag{1.68}
$$

where we have ignored the irreducible four-body correlation. This leads to

$$
\begin{aligned}
& \sum_{1+2+3=K} k_1\,k_2\,k_3\,\bar{\chi}^{(3)}(1|2|3)\,\big(\langle\delta\phi_1\,\delta\phi_2\,\delta\phi_3\,\delta\phi_{K'}\rangle - \langle\delta\phi_1\,\delta\phi_{K'}\rangle\,\langle\delta\phi_2\,\delta\phi_3\rangle\big) \\
& = \sum_{1+2+3=K} k_1\,k_2\,k_3\,\bar{\chi}^{(3)}(1|2|3)\,\delta(1+2+3+K') \\
& \quad \times\Big(\delta(1+2)\,\langle\delta\phi^2\rangle_1\,\langle\delta\phi^2\rangle_3 + \delta(1+3)\,\langle\delta\phi^2\rangle_1\,\langle\delta\phi^2\rangle_2\Big) \\
& = 2\delta(K+K')\sum_1 k\,k_1^2\,\bar{\chi}^{(3)}(1|-1|K)\,\langle\delta\phi^2\rangle_1\,\langle\delta\phi^2\rangle_K\,.
\end{aligned}
\tag{1.69}
$$

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Thus, (1.66) simplifies

$$\begin{aligned} 0 &= k\,\epsilon(\mathbf{k},\omega)\,\langle\delta\phi^2\rangle_{\mathbf{k},\omega}\,\delta(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega') \\ &\quad - i \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} \sum_{\omega_1+\omega_2=\omega} k_1\,k_2\,\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)\,\langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}',\omega'}\rangle \\ &\quad -2\,\delta(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega')\sum_{\mathbf{k}',\omega'} k\,k'^2 \\ &\quad \times \bar{\chi}^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',-\omega'|\mathbf{k},\omega)\,\langle\delta\phi^2\rangle_{\mathbf{k}',\omega'}\,\langle\delta\phi^2\rangle_{\mathbf{k},\omega}. \end{aligned} \tag{1.70}$$

In deriving this result, we have made use of the properties of homogeneous and stationary turbulence (1.25) to write

$$\begin{aligned} \langle\delta\phi_{\mathbf{k},\omega}\,\delta\phi_{\mathbf{k}',\omega'}\rangle &= \delta(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega')\,\langle\delta\phi^2\rangle_{\mathbf{k},\omega}, \langle\delta\phi_{\mathbf{k},\omega}\,\delta\phi_{\mathbf{k}',\omega'}\,\delta\phi_{\mathbf{k}'',\omega''}\,\delta\phi_{\mathbf{k}''',\omega'''}\rangle \\ &= \delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}''')\,\delta(\omega+\omega'+\omega''+\omega''') \\ &\quad \times [\,\delta(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega')\,\langle\delta\phi^2\rangle_{\mathbf{k},\omega}\,\langle\delta\phi^2\rangle_{\mathbf{k}'',\omega''} \\ &\quad + \delta(\mathbf{k}+\mathbf{k}'')\,\delta(\omega+\omega'')\,\langle\delta\phi^2\rangle_{\mathbf{k},\omega}\,\langle\delta\phi^2\rangle_{\mathbf{k}',\omega'} \\ &\quad + \delta(\mathbf{k}'+\mathbf{k}'')\,\delta(\omega'+\omega'')\,\langle\delta\phi^2\rangle_{\mathbf{k},\omega}\,\langle\delta\phi^2\rangle_{\mathbf{k}',\omega'} \\ &\quad + \langle\delta\phi^4\rangle_{\mathbf{k}\omega;\,\mathbf{k}+\mathbf{k}',\omega+\omega';\,\mathbf{k}+\mathbf{k}'+\mathbf{k}'',\omega+\omega'+\omega''}\,], \end{aligned} \tag{1.71}$$

and after having done so, we ignored the irreducible four-body correlation $\langle\delta\phi^4\rangle_{\mathbf{k}\omega;\,\mathbf{k}+\mathbf{k}',\omega+\omega';\,\mathbf{k}+\mathbf{k}'+\mathbf{k}'',\omega+\omega'+\omega''}$ in order to truncate the hierarchy of correlations.

The resultant wave equation (1.70) still contains the three-body correlation $\langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{-\mathbf{k},-\omega}\rangle$, hence, not completely closed yet. To compute this three-body correlation, we return to (1.65) and consider up to second-order nonlinearity,

$$\begin{aligned} 0 = k\,\epsilon(\mathbf{k},\omega)\,\delta\phi_{\mathbf{k},\omega} - i \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} \sum_{\omega_1+\omega_2=\omega} k_1\,k_2\,\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) \\ \times \left(\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2} - \langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\rangle\right). \end{aligned} \tag{1.72}$$

The third-order nonlinear term makes contributions to higher-order corrections only, thus can be ignored at the outset. We impose the iterative solution of (1.72),

$$\delta\phi_{\mathbf{k},\omega} = \delta\phi^{(0)}_{\mathbf{k},\omega} + \delta\phi^{(1)}_{\mathbf{k},\omega} + \cdots \tag{1.73}$$

If we ignore the nonlinear correction in (1.72), and truncate the solution by $\delta\phi_{\mathbf{k},\omega} = \delta\phi^{(0)}_{\mathbf{k},\omega}$, then we have

$$0 = \epsilon(\mathbf{k},\omega)\,\delta\phi^{(0)}_{\mathbf{k},\omega}. \tag{1.74}$$

The solution $\delta\phi^{(0)}_{\mathbf{k},\omega}$ represents a sinusoidal (or plane wave) solution, which is uncorrelated with other spectral components. For such a plane-wave solution, ensemble averages of all odd moments of $\delta\phi^{(0)}_{\mathbf{k},\omega}$ are zero:

$$\langle \delta\phi^{(0)}_{\mathbf{k},\omega}\rangle = 0, \quad \langle \delta\phi^{(0)}_{\mathbf{k},\omega}\,\delta\phi^{(0)}_{\mathbf{k}',\omega'}\,\delta\phi^{(0)}_{\mathbf{k}'',\omega''}\rangle = 0, \ldots \tag{1.75}$$

This is because the plane-wave solution has a phase dependence of the form $\delta\phi^{(0)}_{\mathbf{k},\omega} \sim \hat{\phi}_{\mathbf{k},\omega} e^{i\varphi_{\mathbf{k},\omega}}$, so that odd moments are associated with odd products of sinusoidal functions each with phase $\varphi_{\mathbf{k},\omega}$. When integrated over $\varphi_{\mathbf{k},\omega}$, odd moments thus disappear. When the iterative solution (1.73) is inserted to (1.72), the next order solution emerges:

$$\delta\phi^{(1)}_{\mathbf{k},\omega} = \frac{i}{k\,\epsilon(\mathbf{k},\omega)} \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}\ \sum_{\omega_1+\omega_2=\omega} k_1\,k_2\,\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)$$
$$\times\left(\delta\phi^{(0)}_{\mathbf{k}_1,\omega_1}\,\delta\phi^{(0)}_{\mathbf{k}_2,\omega_2} - \langle\delta\phi^{(0)}_{\mathbf{k}_1,\omega_1}\,\delta\phi^{(0)}_{\mathbf{k}_2,\omega_2}\rangle\right). \tag{1.76}$$

The three-body correlation of interest can be approximately expressed as follows:

$$\langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}',\omega'}\rangle = \langle\delta\phi^{(1)}_{\mathbf{k}_1,\omega_1}\,\delta\phi^{(0)}_{\mathbf{k}_2,\omega_2}\,\delta\phi^{(0)}_{\mathbf{k}',\omega'}\rangle$$
$$+\langle\delta\phi^{(0)}_{\mathbf{k}_1,\omega_1}\,\delta\phi^{(1)}_{\mathbf{k}_2,\omega_2}\,\delta\phi^{(0)}_{\mathbf{k}',\omega'}\rangle + \langle\delta\phi^{(0)}_{\mathbf{k}_1,\omega_1}\,\delta\phi^{(0)}_{\mathbf{k}_2,\omega_2}\,\delta\phi^{(1)}_{\mathbf{k}',\omega'}\rangle$$
$$+\cdots. \tag{1.77}$$

Upon making use of solution (1.76) and substituting for $\delta\phi^{(1)}_{\mathbf{k}_1,\omega_1}$, $\delta\phi^{(1)}_{\mathbf{k}_2,\omega_2}$, and $\delta\phi^{(1)}_{\mathbf{k}',\omega'}$, it can be shown that the result is (the intermediate steps are somewhat tedious but straightforward)

$$\langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}',\omega'}\rangle$$
$$= \frac{2i\,k\,|\mathbf{k}-\mathbf{k}_1|\,\chi^{(2)}(-\mathbf{k}+\mathbf{k}_1,\,-\omega+\omega_1|\mathbf{k},\omega)}{k_1\,\epsilon(\mathbf{k}_1,\omega_1)}\,\langle\delta\phi^2\rangle_{\mathbf{k}-\mathbf{k}_1,\omega-\omega_1}\,\langle\delta\phi^2\rangle_{\mathbf{k},\omega}$$
$$+\frac{2i\,k\,k_1\,\chi^{(2)}(-\mathbf{k}_1,\,-\omega_1|\mathbf{k},\omega)}{|\mathbf{k}-\mathbf{k}_1|\,\epsilon(\mathbf{k}-\mathbf{k}_1,\omega-\omega_1)}\,\langle\delta\phi^2\rangle_{\mathbf{k}_1,\omega_1}\,\langle\delta\phi^2\rangle_{\mathbf{k},\omega} \tag{1.78}$$
$$-\frac{2i\,k_1\,|\mathbf{k}-\mathbf{k}_1|\,\chi^{(2)*}(\mathbf{k}_1,\omega_1|\mathbf{k}-\mathbf{k}_1,\omega-\omega_1)}{k\,\epsilon^*(\mathbf{k},\omega)}\,\langle\delta\phi^2\rangle_{\mathbf{k}_1,\omega_1}\,\langle\delta\phi^2\rangle_{\mathbf{k}-\mathbf{k}_1,\omega-\omega_1},$$

where we have deleted the superscripts "(0)" after everything is said and done, and have made use of the symmetry properties,

$$\epsilon(-\mathbf{k},\,-\omega) = \epsilon^*(\mathbf{k},\omega),$$
$$\chi^{(2)}(-\mathbf{k}_1,\,-\omega_1|-\mathbf{k}_2,\,-\omega_2) = -\chi^{(2)*}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2),$$
$$\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) = \chi^{(2)}(\mathbf{k}_2,\omega_2|\mathbf{k}_1,\omega_1). \tag{1.79}$$

We have not yet discussed these symmetry properties associated with various susceptibilities, although the complex conjugate properties can be easily deduced from definitions (1.61)–(1.64). In the subsequent section we will discuss the symmetries as well as other properties associated with the susceptibilities in detail.

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With (1.78) the wave equation (1.70) is now written as

$$
\begin{aligned}
0 = \epsilon(\mathbf{k},\omega)\,\langle k^2\,\delta\phi^2\rangle_{\mathbf{k},\omega} &+ 2\sum_{\mathbf{k}',\omega'}\Bigg[\,\chi^{(2)}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega') \\
&\times\left(\frac{\chi^{(2)}(-\mathbf{k}+\mathbf{k}',\,-\omega+\omega'|\mathbf{k},\omega)}{\epsilon(\mathbf{k}',\omega')}\,\langle\,|\mathbf{k}-\mathbf{k}'|^2\,\delta\phi^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\right. \\
&+\left.\frac{\chi^{(2)}(-\mathbf{k}',\,-\omega'|\mathbf{k},\omega)}{\epsilon(\mathbf{k}-\mathbf{k}',\omega-\omega')}\,\langle\,k'^2\,\delta\phi^2\rangle_{\mathbf{k}',\omega'}\right)\langle\,k^2\,\delta\phi^2\rangle_{\mathbf{k},\omega} \\
&-\frac{|\chi^{(2)}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')|^2}{\epsilon^*(\mathbf{k},\omega)}\,\langle\,k'^2\,\delta\phi^2\rangle_{\mathbf{k}',\omega'}\,\langle\,|\mathbf{k}-\mathbf{k}'|^2\,\delta\phi^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\Bigg] \\
&-2\sum_{\mathbf{k}',\omega'}\bar{\chi}^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',\,-\omega'|\mathbf{k},\omega)\,\langle\,k'^2\,\delta\phi^2\rangle_{\mathbf{k}',\omega'}\,\langle\,k^2\,\delta\phi^2\rangle_{\mathbf{k},\omega}.
\end{aligned}
\tag{1.80}
$$

This equation is known as the nonlinear spectral balance equation since the linear term, which appears as the first term on the right-hand side of the equality, is balanced by the rest that represents nonlinear response. We may rewrite the (1.80) by noting that the electrostatic potential correlation can be rewritten as the electric field correlation

$$
\langle\,k^2\delta\phi^2\rangle_{\mathbf{k},\omega} = \langle\delta E^2\rangle_{\mathbf{k},\omega},
\tag{1.81}
$$

where $\langle\delta E^2\rangle$ is related to the spectral electric field energy density, $\mathscr{E}_{\rm wave} = \langle\delta E^2\rangle/(8\pi)$. As a consequence, (1.80) is equivalently written as

$$
\begin{aligned}
0 = \epsilon(\mathbf{k},\omega)\,\langle\delta E^2\rangle_{\mathbf{k},\omega} &+ 2\sum_{\mathbf{k}',\omega'}\Bigg[\,\chi^{(2)}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega') \\
&\times\left(\frac{\chi^{(2)}(-\mathbf{k}+\mathbf{k}',\,-\omega+\omega'|\mathbf{k},\omega)}{\epsilon(\mathbf{k}',\omega')}\,\langle\delta E^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\right. \\
&+\left.\frac{\chi^{(2)}(-\mathbf{k}',\,-\omega'|\mathbf{k},\omega)}{\epsilon(\mathbf{k}-\mathbf{k}',\omega-\omega')}\,\langle\delta E^2\rangle_{\mathbf{k}',\omega'}\right)\langle\delta E^2\rangle_{\mathbf{k},\omega} \\
&-\frac{|\chi^{(2)}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')|^2}{\epsilon^*(\mathbf{k},\omega)}\,\langle\delta E^2\rangle_{\mathbf{k}',\omega'}\,\langle\delta E^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\Bigg] \\
&-2\sum_{\mathbf{k}',\omega'}\bar{\chi}^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',\,-\omega'|\mathbf{k},\omega)\,\langle\delta E^2\rangle_{\mathbf{k}',\omega'}\,\langle\delta E^2\rangle_{\mathbf{k},\omega}.
\end{aligned}
\tag{1.82}
$$

To determine the adiabatic time evolution of the spectral wave energy density, we now reinstitute the slow time dependence implicit in the wave-particle resonance denominator:

$$
(\omega-\mathbf{k}\cdot\mathbf{v}+i0+i\partial/\partial t)^{-1}.
$$

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Recall that in deriving the final spectral balance equation (1.82), we had "absorbed" the derivative $i\partial/\partial t$ in the "new definition" of ω – see (1.41). This resulted in turning the differential equation (1.40) into an algebraic equation. After the desired equation (1.82) has now been obtained, we reintroduce the factor $i\partial/\partial t$ in the arguments of the response functions. As a consequence, various dielectric susceptibility response functions become operators in the slow time t of the amplitude evolution. However, $i\partial/\partial t$ was present in the original equation (1.40) only on the left-hand side, while the angular frequency ω appeared on both sides. Consequently, when we reintroduce the slow-time derivative $i\partial/\partial t$ to the angular frequency, we do so by treating this object as a small correction, and we introduce it only to the leading term, which is the linear response function. To reiterate, this whole procedure is heuristic, and as we noted already, the proper way to treat this type of problem is via multiple time scale analysis (Davidson, 1972). In this book, we take the present shortcut method nonetheless. When the slow-time derivative is thus reintroduced, the linear term is modified as

$$\begin{aligned}\epsilon(\mathbf{k},\omega)\,\langle\delta E^2\rangle_{\mathbf{k},\omega} &\to \epsilon\left(\mathbf{k},\omega+i\frac{\partial}{\partial t}\right)\,\langle\delta E^2\rangle_{\mathbf{k},\omega}\\ &\to \left(\epsilon(\mathbf{k},\omega)+\frac{i}{2}\frac{\partial\epsilon(\mathbf{k},\omega)}{\partial\omega}\frac{\partial}{\partial t}\right)\langle\delta E^2\rangle_{\mathbf{k},\omega},\end{aligned} \tag{1.83}$$

where 1/2 in front of the time derivative in the last expression stems from the fact that

$$\frac{\partial\,\delta E_{-\mathbf{k},-\omega}(t)}{\partial t}=\frac{\partial\,\delta E_{\mathbf{k},\omega}(t)}{\partial t}. \tag{1.84}$$

Recall that the slow-time derivative originally affects only $\delta f^a_{\mathbf{k},\omega}$, which automatically implies that $i\partial/\partial t$ is meant to operate on $\delta E_{\mathbf{k},\omega}$ or $\delta E_{-\mathbf{k},-\omega}$ separately, but not on their products. Consequently, when the slow-time derivative is reintroduced, the proper procedure is as follows:

$$\begin{aligned}\epsilon(\mathbf{k},\omega)\,\langle\delta E^2\rangle_{\mathbf{k},\omega} &= \left\langle\delta E_{-\mathbf{k},-\omega}\,\epsilon(\mathbf{k},\omega)\,\delta E_{\mathbf{k},\omega}\right\rangle\\ &\to \left\langle\delta E_{-\mathbf{k},-\omega}\,\epsilon\left(\mathbf{k},\omega+i\frac{\partial}{\partial t}\right)\delta E_{\mathbf{k},\omega}\right\rangle\\ &= \left\langle\delta E_{-\mathbf{k},-\omega}\left(\epsilon(\mathbf{k},\omega)+i\frac{\partial\epsilon(\mathbf{k},\omega)}{\partial\omega}\frac{\partial}{\partial t}\right)\delta E_{\mathbf{k},\omega}\right\rangle\\ &= \epsilon(\mathbf{k},\omega)\left\langle\delta E^2\right\rangle_{\mathbf{k},\omega}\\ &\quad+\frac{i}{2}\frac{\partial\epsilon(\mathbf{k},\omega)}{\partial\omega}\left\langle\delta E_{-\mathbf{k},-\omega}\frac{\partial\delta E_{\mathbf{k},\omega}}{\partial t}+\delta E_{\mathbf{k},\omega}\frac{\partial\delta E_{-\mathbf{k},-\omega}}{\partial t}\right\rangle.\end{aligned}$$

After making use of (1.84), then we readily arrive at (1.83).

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This procedure results in the formal wave kinetic equation

$$
\begin{aligned}
0 &= \epsilon(\mathbf{k},\omega)\,\langle\delta E^2\rangle_{\mathbf{k},\omega} + \frac{i}{2}\frac{\partial\epsilon(\mathbf{k},\omega)}{\partial\omega}\frac{\partial\,\langle\delta E^2\rangle_{\mathbf{k},\omega}}{\partial t} \\
&\quad + 2\sum_{\mathbf{k}',\omega'}\Bigg[\chi^{(2)}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega') \\
&\quad \times\left(\frac{\chi^{(2)}(-\mathbf{k}+\mathbf{k}',-\omega+\omega'|\mathbf{k},\omega)}{\epsilon(\mathbf{k}',\omega')}\,\langle\delta E^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\right. \\
&\quad \left.+\frac{\chi^{(2)}(-\mathbf{k}',-\omega'|\mathbf{k},\omega)}{\epsilon(\mathbf{k}-\mathbf{k}',\omega-\omega')}\,\langle\delta E^2\rangle_{\mathbf{k}',\omega'}\right)\langle\delta E^2\rangle_{\mathbf{k},\omega} \\
&\quad -\frac{|\chi^{(2)}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')|^2}{\epsilon^*(\mathbf{k},\omega)}\,\langle\delta E^2\rangle_{\mathbf{k}',\omega'}\,\langle\delta E^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\Bigg] \\
&\quad -2\sum_{\mathbf{k}',\omega'}\bar{\chi}^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',-\omega'|\mathbf{k},\omega)\,\langle\delta E^2\rangle_{\mathbf{k}',\omega'}\,\langle\delta E^2\rangle_{\mathbf{k},\omega}.
\end{aligned}
\tag{1.85}
$$

This equation is further manipulated by separating the linear dielectric function $\epsilon(\mathbf{k},\omega)$ into real and imaginary parts,

$$
\epsilon(\mathbf{k},\omega) = \mathrm{Re}\,\epsilon(\mathbf{k},\omega) + i\,\mathrm{Im}\,\epsilon(\mathbf{k},\omega), \tag{1.86}
$$

where it is assumed that

$$
|\mathrm{Im}\,\epsilon(\mathbf{k},\omega)| \ll |\mathrm{Re}\,\epsilon(\mathbf{k},\omega)|. \tag{1.87}
$$

This assumption is equivalent to the weak growth/damping approximation. The imaginary part of the derivative, $\partial\,\mathrm{Im}\,\epsilon(\mathbf{k},\omega)/\partial\omega$, which couples with the slow-time derivative $\partial/\partial t$, is also ignored. Before we present the final result, let us invoke another useful symmetry property associated with the second-order nonlinear susceptibility,

$$
\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) = \chi^{(2)}(\mathbf{k}_1+\mathbf{k}_2,\omega_1+\omega_2|-\mathbf{k}_2,-\omega_2), \tag{1.88}
$$

which will be derived in the next section. Then (1.85) can be expressed as

$$
\begin{aligned}
0 &= \frac{i}{2}\frac{\partial\,\mathrm{Re}\,\epsilon(\mathbf{k},\omega)}{\partial\omega}\frac{\partial}{\partial t}\,\langle\delta E^2\rangle_{\mathbf{k},\omega} + \mathrm{Re}\,\epsilon(\mathbf{k},\omega)\,\langle\delta E^2\rangle_{\mathbf{k},\omega} + i\,\mathrm{Im}\,\epsilon(\mathbf{k},\omega)\,\langle\delta E^2\rangle_{\mathbf{k},\omega} \\
&\quad + 2\int d\mathbf{k}'\int d\omega'\,\Bigg[\{\,\chi^{(2)}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\,\}^2 \\
&\quad \times\left(\frac{\langle\delta E^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}{\epsilon(\mathbf{k}',\omega')}+\frac{\langle\delta E^2\rangle_{\mathbf{k}',\omega'}}{\epsilon(\mathbf{k}-\mathbf{k}',\omega-\omega')}\right)\langle\delta E^2\rangle_{\mathbf{k},\omega} \\
&\quad -\frac{|\chi^{(2)}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')|^2}{\epsilon^*(\mathbf{k},\omega)}\,\langle\delta E^2\rangle_{\mathbf{k}',\omega'}\,\langle\delta E^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\Bigg] \\
&\quad -2\int d\mathbf{k}'\int d\omega'\,\bar{\chi}^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',-\omega'|\mathbf{k},\omega)\,\langle\delta E^2\rangle_{\mathbf{k}',\omega'}\,\langle\delta E^2\rangle_{\mathbf{k},\omega}.
\end{aligned}
\tag{1.89}
$$

The real part of this equation,

$$0 = \Bigg\{ \mathrm{Re}\,\epsilon(\mathbf{k},\omega) + 2\,\mathrm{Re}\int d\mathbf{k}' \int d\omega' \,\bigg[\{\, \chi^{(2)}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\,\}^2 \\ \times\left(\frac{\langle \delta E^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}{\epsilon(\mathbf{k}',\omega')} + \frac{\langle \delta E^2\rangle_{\mathbf{k}',\omega'}}{\epsilon(\mathbf{k}-\mathbf{k}',\omega-\omega')}\right) \\ -\bar{\chi}^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',\,-\omega'|\mathbf{k},\omega)\,\langle \delta E^2\rangle_{\mathbf{k}',\omega'}\bigg]\Bigg\}\,\langle \delta E^2\rangle_{\mathbf{k},\omega} \\ -2\,\mathrm{Re}\int d\mathbf{k}' \int d\omega' \,\frac{|\chi^{(2)}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')|^2}{\epsilon^*(\mathbf{k},\omega)}\,\langle \delta E^2\rangle_{\mathbf{k}',\omega'}\,\langle \delta E^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}, \tag{1.90}$$

determines the wave dispersion relation, $\omega = \omega_{\mathbf{k}}$, while the imaginary part,

$$0 = \frac{1}{2}\frac{\partial\,\mathrm{Re}\,\epsilon(\mathbf{k},\omega)}{\partial\omega}\frac{\partial}{\partial t}\,\langle \delta E^2\rangle_{\mathbf{k},\omega} + \mathrm{Im}\,\epsilon(\mathbf{k},\omega)\,\langle \delta E^2\rangle_{\mathbf{k},\omega} \\ +2\,\mathrm{Im}\int d\mathbf{k}' \int d\omega' \,\bigg[\{\, \chi^{(2)}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\,\}^2 \\ \times\left(\frac{\langle \delta E^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}{\epsilon(\mathbf{k}',\omega')} + \frac{\langle \delta E^2\rangle_{\mathbf{k}',\omega'}}{\epsilon(\mathbf{k}-\mathbf{k}',\omega-\omega')}\right)\,\langle \delta E^2\rangle_{\mathbf{k},\omega} \\ -\frac{|\chi^{(2)}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')|^2}{\epsilon^*(\mathbf{k},\omega)}\,\langle \delta E^2\rangle_{\mathbf{k}',\omega'}\,\langle \delta E^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\bigg] \\ -2\,\mathrm{Im}\int d\mathbf{k}' \int d\omega' \,\bar{\chi}^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',\,-\omega'|\mathbf{k},\omega)\,\langle \delta E^2\rangle_{\mathbf{k}',\omega'}\,\langle \delta E^2\rangle_{\mathbf{k},\omega}, \tag{1.91}$$

corresponds to the wave kinetic equation. In general, (1.90) is not quite a "dispersion equation" yet, since in the last term different spectral components are inexorably coupled. However, if we ignore the non-diagonal term, $\langle \delta E^2\rangle_{\mathbf{k}',\omega'}$ $\langle \delta E^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}$, that is, if we approximate (1.90) by

$$0 = \Bigg\{ \mathrm{Re}\,\epsilon(\mathbf{k},\omega) + 2\,\mathrm{Re}\int d\mathbf{k}' \int d\omega' \,\bigg[\{\, \chi^{(2)}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\,\}^2 \\ \times\left(\frac{\langle \delta E^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}{\epsilon(\mathbf{k}',\omega')} + \frac{\langle \delta E^2\rangle_{\mathbf{k}',\omega'}}{\epsilon(\mathbf{k}-\mathbf{k}',\omega-\omega')}\right) \\ -\bar{\chi}^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',\,-\omega'|\mathbf{k},\omega)\,\langle \delta E^2\rangle_{\mathbf{k}',\omega'}\bigg]\Bigg\}\,\langle \delta E^2\rangle_{\mathbf{k},\omega} \\ \equiv \mathrm{Re}\,\tilde{\varepsilon}(\mathbf{k},\omega)\,\langle \delta E^2\rangle_{\mathbf{k},\omega}, \tag{1.92}$$

then (1.92) represents nonlinear dispersion equation in the proper sense. If we are concerned with a situation where the waves excited in the plasma can be characterized by linear dispersion relation, but where we are interested in waves interacting

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with the particles and among themselves via linear and nonlinear wave-particle and wave-wave interactions, then one may simplify the real part of the spectral balance equation, namely, (1.92), by simply ignoring nonlinear mode coupling terms altogether,

$$\mathrm{Re}\,\epsilon(\mathbf{k},\omega)\,\langle\delta E^2\rangle_{\mathbf{k},\omega} = 0. \tag{1.93}$$

In Appendix C, we discuss the result of retaining the nonlinear correction terms in (1.92), but in the main body of this book we focus on linear eigenmodes and nonlinear interactions among the linear modes and the particles.

1.5 Formal Wave Kinetic Equation for Eigenmodes

Linear wave equation (1.93) implies that the angular frequency is a function of wave vector, that is, ω and $\mathbf{k}$ satisfy a "dispersion relation."

$$\omega = \omega_{\mathbf{k}}^{\alpha}, \tag{1.94}$$

where α designates the eigenmode. There could be more than a single solution, hence the superscript α. Equation (1.93) implies that the spectral electric field wave energy density can be represented in terms of the wave intensity

$$\langle\delta E^2\rangle_{\mathbf{k},\omega} = \sum_{\alpha=L,S}\left[\, I_{\mathbf{k}}^{+\alpha}\,\delta(\omega-\omega_{\mathbf{k}}^{\alpha}) + I_{\mathbf{k}}^{-\alpha}\,\delta(\omega+\omega_{\mathbf{k}}^{\alpha})\,\right]. \tag{1.95}$$

In this equation, $I_{\mathbf{k}}^{\pm\alpha}$ represents the intensity of electrostatic waves associated with eigenmode α, propagating in forward/backward ($\pm$) direction. We will discuss the linear wave properties later, but it is well known that linear electrostatic eigenmodes of a uniform, unmagnetized plasma are high-frequency Langmuir wave ($\alpha = L$) and low-frequency ion-sound (or ion acoustic) wave ($\alpha = S$). It is important to distinguish the forward versus backward propagation, as nonlinear interactions of these modes depend on the wave propagation direction.

Inserting (1.95) to the wave kinetic equation (1.91) we have

$$\begin{aligned}
0 &= \sum_{\alpha}\sum_{\sigma=\pm1}\left(\frac{1}{2}\frac{\partial\,\mathrm{Re}\,\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})}{\partial\sigma\omega_{\mathbf{k}}^{\alpha}}\frac{\partial I_{\mathbf{k}}^{\sigma\alpha}}{\partial t} + \mathrm{Im}\,\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})\,I_{\mathbf{k}}^{\sigma\alpha}\right)\delta(\omega-\sigma\omega_{\mathbf{k}}^{\alpha}) \\
&\quad + 2\,\mathrm{Im}\sum_{\alpha}\sum_{\sigma=\pm1}\int d\mathbf{k}'\left(\sum_{\gamma}\sum_{\sigma''=\pm1}\right. \\
&\quad \times \frac{\{\,\chi^{(2)}(\mathbf{k}',\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma}|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})\,\}^2}{\epsilon(\mathbf{k}',\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})}\,I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}
\end{aligned}$$

$$+\sum_{\beta}\sum_{\sigma'=\pm1}\frac{\{\chi^{(2)}(\mathbf{k}',\sigma'\omega^{\beta}_{\mathbf{k}'}|\mathbf{k}-\mathbf{k}',\sigma\omega^{\alpha}_{\mathbf{k}}-\sigma'\omega^{\beta}_{\mathbf{k}'})\}^2}{\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega^{\alpha}_{\mathbf{k}}-\sigma'\omega^{\beta}_{\mathbf{k}'})}I^{\sigma'\beta}_{\mathbf{k}'}\Bigg)\;I^{\sigma\alpha}_{\mathbf{k}}\,\delta(\omega-\sigma\omega^{\alpha}_{\mathbf{k}})$$

$$-2\,\mathrm{Im}\sum_{\beta,\gamma}\sum_{\sigma',\sigma''=\pm1}\int d\mathbf{k}'\,\frac{|\chi^{(2)}(\mathbf{k}',\sigma'\omega^{\beta}_{\mathbf{k}'}|\mathbf{k}-\mathbf{k}',\sigma''\omega^{\gamma}_{\mathbf{k}-\mathbf{k}'})|^2}{\epsilon^*(\mathbf{k},\sigma'\omega^{\beta}_{\mathbf{k}'}+\sigma''\omega^{\gamma}_{\mathbf{k}-\mathbf{k}'})}$$

$$\times I^{\sigma'\beta}_{\mathbf{k}'}\,I^{\sigma''\gamma}_{\mathbf{k}-\mathbf{k}'}\,\delta(\omega-\sigma'\omega^{\beta}_{\mathbf{k}'}-\sigma''\omega^{\gamma}_{\mathbf{k}-\mathbf{k}'})$$

$$-2\,\mathrm{Im}\sum_{\alpha,\beta}\sum_{\sigma,\sigma'=\pm1}\int d\mathbf{k}'\,\bar{\chi}^{(3)}(\mathbf{k}',\sigma'\omega^{\beta}_{\mathbf{k}'}|-\mathbf{k}',-\sigma'\omega^{\beta}_{\mathbf{k}'}|\mathbf{k},\sigma\omega^{\alpha}_{\mathbf{k}})$$

$$\times I^{\sigma'\beta}_{\mathbf{k}'}\,I^{\sigma\alpha}_{\mathbf{k}}\,\delta(\omega-\sigma\omega^{\alpha}_{\mathbf{k}}),\quad(\sigma=\pm1). \tag{1.96}$$

Let us focus on the quantity $[\epsilon(\mathbf{k},\omega)]^{-1}$. If ω lies in the vicinity of linear eigenmode, $\omega\sim\sigma\omega^{\alpha}_{\mathbf{k}}$, where $\mathrm{Re}\,\epsilon(\mathbf{k},\sigma\omega^{\alpha}_{\mathbf{k}})=0$, since we had assumed that $|\mathrm{Im}\,\epsilon(\mathbf{k},\omega)|\ll|\mathrm{Re}\,\epsilon(\mathbf{k},\omega)|$, we may approximately express

$$\begin{aligned}\frac{1}{\epsilon(\mathbf{k},\omega)}&\approx\frac{1}{\mathrm{Re}\,\epsilon(\mathbf{k},\omega)}\\&\approx\sum_{\alpha}\sum_{\sigma=\pm1}\frac{1}{(\omega-\sigma\omega^{\alpha}_{\mathbf{k}}+i0)\,[\partial\mathrm{Re}\,\epsilon(\mathbf{k},\omega)/\partial\omega]_{\omega=\sigma\omega^{\alpha}_{\mathbf{k}}}}\\&=-\sum_{\alpha}\sum_{\sigma=\pm1}\frac{i\pi\,\delta(\omega-\sigma\omega^{\alpha}_{\mathbf{k}})}{\partial\mathrm{Re}\,\epsilon(\mathbf{k},\sigma\omega^{\alpha}_{\mathbf{k}})/\partial\sigma\omega^{\alpha}_{\mathbf{k}}},\end{aligned} \tag{1.97}$$

where we have taken the series expansion,

$$\mathrm{Re}\,\epsilon(\mathbf{k},\omega)\approx\sum_{\sigma}\sum_{\alpha}(\omega-\sigma\omega^{\alpha}_{\mathbf{k}}+i0)\left.\frac{\partial\mathrm{Re}\,\epsilon(\mathbf{k},\omega)}{\partial\omega}\right|_{\omega=\sigma\omega^{\alpha}_{\mathbf{k}}}. \tag{1.98}$$

In the second line of (1.97) we have summed over all possible poles. If we include contributions from those ω's that are sufficiently far away from linear eigenmodes in the complex frequency space, then we must add the principal part contribution to the right-hand side of (1.97) as well:

$$\begin{aligned}\frac{1}{\epsilon(\mathbf{k},\omega)}&=\mathscr{P}\,\frac{1}{\epsilon(\mathbf{k},\omega)}-\sum_{\alpha}\sum_{\sigma=\pm1}\frac{i\pi\,\delta(\omega-\sigma\omega^{\alpha}_{\mathbf{k}})}{\partial\mathrm{Re}\,\epsilon(\mathbf{k},\sigma\omega^{\alpha}_{\mathbf{k}})/\partial\sigma\omega^{\alpha}_{\mathbf{k}}}.\\\frac{1}{\epsilon^*(\mathbf{k},\omega)}&=\mathscr{P}\,\frac{1}{\epsilon^*(\mathbf{k},\omega)}+\sum_{\alpha}\sum_{\sigma=\pm1}\frac{i\pi\,\delta(\omega-\sigma\omega^{\alpha}_{\mathbf{k}})}{\partial\mathrm{Re}\,\epsilon(\mathbf{k},\sigma\omega^{\alpha}_{\mathbf{k}})/\partial\sigma\omega^{\alpha}_{\mathbf{k}}}.\end{aligned} \tag{1.99}$$

Principal value $\mathscr{P}$ is meant to exclude those ω's in the vicinity of linear eigenmodes, $\omega = \pm\omega_{\mathbf{k}}^{\alpha}$.

Making use of (1.99), the wave kinetic equation (1.96) is now expressed as follows:

$$
\begin{aligned}
0 = & \sum_{\alpha}\sum_{\sigma=\pm 1}\left(\frac{1}{2}\frac{\partial\,\mathrm{Re}\,\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})}{\partial\sigma\omega_{\mathbf{k}}^{\alpha}}\frac{\partial I_{\mathbf{k}}^{\sigma\alpha}}{\partial t} + \mathrm{Im}\,\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})\, I_{\mathbf{k}}^{\sigma\alpha}\right) \\
& + 2\,\mathrm{Im}\sum_{\alpha,\beta}\sum_{\sigma,\sigma'=\pm 1}\int d\mathbf{k}'\left(2\,\{\chi^{(2)}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta}|\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta})\}^2\right. \\
& \times\mathscr{P}\frac{1}{\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta})} \\
& \left. -\bar{\chi}^{(3)}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta}|-\mathbf{k}',-\sigma'\omega_{\mathbf{k}'}^{\beta}|\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})\right) I_{\mathbf{k}'}^{\sigma'\beta}\, I_{\mathbf{k}}^{\sigma\alpha} \\
& - 2\,\mathrm{Im}\sum_{\beta,\gamma}\sum_{\sigma',\sigma''=\pm 1}\int d\mathbf{k}'\;|\chi^{(2)}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta}|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})|^2 \\
& \times\mathscr{P}\frac{1}{\epsilon^*(\mathbf{k},\sigma'\omega_{\mathbf{k}'}^{\beta}+\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})}\, I_{\mathbf{k}'}^{\sigma'\beta}\, I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}\,\delta(\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma}) \\
& + 2\pi\sum_{\alpha,\beta,\gamma}\sum_{\sigma,\sigma',\sigma''=\pm 1}\int d\mathbf{k}'\;|\chi^{(2)}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta}|\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta})|^2 \\
& \times\left(\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}\, I_{\mathbf{k}}^{\sigma\alpha}}{\partial\mathrm{Re}\,\epsilon(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta})/\partial\sigma'\omega_{\mathbf{k}'}^{\beta}} + \frac{I_{\mathbf{k}'}^{\sigma'\beta}\, I_{\mathbf{k}}^{\sigma\alpha}}{\partial\mathrm{Re}\,\epsilon(\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})/\partial\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma}}\right. \\
& \left. - \frac{I_{\mathbf{k}'}^{\sigma'\beta}\, I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}}{\partial\mathrm{Re}\,\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})/\partial\sigma\omega_{\mathbf{k}}^{\alpha}}\right)\delta(\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma}).
\end{aligned} \tag{1.100}
$$

In deriving this result, we have invoked the fact that, to the leading order, $\chi^{(2)}$ is purely imaginary so that

$$
\{\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)\}^2 \approx -|\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)|^2, \tag{1.101}
$$

which we imposed for the last term that contains the three-wave resonance delta function condition $\delta(\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})$. This property will be discussed in the subsequent section. In (1.100), the principal part, $\mathscr{P}[1/\epsilon^*(\mathbf{k},\sigma'\omega_{\mathbf{k}'}^{\beta}+\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})]$, and the condition, $\sigma\omega_{\mathbf{k}}^{\alpha}=\sigma'\omega_{\mathbf{k}'}^{\beta}+\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma}$, where $\sigma\omega_{\mathbf{k}}^{\alpha}$ satisfies $\epsilon^*(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})=0$,

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are mutually exclusive. Thus, by definition, this term is zero. This leaves us with

$$\begin{aligned}
&\sum_{\alpha}\sum_{\sigma=\pm 1}\frac{\partial\,\mathrm{Re}\,\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})}{\partial\sigma\omega_{\mathbf{k}}^{\alpha}}\frac{\partial I_{\mathbf{k}}^{\sigma\alpha}}{\partial t} \\
&\quad= -\sum_{\alpha}\sum_{\sigma=\pm 1} 2\,\mathrm{Im}\,\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})\,I_{\mathbf{k}}^{\sigma\alpha} \\
&\qquad -4\sum_{\mathbf{k}'}\sum_{\alpha,\beta}\sum_{\sigma,\sigma'=\pm 1}\mathrm{Im}\Bigg(2\,\{\chi^{(2)}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta}|\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta})\}^2 \\
&\qquad \times \mathscr{P}\frac{1}{\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta})} \qquad\qquad (1.102)\\
&\qquad -\bar{\chi}^{(3)}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta}|-\mathbf{k}',-\sigma'\omega_{\mathbf{k}'}^{\beta}|\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})\Bigg)I_{\mathbf{k}'}^{\sigma'\beta}\,I_{\mathbf{k}}^{\sigma\alpha} \\
&\qquad -4\pi\sum_{\mathbf{k}'}\sum_{\alpha,\beta,\gamma}\sum_{\sigma,\sigma',\sigma''=\pm 1}|\chi^{(2)}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta}|\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta})|^2 \\
&\qquad \times\Bigg(\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}\,I_{\mathbf{k}}^{\sigma\alpha}}{\partial\mathrm{Re}\,\epsilon(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta})/\partial\sigma'\omega_{\mathbf{k}'}^{\beta}}+\frac{I_{\mathbf{k}'}^{\sigma'\beta}\,I_{\mathbf{k}}^{\sigma\alpha}}{\partial\mathrm{Re}\,\epsilon(\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})/\partial\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma}} \\
&\qquad -\frac{I_{\mathbf{k}'}^{\sigma'\beta}\,I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}}{\partial\mathrm{Re}\,\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})/\partial\sigma\omega_{\mathbf{k}}^{\alpha}}\Bigg)\,\delta(\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma}).
\end{aligned}$$

This equation is the formal wave kinetic equation governing the dynamics of linear eigenmodes as they undergo nonlinear interactions among themselves as well as with the plasma particles. At this stage in the development of formalism, however, the result is not practically useful since the various susceptibilities are expressed at a formal level. These quantities are yet to be explicitly calculated in forms that readily lend themselves to further analysis.

1.6 Formal Particle Kinetic Equation

Formal particle kinetic equation (1.40) can be further manipulated by considering the quantity given in (1.103), which follows from (1.52),

$$\begin{aligned}
\langle\delta f_{\mathbf{k},\omega}^{a}\,\delta\phi_{\mathbf{k}',\omega'}\rangle &= \delta(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega')\,\alpha(\mathbf{k},\omega)\,\langle\delta\phi^2\rangle_{\mathbf{k},\omega}\,n_a F_a \\
&\quad+\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}\sum_{\omega_1+\omega_2=\omega}\alpha^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) \\
&\quad\times\langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}',\omega'}\rangle\,n_a F_a \\
&\quad+2\,\delta(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega')\sum_{\mathbf{k}_1}\sum_{\omega_1}\alpha^{(3)}(\mathbf{k}_1,\omega_1|-\mathbf{k}_1,-\omega_1|\mathbf{k},\omega) \\
&\quad\times\langle\delta\phi^2\rangle_{\mathbf{k}_1,\omega_1}\,\langle\delta\phi^2\rangle_{\mathbf{k},\omega}\,n_a F_a. \qquad\qquad (1.103)
\end{aligned}$$

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Upon making use of (1.78), we have

$$
\begin{aligned}
\langle \delta f^{a}_{\mathbf{k},\omega}\,\delta\phi_{\mathbf{k}',\omega'}\rangle = \delta(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega')\,\Bigg\{&\alpha(\mathbf{k},\omega)\,\langle\delta\phi^2\rangle_{\mathbf{k},\omega}\\
&+\sum_{\mathbf{k}_1\omega_1}\frac{2i}{k\,k_1\,|\mathbf{k}-\mathbf{k}_1|}\,\alpha^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}-\mathbf{k}_1,\omega-\omega_1)\\
&\times\Bigg(\frac{\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}-\mathbf{k}_1,\omega-\omega_1)}{\epsilon(\mathbf{k}_1,\omega_1)}\,|\mathbf{k}-\mathbf{k}_1|^2\,k^2\\
&\times\langle\delta\phi^2\rangle_{\mathbf{k}-\mathbf{k}_1,\omega-\omega_1}\,\langle\delta\phi^2\rangle_{\mathbf{k},\omega}\\
&+\frac{\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}-\mathbf{k}_1,\omega-\omega_1)}{\epsilon(\mathbf{k}-\mathbf{k}_1,\omega-\omega_1)}\,k_1^2\,k^2\,\langle\delta\phi^2\rangle_{\mathbf{k}_1,\omega_1}\,\langle\delta\phi^2\rangle_{\mathbf{k},\omega}\\
&-\frac{\chi^{(2)*}(\mathbf{k}_1,\omega_1|\mathbf{k}-\mathbf{k}_1,\omega-\omega_1)}{\epsilon^*(\mathbf{k},\omega)}\,k_1^2\,|\mathbf{k}-\mathbf{k}_1|^2\\
&\times\langle\delta\phi^2\rangle_{\mathbf{k}_1,\omega_1}\,\langle\delta\phi^2\rangle_{\mathbf{k}-\mathbf{k}_1,\omega-\omega_1}\Bigg)\\
&+2\sum_{\mathbf{k}_1,\omega_1}\alpha^{(3)}(\mathbf{k}_1,\omega_1|-\mathbf{k}_1,-\omega_1|\mathbf{k},\omega)\,\langle\delta\phi^2\rangle_{\mathbf{k}_1,\omega_1}\,\langle\delta\phi^2\rangle_{\mathbf{k},\omega}.
\end{aligned}
\tag{1.104}
$$

Substituting (1.104) to the right-hand side of formal particle kinetic equation in (1.40), we obtain

$$
\begin{aligned}
\frac{\partial F_a}{\partial t} = -\frac{ie_a}{m_a}\int d\mathbf{k}\int d\omega\;\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\Bigg\{&\alpha(\mathbf{k},\omega)\,\frac{\langle\delta E^2\rangle_{\mathbf{k},\omega}}{k^2}\\
+2i\int d\mathbf{k}'\int d\omega'\;&\alpha^{(2)}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\,\frac{M(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')}{k\,k'\,|\mathbf{k}-\mathbf{k}'|}\\
+2\int d\mathbf{k}'\int d\omega'\;&\alpha^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',-\omega'|\mathbf{k},\omega)\,\frac{\langle\delta E^2\rangle_{\mathbf{k}',\omega'}\,\langle\delta E^2\rangle_{\mathbf{k},\omega}}{k^2\,k'^2}\Bigg\}\,F_a,
\end{aligned}
\tag{1.105}
$$

where

$$
\begin{aligned}
M(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) = {}&\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)\left(\frac{\langle\delta E^2\rangle_{\mathbf{k}_2,\omega_2}}{\epsilon(\mathbf{k}_1,\omega_1)}+\frac{\langle\delta E^2\rangle_{\mathbf{k}_1,\omega_1}}{\epsilon(\mathbf{k}_2,\omega_2)}\right)\langle\delta E^2\rangle_{\mathbf{k}_1+\mathbf{k}_2,\omega_1+\omega_2}\\
&-\frac{\chi^{(2)*}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)}{\epsilon^*(\mathbf{k}_1+\mathbf{k}_2,\omega_1+\omega_2)}\,\langle\delta E^2\rangle_{\mathbf{k}_1,\omega_1}\,\langle\delta E^2\rangle_{\mathbf{k}_2,\omega_2}.
\end{aligned}
\tag{1.106}
$$

It is instructive to rewrite this equation explicitly by making use of (1.53). Taking the real part only we obtain

$$\frac{\partial F_a}{\partial t} = -\text{Im}\,\frac{e_a^2}{m_a^2}\int d\mathbf{k}\int d\omega\,\frac{\mathbf{k}}{k}\cdot\frac{\partial}{\partial\mathbf{v}}\frac{\langle\delta E^2\rangle_{\mathbf{k},\omega}}{\omega-\mathbf{k}\cdot\mathbf{v}}\frac{\mathbf{k}}{k}\cdot\frac{\partial F_a}{\partial\mathbf{v}}$$
$$+\text{Re}\,\frac{e_a^3}{m_a^3}\int d\mathbf{k}\int d\omega\int d\mathbf{k}'\int d\omega'\,\frac{\mathbf{k}}{k}\cdot\frac{\partial}{\partial\mathbf{v}}\frac{M(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')}{\omega-\mathbf{k}\cdot\mathbf{v}}$$
$$\times\left(\frac{\mathbf{k}'}{k'}\cdot\frac{\partial}{\partial\mathbf{v}}\frac{1}{\omega-\omega'-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}}\frac{(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial}{\partial\mathbf{v}}\right.$$
$$\left.+\frac{(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial}{\partial\mathbf{v}}\frac{1}{\omega'-\mathbf{k}'\cdot\mathbf{v}}\frac{\mathbf{k}'}{k'}\cdot\frac{\partial}{\partial\mathbf{v}}\right)F_a$$
$$+\text{Im}\,\frac{e_a^4}{m_a^4}\int d\mathbf{k}\int d\omega\int d\mathbf{k}'\int d\omega'\,\frac{\mathbf{k}}{k}\cdot\frac{\partial}{\partial\mathbf{v}}\frac{\langle\delta E^2\rangle_{\mathbf{k}',\omega'}\langle\delta E^2\rangle_{\mathbf{k},\omega}}{\omega-\mathbf{k}\cdot\mathbf{v}}\frac{\mathbf{k}'}{k'}\cdot\frac{\partial}{\partial\mathbf{v}}$$
$$\times\frac{1}{\omega-\omega'-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}}\left(\frac{\mathbf{k}'}{k'}\cdot\frac{\partial}{\partial\mathbf{v}}\frac{1}{\omega-\mathbf{k}\cdot\mathbf{v}}\frac{\mathbf{k}}{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right.$$
$$\left.-\frac{\mathbf{k}}{k}\cdot\frac{\partial}{\partial\mathbf{v}}\frac{1}{\omega'-\mathbf{k}'\cdot\mathbf{v}}\frac{\mathbf{k}'}{k'}\cdot\frac{\partial}{\partial\mathbf{v}}\right)F_a. \tag{1.107}$$

We will make explicit use of the spectral wave energy density (1.95) in (1.107), thereby eliminating ω and ω' integrals by virtue of the delta functions. This will be done later in Chapter 4. For now, we treat (1.107) as constituting the formal particle kinetic equation, and as an interlude, we next discuss the properties of various dielectric susceptibility response functions.

1.7 Linear and Nonlinear Susceptibilities

Linear and nonlinear susceptibilities have already been introduced in (1.55)–(1.64). It is instructive to rewrite these response functions by means of partial integrations,

$$\chi_a(\mathbf{k},\omega) = -\omega_{pa}^2\int d\mathbf{v}\,\frac{F_a}{(\omega-\mathbf{k}\cdot\mathbf{v}+i0)^2}, \tag{1.108}$$

$$\chi_a^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) = -\int d\mathbf{v}\,F_a\,\frac{1}{(\omega_1-\mathbf{k}_1\cdot\mathbf{v}+i0)(\omega_2-\mathbf{k}_2\cdot\mathbf{v}+i0)}$$
$$\times\frac{1}{\omega_1+\omega_2-(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{v}+i0}$$
$$\times\left(\frac{k_1^2\,\mathbf{k}_2\cdot(\mathbf{k}_1+\mathbf{k}_2)}{\omega_1-\mathbf{k}_1\cdot\mathbf{v}+i0}+\frac{k_2^2\,\mathbf{k}_1\cdot(\mathbf{k}_1+\mathbf{k}_2)}{\omega_2-\mathbf{k}_2\cdot\mathbf{v}+i0}\right.$$
$$\left.+\frac{(\mathbf{k}_1+\mathbf{k}_2)^2(\mathbf{k}_1\cdot\mathbf{k}_2)}{\omega_1+\omega_2-(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{v}+i0}\right), \tag{1.109}$$

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$$
\begin{aligned}
\bar{\chi}_a^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3) &= \frac{1}{2}\frac{e_a^2}{m_a^2}\frac{\omega_{pa}^2\,[\mathbf{k}_1\cdot(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)]}{k_1\,k_2\,k_3\,|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|} \\
&\times \int d\mathbf{v}\,F_a\,\frac{1}{(\omega_2-\mathbf{k}_2\cdot\mathbf{v}+i0)\,(\omega_3-\mathbf{k}_3\cdot\mathbf{v}+i0)} \\
&\times \frac{1}{[\omega_1+\omega_2+\omega_3-(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{v}+i0]^2} \\
&\times \left[\frac{2}{\omega_1+\omega_2+\omega_3-(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{v}+i0}\right. \\
&\times \left(\frac{3\,[\mathbf{k}_2\cdot(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)]\,[\mathbf{k}_3\cdot(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)]}{\omega_1+\omega_2+\omega_3-(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{v}+i0}\right. \\
&+ \frac{k_2^2\,[\mathbf{k}_3\cdot(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)]}{\omega_2-\mathbf{k}_2\cdot\mathbf{v}+i0} \\
&+ \left.\frac{k_3^2\,[\mathbf{k}_2\cdot(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)]}{\omega_3-\mathbf{k}_3\cdot\mathbf{v}+i0}\right) \\
&+ \frac{1}{\omega_2+\omega_3-(\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{v}+i0} \\
&\times \left(\frac{2\,(\mathbf{k}_2\cdot\mathbf{k}_3)\,[(\mathbf{k}_2+\mathbf{k}_3)\cdot(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)]}{\omega_1+\omega_2+\omega_3-(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{v}+i0}\right. \\
&+ \frac{k_2^2\,[\mathbf{k}_3\cdot(\mathbf{k}_2+\mathbf{k}_3)]}{\omega_2-\mathbf{k}_2\cdot\mathbf{v}+i0} + \frac{k_3^2\,[\mathbf{k}_2\cdot(\mathbf{k}_2+\mathbf{k}_3)]}{\omega_3-\mathbf{k}_3\cdot\mathbf{v}+i0} \\
&+ \left.\left.\frac{(\mathbf{k}_2+\mathbf{k}_3)^2(\mathbf{k}_2\cdot\mathbf{k}_3)}{\omega_2+\omega_3-(\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{v}+i0}\right)\right]. \qquad (1.110)
\end{aligned}
$$

1.7.1 Symmetry Relations

The first useful symmetry property involves the permutation of arguments. The second- and third-order susceptibilities, $\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)$ and $\chi^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3)$, are fully symmetric with respect to permutations of arguments. However, the partial third-order susceptibility $\bar{\chi}^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3)$ is symmetric only with respect to the permutation of the last two sets of arguments:

$$
\begin{aligned}
\chi^{(2)}(\mathbf{k}_2,\omega_2|\mathbf{k}_1,\omega_1) &= \chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2), \\
\bar{\chi}^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_3,\omega_3|\mathbf{k}_2,\omega_2) &= \bar{\chi}^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3), \qquad (1.111) \\
\chi^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3) &= \chi^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_3,\omega_3|\mathbf{k}_2,\omega_2) \\
&= \chi^{(3)}(\mathbf{k}_2,\omega_2|\mathbf{k}_1,\omega_1|\mathbf{k}_3,\omega_3) = \chi^{(3)}(\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3|\mathbf{k}_1,\omega_1) \\
&= \chi^{(3)}(\mathbf{k}_3,\omega_3|\mathbf{k}_2,\omega_2|\mathbf{k}_1,\omega_1) = \chi^{(3)}(\mathbf{k}_3,\omega_3|\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2).
\end{aligned}
$$

The next useful symmetry property involves arguments of opposite signs:

$$\begin{aligned}
\epsilon(-\mathbf{k},\,-\omega) &= \epsilon^*(\mathbf{k},\omega),\\
\chi^{(2)}(-\mathbf{k}_1,\,-\omega_1|-\mathbf{k}_2,\,-\omega_2) &= -\chi^{(2)*}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2), \qquad (1.112)\\
\bar{\chi}^{(3)}(-\mathbf{k}_1,\,-\omega_1|-\mathbf{k}_2,\,-\omega_2|-\mathbf{k}_3,\,-\omega_3) &= \bar{\chi}^{(3)*}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3),\\
\chi^{(3)}(-\mathbf{k}_1,\,-\omega_1|-\mathbf{k}_2,\,-\omega_2|-\mathbf{k}_3,\,-\omega_3) &= \chi^{(3)*}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3).
\end{aligned}$$

These symmetry relations can easily be checked from definitions (1.62)–(1.64) or (1.108)–(1.110). If the small positive imaginary part in the resonant denominator is ignored, then we obtain a useful approximate symmetry relation from (1.109),

$$\begin{aligned}
\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) &= \chi^{(2)}(\mathbf{k}_2,\omega_2|\mathbf{k}_1,\omega_1)\\
&= \chi^{(2)}(\mathbf{k}_1+\mathbf{k}_2,\omega_1+\omega_2|-\mathbf{k}_2,\,-\omega_2), \qquad (1.113)
\end{aligned}$$

which we have already invoked in (1.88).

1.7.2 Linear Dielectric Susceptibility

In the long-wavelength limit ($k^2 \to 0$), the linear dielectric function $\chi^a(\mathbf{k},\omega)$ takes on the limiting form,

$$\chi^a(0,\omega) = -\frac{\omega_{pa}^2}{\omega^2}. \qquad (1.114)$$

To include thermal corrections, we expand the resonant denominator for small argument, $\mathbf{k}\cdot\mathbf{v}/\omega \ll 1$, to obtain

$$\begin{aligned}
\chi^a(\mathbf{k},\omega) \approx &-\frac{\omega_{pa}^2}{\omega^2}\left(1+\frac{3k^2T_a}{m_a\omega^2}\right)\\
&-i\pi\frac{\omega_{pa}^2}{k^2}\int d\mathbf{v}\,\mathbf{k}\cdot\frac{\partial F_a}{\partial\mathbf{v}}\,\delta(\omega-\mathbf{k}\cdot\mathbf{v}), \qquad (1.115)
\end{aligned}$$

where we have defined the kinetic temperature,

$$T_a = m_a\int d\mathbf{v}\,v^2 F_a = \frac{m_a v_{Ta}^2}{2}. \qquad (1.116)$$

Here, $v_{Ta} = \sqrt{2T_a/m_a}$ represents the thermal speed for species a. In deriving (1.115) we assumed that the plasma species a has zero net drift, $\int d\mathbf{v}\;\mathbf{v}\,F_a \approx 0$. The expression (1.115) assumes the so-called fast-wave condition, $\omega \gg kv_{Ta}$. For the opposite case when $\omega \ll kv_{Ta}$ is satisfied (the slow-wave condition), we may approximate the principal part of the resonant denominator by

$$\mathscr{P}\frac{1}{\omega-\mathbf{k}\cdot\mathbf{v}+i0} \approx -\frac{1}{\mathbf{k}\cdot\mathbf{v}}. \qquad (1.117)$$

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If we assume that the distribution F_a is given by a quasi-Maxwellian form so that we may write

$$\frac{\partial F_a}{\partial \mathbf{v}} \approx -\frac{2(\mathbf{k}\cdot\mathbf{v})\, F_a}{v_{Ta}^2}, \tag{1.118}$$

then this leads to the top-and-bottom cancelation of the factor $\mathbf{k}\cdot\mathbf{v}$ within the velocity integral. The resulting linear dielectric susceptibility is

$$\chi^a(\mathbf{k},\omega) = \frac{2\omega_{pa}^2}{k^2 v_{Ta}^2} - i\pi\frac{\omega_{pa}^2}{k^2}\int d\mathbf{v}\,\mathbf{k}\cdot\frac{\partial F_a}{\partial \mathbf{v}}\,\delta(\omega - \mathbf{k}\cdot\mathbf{v}). \tag{1.119}$$

For isotropic thermal equilibrium distribution, the linear dielectric function is given in terms of the plasma dispersion function $Z(z)$, as

$$\epsilon(\mathbf{k},\omega) = 1 - \sum_a \frac{\omega_{pa}^2}{\omega^2}\,\zeta_{\mathbf{k},\omega}^{a2}\, Z'(\zeta_{\mathbf{k},\omega}^a), \tag{1.120}$$

where

$$\zeta_{\mathbf{k},\omega}^a = \frac{\omega}{k v_{Ta}}, \qquad v_{Ta}^2 = \frac{2T_a}{m_a}, \tag{1.121}$$

and

$$\begin{aligned} Z(z) &= \int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}}\frac{e^{-x^2}}{x-z} = 2i\,e^{-z^2}\int_{-\infty}^{iz} dt\, e^{t^2}, \\ Z'(z) &= -2[1 + z\,Z(z)]. \end{aligned} \tag{1.122}$$

The plasma dispersion function and its properties are well known, for example, see Huba (2009). Series and asymptotic expansions of $Z(z)$ are given by

$$\begin{aligned} Z(z) &= i\sqrt{\pi}\,e^{-z^2} - 2z + \frac{4z^3}{3} - \frac{8z^5}{15} + \frac{16z^7}{105} + \cdots, \quad (z^2 < 1), \\ Z(z) &= i\sqrt{\pi}\,\sigma\, e^{-z^2} - \frac{1}{z} - \frac{1}{2z^3} - \frac{3}{4z^5} - \frac{15}{8z^7} + \cdots, \quad (z^2 > 1), \end{aligned} \tag{1.123}$$

where $\sigma = 0$ if $\mathrm{Im}z > 1/\mathrm{Re}z$, $\sigma = 1$ if $|\mathrm{Im}z| < 1/\mathrm{Re}z$, and $\sigma = 2$ if $-\mathrm{Im}z > 1/\mathrm{Re}z$. The plasma dispersion or Fried–Conte function (Fried and Conte, 1961) and its properties are further discussed in Appendix D.

Making use of the asymptotic expansion we obtain an approximate expression for the linear dielectric response function for thermal equilibrium,

$$\epsilon(\mathbf{k},\omega) = 1 - \sum_a \frac{\omega_{pa}^2}{\omega^2}\left(1 + \frac{3k^2 T_a}{m_a \omega^2}\right) + 2i\pi^{1/2} \sum_a \frac{\omega_{pa}^2 \omega}{k^3 v_{Ta}^3} \exp\left(-\frac{\omega^2}{k^2 v_{Ta}^2}\right). \tag{1.124}$$

As it turns out, this expression, derived under the assumption of fast wave condition, is applicable to high-frequency Langmuir waves where both conditions, $\omega \gg k v_{Te}$ and $\omega \gg k v_{Ti}$, are valid. For ion-sound waves, on the other hand, $\omega \ll k v_{Te}$ (slow-wave condition for electrons), while $\omega \gg k v_{Ti}$ (fast wave condition for protons). In this case, we have

$$\epsilon(\mathbf{k},\omega) = 1 + \frac{2\omega_{pe}^2}{k^2 v_{Te}^2} - \frac{\omega_{pi}^2}{\omega^2}\left(1 + \frac{3k^2 T_i}{m_i \omega^2}\right) + 2i\pi^{1/2} \sum_a \frac{\omega_{pa}^2 \omega}{k^3 v_{Ta}^3} \exp\left(-\frac{\omega^2}{k^2 v_{Ta}^2}\right). \tag{1.125}$$

These properties will be used later when we review the linear wave theory.

1.7.3 Second-Order Nonlinear Susceptibility

The following limiting forms for the second-order susceptibility can easily be derived:

$$\chi_a^{(2)}(0,\omega_1|0,\omega_2) = 0,$$
$$\chi_a^{(2)}(\mathbf{k}_1,0|\mathbf{k}_2,0) = -\frac{i e_a}{T_a} \frac{\omega_{pa}^2}{k_1\, k_2\, |\mathbf{k}_1 + \mathbf{k}_2|} \frac{1}{v_{Ta}^2}. \tag{1.126}$$

If the fast-wave condition is satisfied for all frequencies and wave numbers,

$$\omega_1 \gg k_1 v_{Ta}, \quad \omega_2 \gg k_2 v_{Ta}, \quad \omega_1 + \omega_2 \gg |\mathbf{k}_1 + \mathbf{k}_2| v_{Ta}, \tag{1.127}$$

then the approximate second-order nonlinear susceptibility can be obtained by assuming that the temperature is zero ($T_a \to 0$), or equivalently by choosing $F_a(\mathbf{v}) = \delta(\mathbf{v})$:

$$\chi_a^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) = \frac{-i}{2} \frac{e_a}{m_a} \frac{\omega_{pa}^2}{\omega_1\, \omega_2\, (\omega_1 + \omega_2)} \frac{1}{k_1\, k_2\, |\mathbf{k}_1 + \mathbf{k}_2|} \times \left(\frac{k_1^2}{\omega_1} \mathbf{k}_2 \cdot (\mathbf{k}_1 + \mathbf{k}_2) + \frac{k_2^2}{\omega_2} \mathbf{k}_1 \cdot (\mathbf{k}_1 + \mathbf{k}_2) + \frac{(\mathbf{k}_1 + \mathbf{k}_2)^2}{\omega_1 + \omega_2} \mathbf{k}_1 \cdot \mathbf{k}_2\right). \tag{1.128}$$

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If one of the frequencies, say ω_1, does not satisfy the fast-wave condition while ω_2 and $\omega_1 + \omega_2$ are characterized by $\omega_2 \gg k_2 v_{Ta}$ and $\omega_1 + \omega_2 \gg |\mathbf{k}_1 + \mathbf{k}_2| v_{Ta}$, then we may obtain the following expression:

$$\chi_a^{(2)}(\mathbf{k}_1, \omega_1 | \mathbf{k}_2, \omega_2) = \frac{i}{2} \frac{e_a}{m_a} \frac{\omega_{pa}^2}{\omega_2\,(\omega_1 + \omega_2)} \frac{1}{k_1\, k_2\, |\mathbf{k}_1 + \mathbf{k}_2|} \times \mathbf{k}_2 \cdot (\mathbf{k}_1 + \mathbf{k}_2) \int d\mathbf{v}\, \frac{\mathbf{k}_1 \cdot \partial F_a / \partial \mathbf{v}}{\omega_1 - \mathbf{k}_1 \cdot \mathbf{v} + i0}, \tag{1.129}$$

where we have made use of the identity

$$\int d\mathbf{v}\, \frac{F_a}{(\omega_1 - \mathbf{k}_1 \cdot \mathbf{v} + i0)^2} = -\frac{1}{k_1^2} \int d\mathbf{v}\, \frac{\mathbf{k}_1 \cdot \partial F_a / \partial \mathbf{v}}{\omega_1 - \mathbf{k}_1 \cdot \mathbf{v} + i0}, \tag{1.130}$$

and have invoked the fast wave conditions to ignore terms k_2^2/ω and $(\mathbf{k}_1 + \mathbf{k}_2)^2/(\omega_1 + \omega_2)$. The velocity integral is related to the linear response function so that we have

$$\chi_a^{(2)}(\mathbf{k}_1, \omega_1 | \mathbf{k}_2, \omega_2) = \frac{i}{2} \frac{e_a}{m_a} \frac{k_1}{\omega_2\,(\omega_1 + \omega_2)} \frac{\mathbf{k}_2 \cdot (\mathbf{k}_1 + \mathbf{k}_2)}{k_2\, |\mathbf{k}_1 + \mathbf{k}_2|}\, \chi_a(\mathbf{k}_1, \omega_1). \tag{1.131}$$

If we further assume that $\omega_1 \ll k_1 v_{Ta}$, that is, the slow-wave condition for ω_1, then we obtain

$$\chi_a^{(2)}(\mathbf{k}_1, \omega_1 | \mathbf{k}_2, \omega_2) = \frac{i}{2} \frac{e_a}{m_a} \frac{\omega_{pa}^2}{\omega_2\,(\omega_1 + \omega_2)} \frac{\mathbf{k}_2 \cdot (\mathbf{k}_1 + \mathbf{k}_2)}{k_1\, k_2\, |\mathbf{k}_1 + \mathbf{k}_2|} \times \left(\frac{2}{v_{Ta}^2} - i\pi \int d\mathbf{v}\, \mathbf{k}_1 \cdot \frac{\partial F_a}{\partial \mathbf{v}}\, \delta(\omega_1 - \mathbf{k}_1 \cdot \mathbf{v}) \right). \tag{1.132}$$

If ω_2 represents an arbitrary wave, then by following the same steps as in the previous case, we may derive the following result:

$$\begin{aligned} \chi_a^{(2)}(\mathbf{k}_1, \omega_1 | \mathbf{k}_2, \omega_2) &= \frac{i}{2} \frac{e_a}{m_a} \frac{k_2}{\omega_1\,(\omega_1 + \omega_2)} \frac{\mathbf{k}_1 \cdot (\mathbf{k}_1 + \mathbf{k}_2)}{k_1\, |\mathbf{k}_1 + \mathbf{k}_2|}\, \chi_a(\mathbf{k}_2, \omega_2) \\ &\approx \frac{i}{2} \frac{e_a}{m_a} \frac{\omega_{pa}^2}{\omega_1\,(\omega_1 + \omega_2)} \frac{\mathbf{k}_1 \cdot (\mathbf{k}_1 + \mathbf{k}_2)}{k_1\, k_2\, |\mathbf{k}_1 + \mathbf{k}_2|} \\ &\quad \times \left(\frac{2}{v_{Ta}^2} - i\pi \int d\mathbf{v}\, \mathbf{k}_2 \cdot \frac{\partial F_a}{\partial \mathbf{v}}\, \delta(\omega_2 - \mathbf{k}_2 \cdot \mathbf{v}) \right). \end{aligned} \tag{1.133}$$

Finally, if $\omega_1 + \omega_2$ represents the low frequency while ω_1 and ω_2 satisfy the fast-wave condition (which becomes possible if ω_1 and ω_2 have opposite signs), then we have

$$\begin{aligned}
\chi_a^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) &= \frac{i}{2}\frac{e_a}{m_a}\frac{|\mathbf{k}_1+\mathbf{k}_2|}{\omega_1\,\omega_2}\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1\,k_2}\,\chi_a(\mathbf{k}_1+\mathbf{k}_2,\omega_1+\omega_2) \\
&= \frac{i}{2}\frac{e_a}{m_a}\frac{\omega_{pa}^2}{\omega_1\,\omega_2}\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1\,k_2\,|\mathbf{k}_1+\mathbf{k}_2|}\left(\frac{2}{v_{Ta}^2}\right. \\
&\quad \left. - i\pi\int d\mathbf{v}\,(\mathbf{k}_1+\mathbf{k}_2)\cdot\frac{\partial F_a}{\partial\mathbf{v}}\,\delta[\omega_1+\omega_2-(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{v}]\right).
\end{aligned} \tag{1.134}$$

If we ignore the resonant contribution to the second-order nonlinear susceptibility, then it becomes evident that the leading order expression satisfies

$$\{\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)\}^2 \approx -|\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)|^2, \tag{1.135}$$

which we have invoked in (1.101).

1.7.4 Third-Order Nonlinear Susceptibility

To simplify the discussion, herewith we only consider the third-order susceptibility of the form, $\bar{\chi}_a^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',-\omega'|\mathbf{k},\omega)$:

$$\begin{aligned}
\bar{\chi}_a^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',-\omega'|\mathbf{k},\omega) &= -\frac{1}{2}\frac{e_a^2}{m_a^2}\frac{\omega_{pa}^2}{k^2\,k'^2}\int d\mathbf{v}\,\frac{1}{\omega-\mathbf{k}\cdot\mathbf{v}+i0} \\
&\quad\times\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{v}}\left\{\frac{1}{\omega-\omega'-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}+i0}\right. \\
&\quad\times\left[\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{k}'\cdot\partial F_a/\partial\mathbf{v}}{\omega'-\mathbf{k}'\cdot\mathbf{v}+i0}\right)\right. \\
&\quad\left.\left.-\,\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{k}\cdot\partial F_a/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}+i0}\right)\right]\right\},
\end{aligned} \tag{1.136}$$

or equivalently (after partial integrations)

$$\begin{aligned}
\bar{\chi}_a^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',-\omega'|\mathbf{k},\omega) &= \frac{1}{2}\frac{e_a^2}{m_a^2}\frac{\omega_{pa}^2\,(\mathbf{k}\cdot\mathbf{k}')}{k^2\,k'^2} \\
&\quad\times\int d\mathbf{v}\,\frac{F_a}{(\omega-\mathbf{k}\cdot\mathbf{v}+i0)^3(\omega'-\mathbf{k}'\cdot\mathbf{v}+i0)} \\
&\quad\times\left[\frac{2k^2}{\omega-\mathbf{k}\cdot\mathbf{v}+i0}\left(\frac{4\,(\mathbf{k}\cdot\mathbf{k}')}{\omega-\mathbf{k}\cdot\mathbf{v}+i0}\right.\right. \\
&\quad\left.+\,\frac{k'^2}{\omega'-\mathbf{k}'\cdot\mathbf{v}+i0}\right)+\frac{1}{\omega-\omega'-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}+i0}
\end{aligned}$$

$$\times\left(\frac{2\,(\mathbf{k}\times\mathbf{k}')^2+3\,k^2\,[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')]}{\omega-\mathbf{k}\cdot\mathbf{v}+i0}\right.$$
$$\left.+\frac{k'^2\,[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]}{\omega'-\mathbf{k}'\cdot\mathbf{v}+i0}+\frac{|\mathbf{k}-\mathbf{k}'|^2(\mathbf{k}\cdot\mathbf{k}')}{\omega-\omega'-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}+i0}\right)\Bigg]. \tag{1.137}$$

It is straightforward to obtain the following limiting expressions:

$$\bar{\chi}_a^{(3)}(\mathbf{k}',0|-\mathbf{k}',0|\mathbf{k},0)=-\frac{e_a^2}{T_a^2}\frac{\omega_{pa}^2}{k^2\,k'^2}\frac{\mathbf{k}\cdot\mathbf{k}'}{k^2}\frac{1}{v_{Ta}^2},$$
$$\bar{\chi}_a^{(3)}(0,\omega'|0,-\omega'|0,\omega)=0. \tag{1.138}$$

The first expression is obtained from (1.136), while the second result follows from (1.137).

When all three frequencies, ω, ω', and $\omega-\omega'$, satisfy the fast-wave conditions $\omega\gg kv_{Ta}$, $\omega'\gg k'v_{Ta}$, $\omega-\omega'\gg|\mathbf{k}-\mathbf{k}'|v_{Ta}$, then we have

$$\bar{\chi}_a^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',-\omega'|\mathbf{k},\omega)=\frac{1}{2}\frac{e_a^2}{m_a^2}\frac{\omega_{pa}^2}{\omega^3\,\omega'}\frac{\mathbf{k}\cdot\mathbf{k}'}{k^2\,k'^2}\left(\frac{2\,k^2\,k'^2}{\omega\,\omega'}+\frac{k'^2\,[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]}{\omega'\,(\omega-\omega')}\right.$$
$$+\frac{8\,k^2\,(\mathbf{k}\cdot\mathbf{k}')}{\omega^2}+\frac{2\,(\mathbf{k}\times\mathbf{k}')^2+3k^2\,[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')]}{\omega\,(\omega-\omega')}$$
$$\left.+\frac{(\mathbf{k}-\mathbf{k}')^2(\mathbf{k}\cdot\mathbf{k}')}{(\omega-\omega')^2}\right). \tag{1.139}$$

If ω' represents an arbitrary wave frequency, but the fast-wave condition is applicable for other two frequencies, $\omega\gg kv_{Ta}$ and $\omega-\omega'\gg|\mathbf{k}-\mathbf{k}'|\,v_{Ta}$, then we obtain

$$\bar{\chi}_a^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',-\omega'|\mathbf{k},\omega)$$
$$=-\frac{1}{2}\frac{e_a^2}{m_a^2}\frac{\omega_{pa}^2}{\omega^3}\frac{\mathbf{k}\cdot\mathbf{k}'}{k^2\,k'^2}\left(\frac{2\,k^2}{\omega}+\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')}{\omega-\omega'}\right)\int d\mathbf{v}\,\frac{\mathbf{k}'\cdot\partial F_a/\partial\mathbf{v}}{\omega'-\mathbf{k}'\cdot\mathbf{v}+i0}$$
$$=-\frac{1}{2\omega^3}\frac{e_a^2}{m_a^2}\frac{\mathbf{k}\cdot\mathbf{k}'}{k^2}\left(\frac{2\,k^2}{\omega}+\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')}{\omega-\omega'}\right)\chi_a(\mathbf{k}',\omega'). \tag{1.140}$$

Note that this result can be alternatively expressed as

$$\bar{\chi}_a^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',-\omega'|\mathbf{k},\omega)$$
$$=\frac{i}{\omega^3}\frac{e_a}{m_a}\frac{(\mathbf{k}\cdot\mathbf{k}')\,|\mathbf{k}-\mathbf{k}'|}{k\,k'\,[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]}\left\{2k^2(\omega-\omega')+[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]\,\omega\right\}$$
$$\times\,\chi_a^{(2)}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega'). \tag{1.141}$$

Next, let us consider the case when $\omega - \omega'$ represents an arbitrary frequency, while the other two frequencies satisfy the fast-wave condition. In this case, we have

$$\begin{aligned}
&\bar{\chi}_a^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',-\omega'|\mathbf{k},\omega) \\
&= -\frac{1}{2}\frac{e_a^2}{m_a^2}\frac{\omega_{pa}^2}{\omega^3\,\omega'}\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2\,k'^2}\int d\mathbf{v}\,\frac{(\mathbf{k}-\mathbf{k}')\cdot\partial F_a/\partial\mathbf{v}}{\omega-\omega'-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}+i0} \\
&= -\frac{1}{2}\frac{e_a^2}{m_a^2}\frac{|\mathbf{k}-\mathbf{k}'|^2}{\omega^3\,\omega'}\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2\,k'^2}\,\chi_a(\mathbf{k}-\mathbf{k}',\omega-\omega'),
\end{aligned} \tag{1.142}$$

which can be alternatively expressed as

$$\bar{\chi}_a^{(3)}(\mathbf{k}',\omega'|-\mathbf{k}',-\omega'|\mathbf{k},\omega) = \frac{i e_a}{m_a}\frac{\mathbf{k}\cdot\mathbf{k}'}{k\,k'}\frac{|\mathbf{k}-\mathbf{k}'|}{\omega^2}\,\chi_a^{(2)}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega'). \tag{1.143}$$

1.8 Linear Waves and Weak Instabilities

In this section we review the textbook theory of small amplitude electrostatic (linear) waves and weak instabilities operative in unmagnetized plasmas, which includes the electron beam-plasma (or bump-on-tail) instability. Linear dispersion relation is determined from the solvability condition of (1.93),

$$\mathrm{Re}\,\epsilon(\mathbf{k},\omega) = 1 + \mathrm{Re}\sum_a \frac{\omega_{pa}^2}{k^2}\int d\mathbf{v}\,\frac{\mathbf{k}\cdot\partial F_a/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}+i0} = 0. \tag{1.144}$$

Let us assume that isotropic thermal Maxwellian forms represent the bulk electron and ion distributions but a tenuous electron beam may also exist. For Langmuir waves satisfying the fast wave condition for both electrons and ions, $|\omega_{\mathbf{k}}^L|/k\,v_{Te} \gg 1$ and $|\omega_{\mathbf{k}}^L|/k\,v_{Ti} \gg 1$, where v_{Ta} is the thermal speed defined via (1.116), upon making use of (1.115) $\mathrm{Re}\,\epsilon(\mathbf{k},\pm\omega_{\mathbf{k}}^L)$ is approximately given by

$$\mathrm{Re}\,\epsilon(\mathbf{k},\pm\omega_{\mathbf{k}}^L) = 1 - \frac{\omega_{pe}^2}{\omega_{\mathbf{k}}^{L2}}\left(1 + \frac{3k^2T_e}{m_e\omega_{\mathbf{k}}^{L2}}\right), \tag{1.145}$$

where $\omega = \omega_{\mathbf{k}}^L$ denotes the Langmuir wave dispersion relation. By setting $\mathrm{Re}\,\epsilon(\mathbf{k},\pm\omega_{\mathbf{k}}^L)$ equal to zero one readily obtains the following:

$$\omega_{\mathbf{k}}^L = \omega_{pe}\left(1 + \frac{3}{2}k^2\lambda_{De}^2\right), \qquad \omega_{-\mathbf{k}}^L = -\omega_{\mathbf{k}}^L, \tag{1.146}$$

where T_e is the electron bulk temperature and

$$\lambda_{De}^2 = \frac{T_e}{4\pi n e^2} = \frac{v_{Te}^2}{2\omega_{pe}^2}, \tag{1.147}$$

is the square of the electron Debye length. In (1.146) the symmetry property (1.9) is invoked in order to have the relation $\omega^L_{-\mathbf{k}} = -\omega^L_{\mathbf{k}}$.

For ion-sound mode characterized by $|\omega^S_{\mathbf{k}}|/k\, v_{Te} \ll 1$ and $|\omega^S_{\mathbf{k}}|/k\, v_{Ti} \geq 1$, upon combining (1.115) and (1.119) we have

$$\mathrm{Re}\,\epsilon(\mathbf{k}, \pm\omega^S_{\mathbf{k}}) = 1 + \frac{1}{k^2\lambda^2_{De}} - \frac{\omega^2_{pi}}{\omega^{S2}_{\mathbf{k}}}\left(1 + \frac{3k^2 T_i}{m_i \omega^{S2}_{\mathbf{k}}}\right), \tag{1.148}$$

where T_i is the ion (proton) temperature and $\omega^S_{\mathbf{k}}$ denotes the ion-sound wave dispersion relation. Setting $\mathrm{Re}\,\epsilon(\mathbf{k}, \pm\omega^S_{\mathbf{k}})$ equal to zero leads to

$$\omega^S_{\mathbf{k}} = \frac{k c_S\,(1 + 3T_i/T_e)^{1/2}}{(1 + k^2\lambda^2_{De})^{1/2}}, \quad \omega^S_{-\mathbf{k}} = -\omega^S_{\mathbf{k}}, \tag{1.149}$$

where

$$c_S = \sqrt{\frac{T_e}{m_i}} \tag{1.150}$$

is the ion sound (or ion acoustic) speed.

It is useful to evaluate the derivatives of real parts of the dielectric constants,

$$\epsilon'(\mathbf{k}, \pm\omega^L_{\mathbf{k}}) \equiv \frac{\partial\,\mathrm{Re}\,\epsilon(\mathbf{k}, \pm\omega^L_{\mathbf{k}})}{\partial(\pm\omega^L_{\mathbf{k}})} = \frac{2}{(\pm\omega^L_{\mathbf{k}})},$$
$$\epsilon'(\mathbf{k}, \pm\omega^S_{\mathbf{k}}) \equiv \frac{\partial\,\mathrm{Re}\,\epsilon(\mathbf{k}, \pm\omega^S_{\mathbf{k}})}{\partial(\pm\omega^S_{\mathbf{k}})} = \frac{2}{(\pm\omega^L_{\mathbf{k}})}\frac{1}{\mu_{\mathbf{k}}},$$
$$\mu_{\mathbf{k}} = k^3\lambda^3_{De}\left(\frac{m_e}{m_i}\right)^{1/2}\left(1 + \frac{3T_i}{T_e}\right)^{1/2}. \tag{1.151}$$

Figure 1.1 displays the three basic plasma eigenmodes (normal modes) of unmagnetized plasma. These are the transverse modes $\omega^T_{\mathbf{k}} = \sqrt{\omega^2_{pe} + c^2k^2}$, which we did not discuss under the present electrostatic approximation (we will deal with the transverse mode in later chapters), and Langmuir and ion acoustic (or ion sound) modes, which we have already covered.

The instability (or damping) of plasma eigenmodes can be discussed on the basis of the imaginary part of dielectric response function

$$\mathrm{Im}\,\epsilon(\mathbf{k},\omega) = -\sum_a \frac{\pi\omega^2_{pa}}{k^2}\int d\mathbf{v}\,\mathbf{k}\cdot\frac{\partial F_a}{\partial\mathbf{v}}\,\delta(\omega - \mathbf{k}\cdot\mathbf{v}). \tag{1.152}$$

Consider the linear term in the formal wave kinetic equation (1.102),

$$\frac{\partial I^{\sigma\alpha}_{\mathbf{k}}}{\partial t} = -2\,\frac{\mathrm{Im}\epsilon(\mathbf{k},\sigma\omega^\alpha_{\mathbf{k}})}{\partial\mathrm{Re}\epsilon(\mathbf{k},\sigma\omega^\alpha_{\mathbf{k}})/\partial\sigma\omega^\alpha_{\mathbf{k}}}\,I^{\sigma\alpha}_{\mathbf{k}} \equiv 2\gamma^\alpha_{\mathbf{k}}\,I^{\sigma\alpha}_{\mathbf{k}}. \tag{1.153}$$

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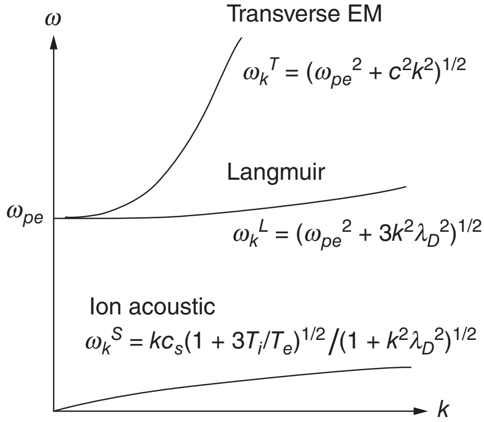

Figure 1.1 Three basic eigenmodes of unmagnetized plasma; transverse mode $\omega_{\mathbf{k}}^{T}$, which we could not discuss under the present electrostatic treatment, and Langmuir and ion acoustic (or ion sound) modes, $\omega_{\mathbf{k}}^{L}$ and $\omega_{\mathbf{k}}^{S}$, respectively.

In this equation we have introduced the linear "growth rate" or equivalently "Landau damping rate," – see Appendix A;

$$\gamma_{\mathbf{k}}^{\alpha} = -\frac{\mathrm{Im}\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})}{\partial\mathrm{Re}\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})/\partial\sigma\omega_{\mathbf{k}}^{\alpha}}. \tag{1.154}$$

Depending on whether the sign of $\gamma_{\mathbf{k}}^{\alpha}$ is positive or negative, the quantity $\gamma_{\mathbf{k}}^{\alpha}$ denotes either the growth rate for instabilities or the damping rate associated with waves.

The gentle electron beam-plasma or bump-on-tail instability will play a crucial role as a test bed for weak turbulence theory to be developed in the present monograph. The weak (or gentle) electron beam-plasma instability has been studied since the beginning of modern plasma physics, and there is a substantive body of literature on the topic, so that it is practically impossible to cite them all, but some selective references are those by Vedenov and Velikhov (1962); Drummond and Pines (1962); Frieman and Rutherford (1964); Bernstein and Engelmann (1966); Dawson and Shanny (1968); Morse and Nielson (1969); Vahala and Montgomery (1970); Roberson et al. (1971); Joyce et al. (1971) among earlier works, and Appert et al. (1976); Ivanov et al. (1976); Grognard (1982); Dum (1990); Muschietti and Dum (1991); Tsunoda et al. (1987); Nishikawa and Cairns (1991); Dum and Nishikawa (1994), to cite some representative papers up to the mid-1990s. Since the decade of 1990s, researches on beam-plasma interaction have moved on to more application-oriented topics, so that problems on pure or fundamental aspects of the beam-plasma instability, especially in relations to linear or quasilinear aspects, seem to have been exhausted. However, as we shall see later in this book, certain fundamental aspects associated with the electron beam-plasma interaction in the

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weak turbulence regime are still being investigated. These relate to the electron acceleration by Langmuir turbulence and the radiation generation.

Let us focus on the Langmuir wave, $\alpha = L$, in (1.154), and restrict ourselves to the forward propagation, $\sigma = +1$. Here, the directionality of forward versus backward is with respect to the beam propagation direction. Making use of (1.151) and (1.152), the growth rate (1.154) for forward-propagating Langmuir wave is given by

$$\gamma_{\mathbf{k}}^{L} = \frac{\pi\omega_{\mathbf{k}}^{L}}{2}\frac{\omega_{pe}^{2}}{k^{2}}\int d\mathbf{v}\,\mathbf{k}\cdot\frac{\partial F_{e}}{\partial\mathbf{v}}\,\delta(\omega_{\mathbf{k}}^{L}-\mathbf{k}\cdot\mathbf{v}), \tag{1.155}$$

where we have omitted the plus sign and have retained the electron terms only in $\mathrm{Im}\,\epsilon(\mathbf{k},\omega)$. For isotropic Maxwellian, $F_e = (\pi v_{Te})^{-3}\exp(-v^2/v_{Te}^2)$, it is obvious that $\gamma_{\mathbf{k}}^{L} < 0$. That is, Langmuir waves for thermal plasma is subject to Landau damping. If, on the other hand, we consider that the electron distribution is composed of a core thermal population plus an energetic but tenuous beam, take for instance, the drifting Gaussian beam distribution,

$$F_e(\mathbf{v}) = \left(1-\frac{n_b}{n_0}\right)\frac{e^{-v^2/v_{Te}^2}}{\pi^{3/2}\,v_{Te}^3} + \frac{n_b}{n_0}\frac{e^{-(\mathbf{v}-\mathbf{V}_b)^2/v_{Tb}^2}}{\pi^{3/2}\,v_{Tb}^3}, \tag{1.156}$$

where $v_{Tb} = \sqrt{2T_b/m_e}$ is the thermal spread (or beam temperature), and where we assume

$$n_b \ll n_0, \tag{1.157}$$

then by assuming that the background population largely determines the real frequency, we may determine the damping (or growth) rate:

$$\begin{aligned}\gamma_{\mathbf{k}}^{L} = -\pi^{1/2}\,\omega_{\mathbf{k}}^{L}\,\frac{\omega_{pe}^{2}}{k^{2}}\Bigg[&\frac{v_\parallel}{v_{Te}^3}\left(1-\frac{n_b}{n_0}\right)e^{-v_\parallel^2/v_{Te}^2}\\ &+\frac{(v_\parallel - V_b)}{v_{Tb}^3}\frac{n_b}{n_0}\,e^{-(v_\parallel-V_b)^2/v_{Tb}^2}\Bigg]_{v_\parallel=\omega_{\mathbf{k}}^{L}/k}.\end{aligned} \tag{1.158}$$

In (1.158) we have decomposed the velocity vector into components perpendicular and parallel to the beam vector, and without loss of generality, we have taken the **k** vector to be directed along the beam.

In the absence of electron beam (1.158) reduces to

$$\begin{aligned}\gamma_{\mathbf{k}}^{L} &= -\pi^{1/2}\,\frac{\omega_{pe}^{2}(\omega_{\mathbf{k}}^{L})^2}{k^3 v_{Te}^3}\exp\left(-\frac{(\omega_{\mathbf{k}}^{L})^2}{k^2 v_{Te}^2}\right)\\ &= -\frac{\pi^{1/2}\omega_{pe}^4}{k^3 v_{Te}^3}\left(1+\frac{3k^2 v_{Te}^2}{2\omega_{pe}^2}\right)\exp\left(-\frac{3}{2}-\frac{v_{Te}^2}{k^2\omega_{pe}^2}\right).\end{aligned} \tag{1.159}$$

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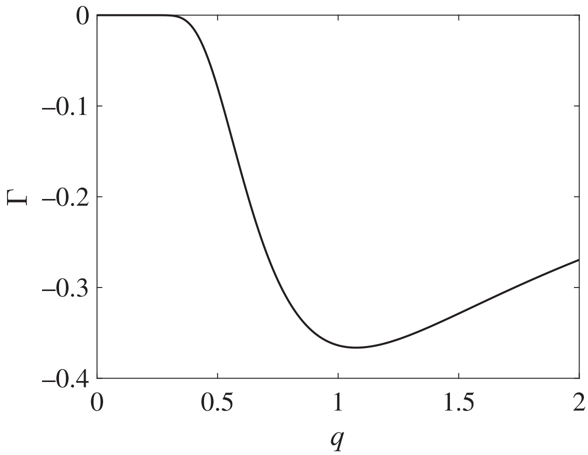

Figure 1.2 Normalized Landau damping rate Γ versus normalized wave number q.

This is the Landau damping rate for thermal plasma. Plotted in Figure 1.2 is the normalized Landau damping rate $\Gamma = \gamma_{\mathbf{k}}^{L}/\omega_{pe}$ versus normalized wave number $q = kv_{Te}/\omega_{pe}$. Notice that the damping rate exponentially decreases in magnitude for $k \to 0$.

Suppose that the background electrons are cold, $v_{Te} \ll v_{Tb}$. In such a case, the damping rate associated with the background component can be ignored, and the beam electrons determine the net growth/damping rate only,

$$\gamma_{\mathbf{k}}^{L} \approx \pi^{1/2} \frac{n_b}{n_0} \frac{\omega_{pe}^3}{k^3 v_{Tb}^3} (kV_b - \omega_{pe}) \exp\left(-\frac{(kV_b - \omega_{pe})^2}{k^2 v_{Tb}^2}\right). \tag{1.160}$$

This shows that $\gamma_{\mathbf{k}}^{L} > 0$ over the unstable range of k corresponding to

$$k > \frac{\omega_{pe}}{V_b}. \tag{1.161}$$

The instability of Langmuir wave driven by a gentle (or weak) electron beam is called the bump-on-tail instability, and (1.160) represents the approximate growth rate for the said instability.

In the low-frequency regime similar processes involving ion-acoustic wave with its associated damping phenomena as well as instabilities may be operative. It is a straightforward exercise to obtain the Landau damping rate for ion acoustic wave,

$$\gamma_{\mathbf{k}}^{S} = -\,\omega_{pi} \frac{\pi^{1/2}}{2^{3/2}} \frac{k\lambda_D (1 + 3T_i/T_e)}{(1 + k^2\lambda_D^2)^2} \left[\sqrt{\frac{m_e}{m_i}} \exp\left(-\frac{m_e}{2m_i} \frac{1 + 3T_i/T_e}{1 + k^2\lambda_D^2}\right) + \left(\frac{T_e}{T_i}\right)^{3/2} \exp\left(-\frac{T_e}{2T_i} \frac{1 + 3T_i/T_e}{1 + k^2\lambda_D^2}\right)\right]. \tag{1.162}$$

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If we approximate that $m_e/m_i \approx 0$, and introduce the dimensionless variables,

$$\Gamma = \frac{2^{3/2}}{\pi^{1/2}} \frac{\gamma_k}{\omega_{pi}}, \qquad \kappa = k\lambda_D, \qquad \tau = \frac{T_i}{T_e}, \tag{1.163}$$

then we have

$$\Gamma = -\frac{\kappa\,(1+3\tau)}{(1+\kappa^2)^2} \frac{1}{\tau^{3/2}} \exp\left(-\frac{1}{2\tau} \frac{1+3\tau}{1+\kappa^2}\right). \tag{1.164}$$

Damping generally increases for increasing κ, so let us focus on small κ behavior. If we consider Γ/κ for $\kappa \ll 1$, we have

$$\frac{\Gamma}{\kappa} \approx -\frac{1+3\tau}{\tau^{3/2}} \exp\left(-\frac{1+3\tau}{2\tau}\right). \tag{1.165}$$

Below in Figure 1.3 is the plot of $-\Gamma/\kappa$ versus $\tau = T_i/T_e$ in horizontal logarithmic scale. As one can see, if $T_i \ll T_e$ then the damping rate becomes exponentially small, and ion sound wave may propagate in a plasma undamped. For $T_i \ll T_e$, the ion-sound speed $c_S^2 = T_e/m_i$ is much higher than ion thermal speed $v_{Ti}^2 = 2T_i/m_i$, but lower than electron thermal speed $v_{Te}^2 = 2T_e/m_e$, so that

$$\sqrt{\frac{2T_i}{m_i}} < \frac{\omega_{\mathbf{k}}^S}{k} < \sqrt{\frac{2T_e}{m_e}}. \tag{1.166}$$

In this case the ion-sound mode does not suffer Landau damping by ions as the wave phase speed is sufficiently higher than ion thermal speed. On the other hand,

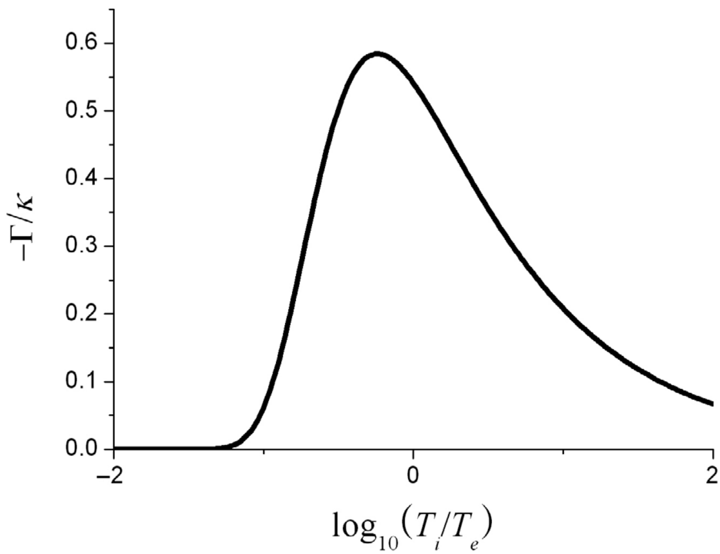

Figure 1.3 Normalized ion acoustic wave damping rate $-\Gamma/\kappa$ versus T_i/T_e.

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the wave also does not suffer from electron Landau damping since the electron distribution is practically constant over the velocity range corresponding to the ion acoustic speed. This shows that the excitation and persistence of ion sound wave requires hot electrons. Note that the ion sound damping rate is maximum for $T_i \sim T_e$.

Ion acoustic mode can become unstable when there is a net mild drift between the ions and electrons. Let us consider the stationary ions and drifting Gaussian electrons (without loss of generality, we assume the electron drift direction to be along z axis), where the electron drift speed is significantly lower than electron thermal speed,

$$F_i(\mathbf{v}) = \frac{1}{\pi^{3/2}\, v_{Ti}^3} \exp\left(-\frac{v^2}{v_{Ti}^2}\right),$$
$$F_e(\mathbf{v}) = \frac{1}{\pi^{3/2}\, v_{Te}^3} \exp\left(-\frac{(\mathbf{v} - V_e\hat{\mathbf{z}})^2}{v_{Te}^2}\right). \tag{1.167}$$

Then, assuming $\mathbf{k} = k\hat{\mathbf{z}}$, and for $T_e/T_i \gg 1$ and $m_e/m_i \ll 1$, we obtain the growth rate for ion acoustic instability,

$$\gamma_k^S = \frac{\pi^{1/2}}{2^{3/2}} \left(\frac{m_e}{m_i}\right)^{1/2} \left(\frac{V_e}{c_S} - \frac{1}{\sqrt{1 + k^2\lambda_{De}^2}}\right) \frac{k^2 c_S^2}{(1 + k^2\lambda_{De}^2)^2}. \tag{1.168}$$

If electron drift speed V_e is higher than the ion acoustic speed c_S, the ion sound mode becomes unstable.

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