ON A THEOREM OF OSIMA AND NAGAO

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1. Introduction. If we define the *weight* b of a Young diagram containing n nodes to be the number of removable p-hooks where $n = a + bp$, then three fundamental theorems stand out in the modular representation theory of the symmetric group S_n.

1.1 *Two irreducible representations of S_n belong to the same block if and only if they have the same p-core.*

This has been proved in various ways (**1**; **5**; **7**).

1.2 *The number l_b of ordinary irreducible representations in a block of weight b is independent of the p-core and is given by*

$$l_b = \sum_{b_1,\dots b_p} p_{b_1} p_{b_2} \dots p_{b_p} \qquad \left(\sum_1^p b_i = b,\ 0 \leqslant b_i \leqslant b\right).$$

The enumeration here is based on the 1-1 correspondence holding (**5**;**8**) between the representations $[\alpha]$ with a given p-core and the associated star diagrams $[\alpha]_p{}^*$.

1.3 *The number l'_b of modular irreducible representations (indecomposables of the regular representation of S_n)*
  (i) *is independent of the p-core, and*
  (ii) *is given by*

$$l'_b = \sum_{b_1\dots b_{p-1}} p_{b_1} p_{b_2} \dots p_{b_{p-1}} \qquad \left(\sum_1^{p-1} b_i = b,\ 0 \leqslant b_i \leqslant b\right).$$

Theorem 1.3 (ii) was recently proven by Osima (**6**) assuming 1.3 (i) (**8**); Nagao (**4**) obtained 1.3 (i) and (ii) directly. We give here another version of Osima's proof which yields, in addition, generating functions for the number of p-cores containing a nodes and the number of blocks (**1**) to which the representations of S_n belong.

2. Proof of 1.3(ii). The partition generating function

$$(2.1)\qquad \mathscr{P}(x) = 1 + p_1x + p_2x^2 + p_3x^3 + \dots$$
$$= \{(1-x)(1-x^2)(1-x^3)\dots\}^{-1}$$

is well known (**2**, p. 272). It follows from 1.2 that

$$(2.2)\qquad \mathscr{L}(x) = 1 + l_1x + l_2x^2 + \dots = [\mathscr{P}(x)]^p.$$

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If we write

$$\mathscr{C}(x) = 1 + c_1 x + c_2 x^2 + \ldots, \tag{2.3}$$

when c_a is the number of p-cores containing a nodes, then we may enumerate the ordinary representations of S_n lying in all the blocks in the following manner:

$$\mathscr{C}(x)\,\mathscr{L}(x^p) = \mathscr{C}(x)\,[\mathscr{P}(x^p)]^p = \mathscr{P}(x), \tag{2.4}$$

using 2.2 and the fact that $n = a + bp$. On the other hand, assuming 1.3 (i), we may write

$$\mathscr{L}'(x) = 1 + l'_1 x + l'_2 x^2 + \ldots. \tag{2.5}$$

Since the total number of modular irreducible representations is equal to the number of p-regular classes of S_n, we have

$$\mathscr{C}(x)\,\mathscr{L}'(x^p) = \mathscr{P}(x)/\mathscr{P}(x^p). \tag{2.6}$$

From 2.4 and 2.6 it follows immediately that

$$\mathscr{L}'(x^p) = [\mathscr{P}(x^p)]^{p-1}, \tag{2.7}$$

or

$$\mathscr{L}'(x) = [\mathscr{P}(x)]^{p-1}, \tag{2.8}$$

which is precisely the relation 1.3 (ii).

3. The number of p-regular classes. We can say a little more, however. Setting

$$\mathscr{M}(x) = 1 + m_1 x + m_2 x^2 + \ldots, \tag{3.1}$$

where m_n is the number of distinct blocks associated with S_n, we have

$$m_i = c_i + c_{i-p} + c_{i-2p} + \ldots, \tag{3.2}$$

so that

$$\mathscr{M}(x) = \mathscr{C}(x)/(1 - x^p) = \mathscr{P}(x)/(1 - x^p)[\mathscr{P}(x^p)]^p, \tag{3.3}$$

from 2.4.

In this connection it is worth remarking that the generating function on the right hand side of 2.6, namely,

$$\frac{\mathscr{P}(x)}{\mathscr{P}(x^p)} = \frac{(1 - x^p)(1 - x^{2p})\ldots}{(1 - x)(1 - x^2)\ldots}, \tag{3.4}$$

can be interpreted in two ways. We may cancel each factor of $\mathscr{P}(x^p)$ with an equal factor of $\mathscr{P}(x)$ and conclude that $\mathscr{P}(x)/\mathscr{P}(x^p)$ generates the number of partitions of n into summands not divisible by p, which is the number of p-regular classes. Or we may divide the kth factor $(1 - x^k)^{-1}$ of $\mathscr{P}(x)$ into the kth factor $(1 - x^{kp})^{-1}$ of $\mathscr{P}(x^p)$ and generate the number of partitions into summands no one of which appears as many as p times. Hence we have:

3.5 *The number of p-regular classes of S_n is equal to the number of partitions of n in which no summand appears as many as p times.*

This result is of interest in the study of the indecomposables of the regular representation of S_n; such partitions may indeed characterize them.

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