2. RESULTS TO DATE IN THE NUMERICAL DEVELOPMENT OF HARMONIC SERIES FOR THE CO-ORDINATES OF THE MOON

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The problem is to find for the Moon expressions of the form

$$\begin{aligned} x &= \Sigma x_\xi \cos \xi \qquad & \xi &= \{p_\xi g + q_\xi c + r_\xi m + s_\xi (1 - m)\} nt + B_\xi \\ y &= \Sigma y_\xi \sin \xi & &= A_\xi nt + B_\xi \\ z &= \Sigma z_\xi \sin \xi \end{aligned} \tag{1}$$

that satisfy identically the differential equations

$$xDy - yDx + (x^2 + y^2) = D^{-1}\left(x\frac{\partial\Omega}{\partial y} - y\frac{\partial\Omega}{\partial x}\right) + C_1$$

$$\tfrac{1}{2}D^2(r^2) - \tfrac{1}{2}\{(Dx)^2 + (Dy)^2 + (Dz)^2\} + \tfrac{1}{2}(x^2 + y^2) =$$

$$\Omega + \left[r\frac{\partial\Omega}{\partial r}\right] - D^{-1}\left(\frac{1}{n}\frac{\partial\Omega}{\partial t}\right) + 2D^{-1}\left(x\frac{\partial\Omega}{\partial y} - y\frac{\partial\Omega}{\partial x}\right) + C_2 \tag{2}$$

$$xD^2z + z(-D^2x + 2Dy + x) = x\frac{\partial\Omega}{\partial z} - z\frac{\partial\Omega}{\partial x}$$

$$-\tfrac{1}{2}D^2(r^2) + 2(\text{LHS 2nd eq.} - \text{LHS 1st eq.}) + \left(\frac{a}{aa_0}\right)^3\frac{1}{r} = \left[r\frac{\partial\Omega}{\partial r}\right]$$

These are the Euler-Hill-Brown equations.

The method of solution is to substitute an approximate solution of form (1) into equations (2) and to obtain from the residuals corrections δx_ξ, δy_ξ, δz_ξ; δc, δg. The corrections are applied to the approximate solution and the process repeated.

At the last meeting of the IAU we reported that we had substituted, with the IBM 650, Brown's solution into the differential equations and that we were starting the solution of the variation equations. The solution of the variation equations has proved more troublesome than we hoped. We shall discuss this problem briefly.

By neglecting only the squares of the corrections δx_ξ, δy_ξ, δz_ξ, δc, δg it is possible to obtain a set of variation equations of considerable accuracy. We prepared a program for the IBM 650 that gives these coefficients for any required argument. This was done for the results of the IBM 650 substitution.

It was decided to solve this set of 3500 equations in as many unknowns by successive approximations. The approximate values for the unknowns are determined from limited sets of approximate variation equations.

The following approximate equations are obtained by neglecting powers of $(x - 1)$, y, z above the first

$$\delta x_\xi = \frac{2A_\xi x_\xi \delta A_\xi - F_\xi}{1 - A^2_\xi}, \text{ where } \delta A_\xi = p_\xi \delta g + r_\xi \delta c$$

$$\delta y_\xi = -\frac{y_\xi \delta A_\xi + E_\xi + 2\delta x_\xi}{A_\xi} \qquad \delta z_\xi = \frac{2A_\xi z_\xi \delta A_\xi - G_\xi}{1 - A^2_\xi} \tag{3}$$

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These equations are obviously indeterminate

$$\text{in } y \text{ for } A_\xi \approx 0 \text{ and in } x, y \text{ and } z \text{ for } A_\xi \approx 1.$$

However, since many of the divisors are not very small we tried the iterative process with these equations. The results indicated that a more sophisticated treatment of the small divisors was necessary.

Figures 1 and 2 show the relations between terms with small divisors. In Figure 1 we have arranged a group of related arguments according to multiples of l and F. Since l and F have motions of 0·9915 and 1·004 respectively, any argument with a small motion will be associated with others as shown in the diagram. Alternate columns represent arguments for x, y terms and for z terms.

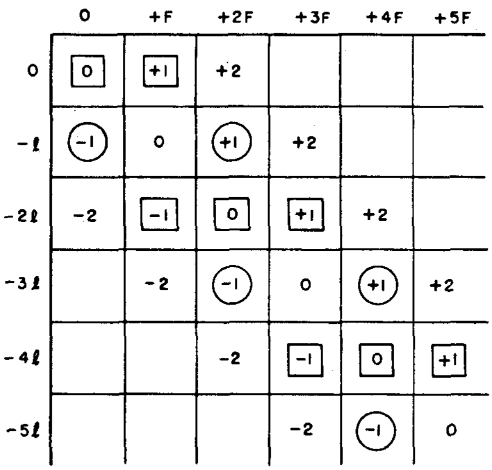

Fig. 1

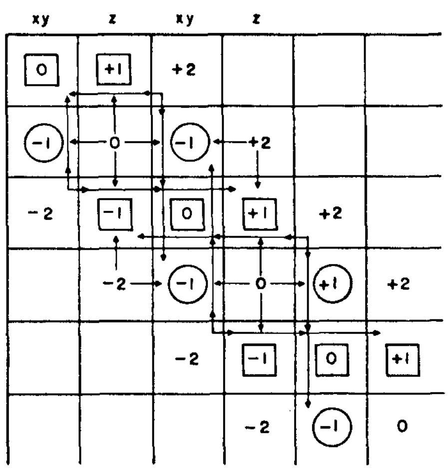

Fig. 2

Figure 2 shows the effect of nearby terms on terms with small divisors. We must also consider the arguments differing by $\pm 2D$ from those on the diagram; these can be thought of as occupying the layers above and below the diagram in a 3-dimensional lattice. Arguments differing by D and by l' are not troublesome because of the small parameters α and e' that accompany them.

To improve the equations we have used a 3-dimensional generalization of a device used by Brown in the one-dimensional case, namely the elimination of the neighboring arguments that do not have small divisors but that strongly influence those that do. The remaining core contains about 20 terms closely inter-dependent but with good separation from the rest of the matrix. We invert the matrix of the equations in the core. For the solution of the full set of variation equations we use for each approximation the values for the terms without small divisors from the equations similar to equations (3) and for the terms with small divisors the values from the inverted matrices.

In this manner we have been able to reduce the residuals in a systematic manner which looks reasonable. This is a necessary but not a sufficient condition. In order to evaluate the process it will be necessary to substitute the corrected solution into the differential equations at least once and to examine the behavior of the residuals.

As planned initially we have coded the substitution for a larger and faster machine, the

IBM 7090. The substitution involves the repeated use of two programs. The first run of the first program gives all second-order product series of the lunar co-ordinates and the left-hand sides of equations (1) - (3); subsequent runs give successively the product series of orders 3, 4, ... The other program computes the Ω terms from the lunar product series. We have used the programs to compute the LHS and the Ω_2 terms.

The new program gives:

(i) an independent check of the IBM 650 substitution;

(ii) a complete substitution in a few days;

(iii) greater over-all accuracy;

(iv) increased accuracy for arguments with small divisors.

The present coding aims at an accuracy of $1'' \times 10^{-5}$ for terms without small divisors, and for an omitted term to be as large as $0''\cdot001$ the period would be greater than 2000 years. If the solution to this precision appears to warrant higher precision the coding could be easily modified and the question would be a simple one of economics.

Among the more interesting parts of this work is the determination of the motions of the perigee and node. In view of the fact that the IBM 7090 substitution is almost ready we have decided not to prepare semi-definitive values of these quantities even though we feel that this could be done with a reasonable amount of work from the results at hand. Present indications are that the motion of the node given by Brown in the Memoirs (**1**) will be changed very little while the motion of the perigee may be increased by thirty seconds per century. Terms not included in the Memoirs and discussed later by Brown (**2**) account for about half the expected change.

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3. OUTLINE OF AN APPLICATION OF VON ZEIPEL'S METHOD TO THE LUNAR THEORY

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The main problem of the lunar theory has been solved by many investigators by various methods. Yet the problem remains of sufficient interest to encourage further efforts. An investigation in progress at the Yale Observatory has as its principal aims: (*a*) evaluating the effectiveness of the von Zeipel method for obtaining the solution; (*b*) examining the causes of the slow convergence according to powers of n'/n in Delaunay's theory and, if possible, finding a remedy for this defect.

Applications of von Zeipel's method have been presented in several recent papers (**1**, **2**); hence only a minimum of detailed explanation will be given here. The principle of the method is to eliminate the periodic terms in a single transformation or a sequence of transformations with the ultimate object of obtaining a final Hamiltonian that is a function of the variables corresponding to Delaunay's L, G, H.

In dealing with the lunar theory it may be advisable to start with the problem in which the inclination of the Moon's orbit and the eccentricity of the Sun's orbit are ignored, and in which the disturbing function is reduced to its principal part. The third limitation amounts to putting the ratio of a/a' equal to zero. The solution of the problem so simplified should yield those