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On varieties of soluble groups

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We show that, under certain conditions, a soluble variety of groups which does not contain the variety of all metabelian groups is a finite exponent by nilpotent by finite exponent variety.

All varieties considered are varieties of groups. For notation and basic results we refer to Hanna Neumann's book [7], with the following exceptions: we shall use doubly underlined Roman capitals, rather than German capitals, for varieties; we shall use $\underline{\underline{V}}(G)$ for the verbal subgroup of a group G corresponding to the variety $\underline{\underline{V}}$; and we shall not reserve G, H for relatively free groups nor F for an absolutely free group.

The following result will be proved:

THEOREM. *Suppose* $\underline{\underline{V}}$ *is a soluble variety which does not contain* $\underline{\underline{A}}^2$ *as a subvariety and which has the following property:*

i) all subvarieties of $\underline{\underline{V}} \wedge \underline{\underline{A}}\underline{\underline{N}}_2\underline{\underline{A}}$ *can be generated by the finite groups they contain.*

Then $\underline{\underline{V}} \leq \underline{\underline{B}}_n\underline{\underline{N}}_c\underline{\underline{B}}_n$ *for some natural numbers* n, c .

Similar results were obtained in [10] when $\underline{\underline{V}}$ is a nilpotent by abelian variety and in [2] when $\underline{\underline{V}}$ is a metanilpotent variety; the restriction *i)* being, in each case, unnecessary. We shall, in Proposition 3, give a different and somewhat shorter proof of these results. We note that the problem of the existence of soluble varieties not satisfying *i)*

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seems, at present, to be open.

After building up some necessary machinery in Section 1, we shall suppose $\underline{\underline{V}}$ to be a counterexample to the theorem. In Section 2 we shall show that we may then suppose that $\underline{\underline{V}}$ is a minimal counterexample and that it is contained in some variety of the type $\underline{\underline{A}}\left(\underline{\underline{B}}_{p^t} \wedge \underline{\underline{N}}_2\right)\underline{\underline{A}}$, where p is prime. Finally, in Section 3, we shall investigate the properties of $\underline{\underline{V}}$ and, in particular, the residual properties of a free group of $\underline{\underline{V}}$, to obtain the required contradiction.

For brevity in the following, we shall say that $\underline{\underline{V}}$ is "of the type (A)" if $\underline{\underline{V}}$ satisfies the conclusion of the theorem, that is if, for some natural numbers n, c, $\underline{\underline{V}} \leq \underline{\underline{B}}_n\underline{\underline{N}}_c\underline{\underline{B}}_n$.

1. Preliminary results

We shall need the classification of metabelian varieties of exponent zero due to Kovács and Newman and given as Theorems 6.1.1 and 6.1.2 of [1]. We quote these here for convenience. Call a variety torsion-free if its free groups are torsion-free.

PROPOSITION 1. *Let* $\underline{\underline{V}}$ *be a proper subvariety of* $\underline{\underline{A}}^2$. *Then there exists a unique torsion-free subvariety* $\underline{\underline{T}}$ *and a unique natural number* u *such that*

$$\underline{\underline{V}} = \underline{\underline{T}} \vee \underline{\underline{A}}_u\underline{\underline{A}} \vee \underline{\underline{P}}$$

where $\underline{\underline{P}}$ *has finite exponent.*

PROPOSITION 2. *The varieties* $\underline{\underline{N}}_c\underline{\underline{A}}_s \wedge \underline{\underline{A}}^2$ $(c, s \geq 1)$ *are torsion-free and join-irreducible. Every torsion-free subvariety of* $\underline{\underline{A}}^2$ *can be uniquely expressed as an irredundant join of some of these torsion-free join-irreducibles.*

We use a special case of Proposition 1 to deduce:

LEMMA 1. *Suppose* $\underline{\underline{V}}$ *is a variety such that* $\underline{\underline{A}}_p\underline{\underline{A}} \nleq \underline{\underline{V}}$ *for some prime* p. *Let* $G \in \underline{\underline{V}}$ *and let* H *be a normal, elementary abelian,* p*-subgroup of* G. *Then* $G/C_G(H)$ *has finite exponent bounded in terms of*

$\underline{\underline{V}}$ *and* p *alone.*

Proof. If $\underline{\underline{V}}$ has finite exponent the result is trivial - so we may suppose this is not the case.

Let $x \in G$ and $K = gp(H, x)$. Then $K \in \underline{\underline{A}}_p\underline{\underline{A}} \wedge \underline{\underline{V}}$. Since $\underline{\underline{A}}_p\underline{\underline{A}} \wedge \underline{\underline{V}} \neq \underline{\underline{A}}_p\underline{\underline{A}}$, we may use Proposition 1 to show that

$$\underline{\underline{A}}_p\underline{\underline{A}} \wedge \underline{\underline{V}} = \underline{\underline{A}} \vee \underline{\underline{P}} \leq [\underline{\underline{A}}, \underline{\underline{P}}] ,$$

where $\underline{\underline{P}}$ has finite exponent n, say.

Then $x^n \in \underline{\underline{P}}(K)$ and, if $h \in H$, $[h, x^n] \in \underline{\underline{A}}(K)$. Thus $[h, x^n, x^n] \in [\underline{\underline{A}}, \underline{\underline{P}}](K) = \{1\}$. Thus $[h, px^n] = 1$. However, since H is abelian of exponent p, $[h, px^n] = [h, x^{pn}] = 1$.

Thus, for all x in G, $x^{pn} \in C_G(H)$ and so $G/C_G(H)$ has finite exponent bounded by pn where pn depends on p and $\underline{\underline{V}}$ only.

We shall need a straightforward extension of this lemma to the case where H is soluble of finite exponent. We give this as:

LEMMA 2. *Suppose that* $\underline{\underline{V}}$ *is a variety,* $G \in \underline{\underline{V}}$, *and that* H *is a soluble normal subgroup of* G *of finite exponent* m. *Suppose also that* $\underline{\underline{A}}_p\underline{\underline{A}} \nleq \underline{\underline{V}}$ *for every prime divisor* p *of* m. *Then* $G/C_G(H)$ *has finite exponent bounded in terms of* m *and* $\underline{\underline{V}}$ *only.*

Proof. H, being a soluble group of finite exponent, has a finite characteristic series, with elementary abelian factors. Suppose

$$\{1\} = H_0 < H_1 < \ldots < H_{k-1} < H_k = H$$

is such a series of minimal length. We use induction on k. If $k = 1$, we may apply Lemma 1. Otherwise, if $x \in G$, we may suppose that x^n centralises H_{k-1} for some natural number n. Also, using Lemma 1, and, if necessary, increasing the n chosen previously, we may assume that x^n centralises H_k/H_{k-1}. Then it is easy to check that x^{mn} centralises $H_k = H$. Thus $G/C_G(H)$ has finite exponent and it is again easy to check

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that it is bounded in terms of m and $\underline{\underline{V}}$ only.

The following lemma, which appears in [*10*], may be regarded as the 'torsion-free counterpart' of a theorem of Hall [*3*]. Since the result in the lemma appears to be well known we shall not digress by offering a proof. If N is a torsion-free nilpotent group and $H \le N$, we define the isolator of H in N as

$$\{x : x \in N \text{ and } x^n \in H \text{ for some natural number } n\}$$

and denote it by $I_N(H)$. Then $I_N(H)$ is a subgroup of N.

LEMMA 3. *Let G be a group and N be a torsion-free, nilpotent, normal subgroup of G, and suppose that $G/I_N(N')$ is nilpotent. Then G is nilpotent.*

LEMMA 4. *Let $\underline{\underline{V}}$ be a soluble variety such that $\underline{\underline{A}}^2 \nleq \underline{\underline{V}}$. Then there is a bound on the class of torsion-free nilpotent groups in $\underline{\underline{V}}$.*

It is not difficult to see that this result may be proved using Lemma 3 and the information about metabelian varieties obtainable from Proposition 2. It may also be deduced, however, from Corollary 1 of [5] and the fact that $\cup\{\underline{\underline{A}}_p\underline{\underline{A}}_p : p \text{ is prime}\} = \underline{\underline{A}}^2$. (The latter fact may be deduced from Propositions 1 and 2.)

We shall need the concept of the verbal Fitting subgroup of a group G, which we define as the product of the nilpotent verbal subgroups of G. Now, in a relatively free group of infinite rank, the centraliser of a verbal subgroup is verbal (the proof is an elementary generalisation of 2.3 of [*8*]). Thus, in a soluble relatively free group of infinite rank, the verbal Fitting subgroup contains its centraliser (the proof, in this case, being similar to that for the Fitting subgroup; see, for example, 1.53 of [*9*]).

2. Development of a minimal counterexample

Suppose that $\underline{\underline{V}}$ is a counterexample to the theorem. We claim that there is then a minimal counterexample which is a subvariety of $\underline{\underline{V}}$ and hence also satisfies *i)*. For, let

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$$\underline{\underline{V}} \geq \underline{\underline{W}}_1 \geq \underline{\underline{W}}_2 \geq \ldots \geq \underline{\underline{W}}_\alpha \geq \ldots$$

be a possibly transfinite descending chain of counterexamples. By Zorn's Lemma, it suffices to prove that the intersection, $\underline{\underline{W}}$ say, of this chain must also be a counterexample. If not, $\underline{\underline{W}}$ is of the type (A) ; say, $\underline{\underline{W}} \leq \underline{\underline{B}}_n \underline{\underline{N}}_c \underline{\underline{B}}_n$. Since $\underline{\underline{W}}$ is soluble, we may equally well suppose that $\underline{\underline{W}} \leq \underline{\underline{A}}_k^l \underline{\underline{N}}_c \underline{\underline{A}}_k^l$ for some natural numbers k, l . However, by a repeated use of the result in [4] that, if $\underline{\underline{U}}$ is nilpotent and $\underline{\underline{V}}$ is finitely based then $\underline{\underline{UV}}$ is finitely based, we see that $\underline{\underline{A}}_k^l \underline{\underline{N}}_c \underline{\underline{A}}_k^l$ may be defined by a single law, u , say. Let X be the word group and let $W_\alpha = \underline{\underline{W}}_\alpha(X)$, $W = \underline{\underline{W}}(X)$. Then $W = \bigcup_\alpha W_\alpha$. However, since $u \in W$, there exists an α such that $u \in W_\alpha$. But then $\underline{\underline{W}}_\alpha \leq \underline{\underline{A}}_k^l \underline{\underline{N}}_c \underline{\underline{A}}_k^l$, a contradiction which proves our claim.

We shall, for the rest of this proof, suppose $\underline{\underline{V}}$ to be a minimal counterexample and fix the notation $G = F_\infty(\underline{\underline{V}})$. We shall write $y_1, \ldots, y_n, \ldots$ for a (relatively) free generating set of G and N for the verbal Fitting subgroup of G . We note that any subgroup $\underline{\underline{B}}_n(G)$ will generate $\underline{\underline{V}}$ and that G can have no non-trivial verbal torsion subgroups. In particular, all nilpotent verbal subgroups of G are torsion-free. Consequently, by Lemma 4, there is a bound on their class. Thus N is nilpotent.

Before proceeding with the main part of the proof, we shall, as promised, prove the theorem in the case that $\underline{\underline{V}}$ is metanilpotent (that is $\underline{\underline{V}} \leq \underline{\underline{N}}_c \underline{\underline{N}}_d$ for some natural numbers c, d). We shall use this result in subsequent proofs.

PROPOSITION 3. *The theorem is true, regardless of i), for metanilpotent varieties.*

Proof. Since G is metanilpotent, G/N is nilpotent. Let $Z \leq G$ be such that Z/N is the centre of G/N . Then Z is verbal in G . Denote $I_N(N')$ by I ; then I is fully invariant, and so verbal in G . Now Z/I is a metabelian group and, since $Z' \leq N$, the derived group

$Z'I/I$ is torsion-free. It follows, from the descriptions in Propositions 1 and 2, that $Z/I \in \underline{\underline{N}}_c\underline{\underline{A}}_s$ for some natural numbers c, s .

Let $M \leq G$ be such that $M/I = \underline{\underline{A}}_s(Z/I)$. Then M is verbal in G and M/I is nilpotent. Thus, since N/I is nilpotent, MN/I is nilpotent. It follows, by Lemma 4, that MN is nilpotent. Since it is also verbal it is contained in the verbal Fitting subgroup N . Thus $M \leq N$.

Hence $Z/N \in \underline{\underline{A}}_s$ and so, since the centre of G/N has finite exponent, G/N does also (see, for example, 1.62 of [9]). Thus G is nilpotent by finite exponent and so $\underline{\underline{V}}$ cannot be a counterexample. The proof is complete.

We shall now show that $\underline{\underline{V}} \leq \underline{\underline{A}}\left(\underline{\underline{B}}_{p^t} \wedge \underline{\underline{N}}_2\right)\underline{\underline{A}}$ for some prime p and natural number t . We shall accomplish this in a number of steps.

2.A. $\underline{\underline{V}} \leq \underline{\underline{N}}_c\underline{\underline{B}}_n\underline{\underline{A}}$ *for some natural numbers* n, c .

Proof. G' certainly generates a proper subvariety of $\underline{\underline{V}}$ since it is of lower solubility length than G . Thus $\mathrm{Var}(G')$ is of the type (A). Also, any torsion verbal subgroup of G' is also a torsion verbal subgroup of G and so is trivial. Thus G' is nilpotent by finite exponent and the result follows.

2.B. $\underline{\underline{V}} \leq \underline{\underline{N}}_c\left(\underline{\underline{B}}_n \wedge \underline{\underline{N}}_2\right)\underline{\underline{A}}$.

Proof. Firstly we note that $N \leq G'$; for otherwise $\mathrm{Var}(G/G') = \underline{\underline{A}} \nleq \mathrm{Var}(G/N)$ and so the latter has finite exponent. But then G is nilpotent by finite exponent and $\underline{\underline{V}}$ cannot be the counterexample we supposed.

We have shown in 2.A that $\underline{\underline{V}} \leq \underline{\underline{N}}_c\underline{\underline{B}}_n\underline{\underline{A}}$ and so G'/N is soluble of finite exponent. Thus there is a series

$$N = B_1 < B_2 < \ldots < B_{k-1} < B_k = G'$$

where

1) each B_i is verbal in G ,

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2) B_i/B_{i-1} is elementary abelian, of exponent $p(i)$ say, and

3) k is the least natural number for which such a series exists.

If $k < 2$, then 2.B is immediately true, and so we may suppose that $k \geq 2$. Also, if $x \in G$, and $gp(B_{k-1}, x) = H$, say, then H does not generate $\underline{\underline{V}}$ because of 3).

Thus $\mathrm{Var}(H)$ is of the type (A). Suppose, however, that T is a torsion verbal subgroup of H . Then $T \cap N = \{1\}$ and so $[T, N] = \{1\}$. But N contains its centraliser in G and is torsion-free and so $T = 1$. Thus H is nilpotent by finite exponent and therefore it is not difficult to check that $\underline{\underline{A}}_{p(i)}\underline{\underline{A}} \nleq \mathrm{Var}(H)$ $(1 \leq i \leq k-1)$. We now apply Lemma 2 to show that, for some natural number m , x^m centralises B_{k-1}/N and so, since x was arbitrary in G , $\underline{\underline{B}}_m(G)$ centralises B_{k-1}/N .

Hence $\left(\underline{\underline{B}}_m(G)N \cap G'\right)/N$ is nilpotent of class at most 2 and, of course, $\underline{\underline{B}}_m(G)N/\left(\underline{\underline{B}}_m(G)N \cap G'\right)$ is abelian. But $\underline{\underline{B}}_m(G)N$, even $\underline{\underline{B}}_m(G)$, generates $\underline{\underline{V}}$ and we have proved 2.B.

2.C. $\underline{\underline{V}} \leq \underline{\underline{N}}_c\left(\underline{\underline{B}}_{p^t} \wedge \underline{\underline{N}}_2\right)\underline{\underline{A}}$ *for some prime power divisor* p^t *of* n .

Proof. Owing to 2.B, G'/N is nilpotent of finite exponent and therefore it is the direct product of its Sylow subgroups - say

$$G'/N = P_1/N \times \ldots \times P_k/N$$

where P_i/N is a $p(i)$ group $(1 \leq i \leq k)$. Our claim amounts to $k = 1$; suppose this is not the case. Then, if $x \in G$, $K_i = gp(P_i, x)$ generates a proper subvariety of $\underline{\underline{V}}$.

Using a method similar to that used in the proof of 2.B, we show that, for each i $(1 \leq i \leq k)$, $\underline{\underline{B}}_{m(i)}$ centralises P_i/N for some natural number $m(i)$. Then if m is a common multiple of $m(1), \ldots, m(k)$, $\underline{\underline{B}}_m(G)$ centralises P_i/N for all i $(1 \leq i \leq k)$ and so centralises G'/N .

Thus $\underline{\underline{B}}_m(G)N/N$ is nilpotent of class at most 2 and $\underline{\underline{B}}_m(G)N$ is

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metanilpotent. Since $\underline{\underline{B}}_m(G)N$ generates $\underline{\underline{V}}$, Proposition 3 shows that $\underline{\underline{V}}$ cannot be a counterexample. Hence $k > 1$ yields a contradiction and 2.C is proved.

2.D. $\underline{\underline{V}} \le \underline{\underline{A}}\left(\underline{\underline{B}}_{p^t} \wedge \underline{\underline{N}}_2\right)\underline{\underline{A}}$.

Proof. We have shown in 2.C that $G/N \in \left(\underline{\underline{B}}_{p^t} \wedge \underline{\underline{N}}_2\right)\underline{\underline{A}}$. It thus suffices to prove that N is abelian: suppose not. Then, if we denote $I_N(N')$ by I , I is non-trivial. Since I is verbal, G/I generates a proper subvariety of $\underline{\underline{V}}$ which is thus of the type (A) . Also, an application of Lemma 3 shows that N/I is the verbal Fitting subgroup of G/I and so is self-centralising. Hence, if T/I is a verbal torsion subgroup of G/I , $T \cap N \le I$ - since N/I is torsion-free - and so $[T, N] \le I$. Thus T/I is trivial. Hence G/I is nilpotent by finite exponent and G is metanilpotent by finite exponent. We may now use Proposition 3 to show that $\underline{\underline{V}}$ is not a counterexample, contrary to our supposition. Therefore N is abelian and 2.D is proved.

In the remaining section we shall abbreviate $\left(\underline{\underline{B}}_{p^t} \wedge \underline{\underline{N}}_2\right)\underline{\underline{A}}$ by $\underline{\underline{T}}$. Hence $\underline{\underline{V}} \le \underline{\underline{AT}}$.

3. Proof of the theorem

3.A. $\underline{\underline{V}}$ *is generated by a finitely generated group.*

Proof. Put $A_0 = \underline{\underline{T}}(G)$, $F_k = F_k(\underline{\underline{V}})$ and $A_k = \underline{\underline{T}}(F_k)$ $(k = 1, 2, 3, \ldots)$. Then A_0 is an abelian verbal subgroup of G and, since G has no verbal torsion subgroups, A_0 is torsion-free. Thus each A_k is torsion-free since it is embedded in A_0 .

If $\underline{\underline{V}}$ were not generated by a finitely generated group, then each F_k would generate a proper subvariety of $\underline{\underline{V}}$, which would then be of the type (A) - say $F_k \in \underline{\underline{B}}_{n(k)}\underline{\underline{N}}_{c(k)}\underline{\underline{B}}_{n(k)}$ $(k = 1, 2, 3, \ldots)$. We claim that we may even suppose $c(k)$ to be independent of k . For, putting $B_k = \underline{\underline{B}}_{n(k)}\left(F_k\right)$

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and $N_k = \underline{\underline{N}}_{c(k)}(B_k)$, B_k/N_k is a finitely generated nilpotent group. Thus, the torsion subgroup, which we denote by T_k/N_k , is finite. We now have a normal series of F_k ,

$$\{1\} \le T_k \le B_k \le F_k ,$$

in which T_k is of finite exponent, B_k/T_k is torsion-free nilpotent and F_k/B_k is of finite exponent. By Lemma 4, the class of B_k/T_k is bounded and so, with a suitable adjustment to $n(k)$, we may suppose that $c(1) = \ldots = c(k) = \ldots = c$, say.

Now N_k is torsion while A_k is torsion-free and so $N_k \cap A_k = 1$. Hence

$$F_k \in \underline{\underline{N}}_c\underline{\underline{B}}_{n(k)} \vee \underline{\underline{T}} \le \underline{\underline{N}}_d\underline{\underline{B}}_{n(k)} \vee \underline{\underline{N}}_d\underline{\underline{A}} \le \underline{\underline{N}}_d\left[\underline{\underline{B}}_{n(k)}, A\right] ,$$

where $d = c$ if $c > 1$ and $d = 2$ if $c = 1$.

It is a consequence of the proof of 3.1 of [4] that $\underline{\underline{N}}_d\left[\underline{\underline{B}}_{n(k)}, \underline{\underline{A}}\right]$ has as a basis for its laws the word

$$W_k(x_1, \ldots, x_{3d+3}) = [b_1, \ldots, b_{d+1}]$$

where

$$b_i = \left[x_{3i-2}, x_{3i-1}, x_{3i}^{n(k)}\right] .$$

Thus, if F_{3d+3} does not generate $\underline{\underline{V}}$,

$$W_{3d+3}(y_1, \ldots, y_{3d+3}) = 1$$

is true in F_{3d+3} (we regard F_k as having free generators $y_1, \ldots, y_k$). But then $W_{3d+3}(y_1, \ldots, y_{3d+3}) = 1$ is also true in G and so the law W_{3d+3} holds in $\underline{\underline{V}}$, that is, $\underline{\underline{V}} \le \underline{\underline{N}}_d\left[\underline{\underline{B}}_{n(k)}, \underline{\underline{A}}\right]$. Hence $\underline{\underline{V}} \le \underline{\underline{N}}_d\underline{\underline{N}}_2\underline{\underline{B}}_{n(k)}$ and so, by Proposition 3, $\underline{\underline{V}}$ could not be a counterexample. This completes the proof of 3.A.

We shall now suppose that $\underline{\underline{V}}$ is generated by one of its finitely generated relatively free groups F , and put $A = \underline{\underline{T}}(F)$.

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3.B. $[a, cx^m] = 1$ *for all* $x \in F$, $a \in A$, *where* c, m *are natural numbers depending on* $\underline{\underline{V}}$ *only.*

Proof. Let $x \in F$ and put $H = gp(A, x)$. Then H is metabelian and so $H \in \underline{\underline{V}} \wedge \underline{\underline{A}}^2$. But, by Propositions 1 and 2, $\underline{\underline{V}} \wedge \underline{\underline{A}}^2$ is of the type (A) - say $\underline{\underline{V}} \wedge \underline{\underline{A}}^2 \leq \underline{\underline{B}}_m \underline{\underline{N}}_c \underline{\underline{B}}_m$. Thus, if $a \in A$,

$$[a^m, cx^m]^m = 1 .$$

It follows, using the fact that A is a torsion-free abelian normal subgroup of F , that $[a, cx^m] = 1$.

3.C. *Suppose* H/K *is an F-normal q-elementary abelian factor of* A . *Then, for some natural number* $r(q)$, *depending only on* q *and* $\underline{\underline{V}}$, $\underline{\underline{B}}_{r(q)}(F)$ *centralises* H/K .

Proof. Suppose, then, that $h \in H$, $x \in F$. Since $H \leq A$, we may apply 3.B to show $[h, cx^m] = 1$ and, trivially, $[h, cx^m] \in K$. Let j be the least natural number such that $q^j \geq c$. Then, also trivially, $[h, q^j x^m] \in K$. Thus, since H and K are normal in F and H/K is of exponent q ,

$$\left[h, x^{q^j m}\right] \in K .$$

Hence, since x was arbitrary in F , putting $r(q) = q^j m$ gives the required result. Since $r(q)$ depends on q, c, m only and c, m depend on $\underline{\underline{V}}$ only, $r(q)$ depends only on q and $\underline{\underline{V}}$, as required.

The next three parts of this section will obtain information on A as an abelian group so that in 3.G we may extract a relevant property of $\mathrm{Aut}(A)$. We define D_n as $\bigcap_j \underline{\underline{B}}_{n^j}(A)$ where n is a natural number and the intersection is taken over all natural numbers j ; we also define D as $\bigcap_n D_n$ where the intersection is again taken over all natural numbers n . It is an elementary fact for all torsion-free abelian groups A that A/D_n and A/D are torsion-free.

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3.D. *If* q *is a prime and* $q \neq p$ $\left(\textit{recall } \underline{\underline{T}} = \left(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_{p^t}\right)\underline{\underline{A}}\right)$, *then* $D_q \neq \{1\}$.

Proof. Write $H = \underline{\underline{B}}_{r(q)}(F)$ (where $r(q)$ is the natural number found in 3.C) and $B = \underline{\underline{T}}(H)$. Then H is verbal in F, B is verbal in H, and $\underline{\underline{B}}_{q^j}(B)$ is verbal in B $(j = 0, 1, 2, \ldots)$. (We shall, in the following, ease the notation by writing q^jB rather than $\underline{\underline{B}}_{q^j}(B)$ - and similarly for corresponding subgroups of A.) In particular $\{q^jB/q^{j+1}B : j = 0, 1, 2, \ldots\}$ are F-verbal, and so F-normal, factors of A (since $B \leq A$). Thus they are centralised by H.

Hence

$$H' \geq B \geq qB \geq \ldots \geq q^jB$$

gives a descending series of H'/q^jB with first factor nilpotent and all other factors central, showing that H'/q^jB is nilpotent. Since H'/q^jB is also of finite exponent p^iq^j, it is the direct product of its Sylow subgroups - S_p, S_q say. But $S_q = B/q^jB$ and

$$S_p \cong \left(H'/q^jB\right)/S_q = \left(H'/q^jB\right)/\left(B/q^jB\right) \cong H'/B .$$

Thus S_q is abelian and S_p has class at most 2. Hence H'/q^jB has class at most 2 : thus

$$[H'', H'] \leq q^jB \quad (j = 0, 1, 2, \ldots) .$$

If $D_q = \{1\}$, then, since $B \leq A$, $\bigcap_j q^jB = \{1\}$. Thus $[H'', H'] \leq \bigcap_j q^jB = \{1\}$ and $H \in \underline{\underline{N}}_2\underline{\underline{A}}$. Since H generates $\underline{\underline{V}}$, Proposition 3 shows that $\underline{\underline{V}}$ could not be a counterexample as we supposed. The proof of 3.D is complete.

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3.E. $D_p = D$.

Proof. Trivially $D \le D_p$ and so it remains to prove that $D_p \le D$. We shall show that A/D_q $(q \ne p)$ is free abelian of finite rank. Then we have

$$\bigcap_i p^i(A/D_q) = D_q/D_q$$

or $\bigcap_i p^i A \le D_q$, where the intersection is again taken over all natural numbers i . Hence $D_p \le D_q$ for all primes q and so $D_p \le \bigcap_q D_q = D$ where this intersection is taken over all primes q .

Now D_q $(q \ne p)$ is non-trivial and verbal in F . Thus F/D_q generates a proper subvariety of $\underline{\underline{V}}$, which is therefore of the type (A); say,

$$F/D_q \in \underline{\underline{B}}_n \underline{\underline{N}}_c \underline{\underline{B}}_n .$$

If M is the subgroup of F with $M/D_q = \underline{\underline{N}}_c\underline{\underline{B}}_n(F/D_q)$ then $M/D_q \in \underline{\underline{B}}_n$ while A/D_q is torsion-free, so $A \cap M = D_q$ and $A/D_q \cong AM/M$. But $F/M \in \underline{\underline{N}}_c\underline{\underline{B}}_n$ and is finitely generated, so it is polycyclic and $AM/M \cong A/D_q$ is also finitely generated. Thus A/D_q is free abelian of finite rank.

3.F. $|A:pA|$ *is finite.*

Proof. We have shown, in 3.C, that, for some natural number $r(p)$, $\underline{\underline{B}}_{r(p)}(F)$ centralises A/pA . It is not difficult to extend this to the result that, for each natural number j , $\underline{\underline{B}}_{r(p^j)}(F)$ with $r(p^j) = p^{j-1}r(p)$ centralises A/p^jA (the method is similar to that used in Lemma 2).

Denote $A.\underline{\underline{B}}_{r(p^j)}(F)/p^jA$ by H_j . Then A/p^jA is central in H_j . Since $\underline{\underline{T}}\left(A.\underline{\underline{B}}_{r(p^j)}(F)\right) \le \underline{\underline{T}}(F) = A$, $\underline{\underline{T}}(H_j) \le A/p^jA$, and so

$H_j \in [\underline{\underline{E}}, \underline{\underline{T}}] \leq \underline{\underline{N}}_3\underline{\underline{A}} \wedge \underline{\underline{V}}$. Thus, by Proposition 3, $\mathrm{Var}(\{H_j : j = 1, 2, \ldots\})$ is of the type (A) - say $H_j \in \underline{\underline{B}}_n\underline{\underline{N}}_c\underline{\underline{B}}_n$ $(j = 1, 2, 3, \ldots)$.

Suppose $P_j \leq F$ is such that $P_j/p^jA = \underline{\underline{N}}_c\underline{\underline{B}}_n(H_j)$ and suppose that $n = p^\alpha k$; $(p^\alpha, k) = 1$. We claim that $P_{\alpha+1} \cap A \leq pA$. For, suppose $a \in P_{\alpha+1} \cap A$. Then $a^n = \left(a^{p^\alpha}\right)^k \in p^{\alpha+1}A$ and so $a^{p^\alpha} \in p^{\alpha+1}A$. Hence, for some $b \in A$, $a^{p^\alpha} = b^{p^{\alpha+1}}$. Since A is torsion-free, $a = b^p$, that is $a \in pA$ as required.

Now $AP_{\alpha+1}/P_{\alpha+1}$ is a subgroup of $H_{\alpha+1}/P_{\alpha+1}$ which, as before, is a polycyclic group. Thus $A/(A \cap P_{\alpha+1}) \cong AP_{\alpha+1}/P_{\alpha+1}$ is finitely generated. Since $pA \geq A \cap P_{\alpha+1}$, A/pA is also finitely generated and so is finite.

3.G. The conclusion of this section, which we state as an independent lemma, is due to L.G. Kovács.

LEMMA 5. *Suppose that* A *is a torsion-free abelian group, that* $\bigcap_j p^jA = 1$ *for some prime* p *, and that* A/pA *is finite. Then every* p*-group of automorphisms of* A *is finite.*

Proof. First observe that if A/pA is finite, so is A/p^2A . There is a natural homomorphism from the automorphism group of A to that of A/p^2A : it is clearly sufficient to show that the kernel of this homomorphism contains no automorphisms of order p ; that is, that if $\beta^p = 1$ and $A(\beta-1) \leq p^2A$ for some automorphism β of A then $\beta = 1$. We show that if $A(\beta-1) \leq p^k$ for some $k > 1$ then also $A(\beta-1) \leq p^{k+1}A$; it will then follow that $A(\beta-1) \leq \bigcap_k p^kA = 1$ and so $\beta = 1$. To this end, let $\beta^p = 1$, $k > 1$, and $A(\beta-1) \leq p^kA$. Working in the endomorphism ring of A ,

$$1 = \beta^p = (1+(\beta-1))^p = 1 + p(\beta-1) + \sum_{i=2}^{p} \binom{p}{i}(\beta-1)^i ,$$

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hence

$$p(\beta-1) = -\sum_{i=2}^{p} \binom{p}{i} (\beta-1)^i$$

and so $pA(\beta-1) \leq A(\beta-1)^2 \leq p^{2k}A \leq p^{k+2}A$. Since A is torsion-free, this implies that $A(\beta-1) \leq p^{k+1}A$ as claimed.

3.H. The proof of the theorem now follows easily. Firstly, we note that condition *i)* implies that $F_\infty(\underline{\underline{V}})$, and so A , is residually finite (17.81 of [7]). Thus $D = 1$ and so, by 3.E, $D_p = 1$. (We note that this is the first time we have used condition *i)*.) Then A satisfies the conditions of Lemma 5.

Let C be the centraliser of A in F and put $H = F'/(F' \cap C)$. Then $H \cong F'C/C$, and $F'C/C$ is a p-group of automorphisms of A : so, by Lemma 5, H is finite. Now H is the derived group of $F/(F' \cap C)$ and so, by 5.41 of [6], the centre of the latter group has finite index. Since $F' \cap C \in \underline{\underline{N}}_3$, we then have that $F \in \underline{\underline{N}}_3\underline{\underline{A}}\underline{\underline{B}}_m$ for some natural number m and so, by Proposition 3, $\underline{\underline{V}}$ cannot be a counterexample.

With this final contradiction, the theorem is proved.

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