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# A CHARACTERIZATION OF THE VERONESE VARIETIES*

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Let $P^m(C)$ be the complex projective space of dimension $m$. In a previous paper [2] we have proved

THEOREM A. *Let $f$ be a Kaehlerian immersion of a connected, complete Kaehler manifold $M^n$ of dimension $n$ into $P^m(C)$. If the image $f(\tau)$ of each geodesic $\tau$ in $M^n$ lies in a complex projective line $P^1(C)$ of $P^m(C)$, then $f(M^n)$ is a complex projective subspace of $P^m(C)$, and $f$ is totally geodesic.*

In the present note, we shall first provide a much simpler geometric proof of this result and then give a characterization of the Veronese varieties by means of the notion of circles in $P^m(C)$. Generally, a curve $x(t)$ with arc-length parameter $t$ in a Riemannian manifold is called a circle if there exists a field of unit vectors $Y_t$ along the curve, which, together with the unit tangent vectors $X_t$, satisfies the differential equations

$$\nabla_t X_t = kY_t \quad \text{and} \quad \nabla_t Y_t = -kX_t ,$$

where $k$ is a positive constant (see [4]).

By the Veronese variety we mean the imbedding of $P^n(C)$ into $P^m(C)$, where $m = n(n+3)/2$, which is defined as follows. Let $S^{2n+1}$ be the unit sphere in the complex vector space $C^{n+1}$ with the standard hermitian inner product $(z, w)$ and corresponding real inner product $\langle z, w\rangle = \mathrm{Re}\,(z, w)$. On the other hand, the set of all complex symmetric matrices of degree $n+1$ can be considered as the vector space $C^{m+1}$, where $m = n(n+3)/2$, in which the standard hermitian inner product can be expressed by

$$(A, B) = \text{trace}\, A\bar{B} , \qquad A, B \in C^{m+1} .$$

The mapping $v$ which takes $x \in C^{n+1}$ into $x\,{}^t x \in C^{m+1}$ maps $S^{2n+1}$ into the

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unit sphere $S^{2m+1}$ of $C^{m+1}$, and induces a holomorphic imbedding of $P^n(C)$ into $P^m(C)$. If we choose the Fubini-Study metrics of constant holomorphic curvature $c\,(>0)$ for $P^m(C)$ and $c/2$ for $P^n(C)$, then the imbedding is isometric. This is what we call the Veronese imbedding.

We now state our new result.

THEOREM B. *Let $f$ be a Kaehlerian immersion of a connected, complete Kaehler manifold $M^n$ of dimension $n$ into $P^m(C)$ with Fubini-Study metric. The image $f(\tau)$ of each geodesic $\tau$ in $M^n$ is a circle in $P^m(C)$ if and only if $f$ is congruent (by a holomorphic isometry of $P^m(C)$) to $i \circ v$, where $v$ is the Veronese imbedding of $P^n(C)$ into $P^{m'}(C)$, with $m' = n(n+3)/2$, and $i$ is the totally geodesic inclusion of $P^{m'}(C)$ into $P^m(C)$.*

## 1. Simpler proof of Theorem A.

Let $x_0$ be a point of $M^n$ and let $M^*$ be the complete totally geodesic complex submanifold (namely, $n$-dimensional projective subspace $P^n(C)$) through the point $f(x_0)$ and tangent to $f(M^n)$, that is, the tangent space $T_{f(x_0)}(M^*)$ equals $f_*(T_{x_0}(M^n))$, where $f_*$ denotes the differential of $f$.

Let $\tau$ be an arbitrary geodesic in $M^n$ starting at $x_0$. By assumption, there is a complex projective line $P^1(C)$ which contains $f(\tau)$. If $X$ denotes the initial tangent vector of $\tau$ at $x_0$, then $f_*(X)$ is tangent to $P^1(C)$. If we denote by $J$ the complex structure of $P^m(C)$ as well as that of $M^n$, then the vector $Jf_*(X) = f_*(JX)$ is tangent to $P^1(C)$. It follows that $T_{f(x_0)}(P^1(C))$ is spanned by $f_*(X)$ and $f_*(JX)$. On the other hand, these two vectors are contained in $f_*(T_{x_0}(M^n)) = T_{f(x_0)}(M^*)$. Thus $T_{f(x_0)}(P^1(C)) \subset T_{f(x_0)}(M^*)$. Since $P^1(C)$ and $M^*$ are totally geodesic in $P^m(C)$, it follows that $P^1(C)$ is contained in $M^*$; thus $f(\tau)$ is contained in $M^*$. Since $\tau$ is an arbitrary geodesic in $M$, we have $f(M) = M^*$.

## 2. Veronese imbedding.

We shall show that the Veronese imbedding $v$ of $P^n(C)$ into $P^m(C)$ with $m = n(n+3)/2$ has the property that the image of each geodesic in $P^n(C)$ is a circle in $P^m(C)$. This property does not depend on the choice of a positive constant $c$ which we choose for the holomorphic sectional curvature of $P^m(C)$ (and that of $P^n(C)$ will be $c/2$). We recall how geometry of $P^m(C)$ is related to that of $S^{2m+1}$. The standard fibration $\pi: S^{2m+1} \to P^m(C)$ is a principal $S^1$-bundle. It has a connection whose horizontal subspaces $Q_x$, $x \in S^{2m+1}$, are given by

$$Q_x = \{X \in \boldsymbol{C}^{m+1}; \langle X, x\rangle = \langle X, ix\rangle = 0\} .$$

The projection $\pi_*$ maps $Q_x$ isomorphically onto the tangent space $T_u(P^m(\boldsymbol{C}))$, where $u = \pi(x)$. If we let

$$g(\pi_* X, \pi_* Y) = (4/c)\langle X, Y\rangle , \qquad X, Y \in Q_x ,$$

then $g$ is the Fubini-Study metric with holomorphic sectional curvature $c$ for $P^m(\boldsymbol{C})$. We shall choose $c = 4$ (to simplify constant factors in the computations that follow). Let us denote by $\nabla'$ the Riemannian connection for $S^{2m+1}$ and by $\tilde{\nabla}$ the Kaehlerian connection for $P^m(\boldsymbol{C})$. We formulate the relationship between $\nabla'$ and $\tilde{\nabla}$ (see [3], Proposition 3) in the following form. A curve in $S^{2m+1}$ is said to be horizontal if its tangent vectors are horizontal.

LEMMA 1. *Let $x_t$ be a horizontal curve in $S^{2m+1}$ and $u_t = \pi(x_t)$. If $Z_t$ is a horizontal vector field along $x_t$ and if $W_t = \pi_*(Z_t)$, then $\tilde{\nabla}_t W_t = \pi_*(\nabla'_t Z_t)$.*

LEMMA 2. *If $x_t$ is a horizontal curve in $S^{2m+1}$ with arc-length parameter $t$, then $\nabla'_t X_t$, where $X_t$ denotes the tangent vector, is horizontal.*

*Proof.* We have

$$\nabla'_t X_t = dX/dt + x_t .$$

Since $x_t$ is horizontal, we have $\langle X_t, ix_t\rangle = 0$ and hence

$$\langle dX/dt, ix_t\rangle + \langle X_t, iX_t\rangle = 0 .$$

But $\langle X_t, iX_t\rangle = 0$ so that $\langle dX/dt, ix_t\rangle = 0$. Thus we obtain

$$\langle \nabla'_t X_t, ix_t\rangle = \langle dX/dt, ix_t\rangle + \langle x_t, ix_t\rangle = 0 .$$

LEMMA 3. *If $x_t$ is a circle in $S^{2m+1}$ which is furthermore a horizontal curve, then $u_t = \pi(x_t)$ is a circle in $P^m(\boldsymbol{C})$.*

*Proof.* We have a field of unit vectors $Y_t$ along $x_t$ such that

$$\nabla'_t X_t = kY_t \quad \text{and} \quad \nabla'_t Y_t = -kX_t ,$$

where $k$ is a positive constant and $X_t$ is the tangent vector. By Lemma 2, $\nabla'_t X_t$ and hence $Y_t$ are horizontal. The tangent vector of $u_t$ is given by $U_t = \pi_*(X_t)$. Consider the field of unit normal vectors $V_t = \pi_*(Y_t)$;

note that $\pi_*$ is isometric from $Q_x$ to $T_{\pi(x)}(P^m(C))$. By Lemma 1, we have

$$\tilde{\nabla}_t U_t = \pi_*(\nabla'_t X_t) = \pi_*(kY_t) = kV_t$$

and, similarly,

$$\tilde{\nabla}_t V_t = \pi_*(\nabla'_t Y_t) = \pi_*(-kX_t) = -kU_t .$$

Thus $u_t$ is a circle in $P^m(C)$.

Now we shall prove our assertion about the Veronese imbedding. We observe that the unitary group $U(n+1)$ acts naturally on $S^{2n+1}$ and $P^n(C)$ as a group of isometries. Each geodesic $\tau$ in $P^n(C)$ is congruent by a transformation belonging to $U(n+1)$ to the curve with homogeneous coordinates $(\cos t, \sin t, 0, \cdots, 0)$. On the other hand, we can let $U(n+1)$ act on the space $C^{m+1}$ of all complex symmetric matrices of degree $n+1$ by $Z \to AZ\,{}^tA$, where $Z \in C^{m+1}$ and $A \in U(n+1)$. This action preserves inner product in $C^{m+1}$ and thus induces the action of $U(n+1)$ on $S^{2m+1}$ and $P^m(C)$ as a group of isometries. Now the Veronese imbedding $v$ is equivariant relative to the actions of $U(n+1)$ on $P^n(C)$ and on $P^m(C)$.

It is thus sufficient to prove the following. Let $\tau$ be the geodesic $w_t$ in $P^n(C)$ given by $w_t = \pi(z_t)$, where $z_t = (\cos(t/\sqrt{2}), \sin(t/\sqrt{2}), 0, \cdots, 0)$ is a curve on $S^{2n+1}$. Since the holomorphic sectional curvature of $P^n(C)$ has been chosen to be 2, we have

$$\|dw/dt\|^2 = 2\,\|dz/dt\|^2 = 1 ,$$

which shows that $t$ is the arc-length parameter for the geodesic $w_t$. Let

$$x_t = v(z_t) , \qquad u_t = v(w_t) \quad \text{so that} \quad u_t = \pi(x_t) .$$

We wish to show that $u_t$ is a circle in $P^m(C)$. The curve $x_t$ on $S^{2m+1}$ can be represented simply by the first $2 \times 2$ block of the form

$$\begin{bmatrix} \cos^2(t/\sqrt{2}) & \sin(t/\sqrt{2})\cos(t/\sqrt{2}) \\ \sin(t/\sqrt{2})\cos(t/\sqrt{2}) & \sin^2(t/\sqrt{2}) \end{bmatrix}$$

since the other components are all 0. The tangent vectors $X_t$ of the curve $x_t$ are represented in the same sense by

$$X_t = (1/\sqrt{2})\begin{bmatrix} -\sin(\sqrt{2}\,t) & \cos(\sqrt{2}\,t) \\ \cos(\sqrt{2}\,t) & \sin(\sqrt{2}\,t) \end{bmatrix} .$$

Since $\langle X_t, ix_t\rangle = 0$, $x_t$ is a horizontal curve in $S^{2m+1}$. If we show that it is a circle in $S^{2m+1}$, then Lemma 3 implies that $u_t = \pi(x_t)$ is a circle in $P^m(\boldsymbol{C})$.

We have

$$dX/dt = \begin{bmatrix} -\cos(\sqrt{2}\,t) & -\sin(\sqrt{2}\,t) \\ -\sin(\sqrt{2}\,t) & \cos(\sqrt{2}\,t) \end{bmatrix}.$$

The vector

$$\nabla'_t X_t = dX/dt + x_t$$

is also horizontal (since its components are real) and has length 1, because

$$\begin{aligned} &\langle dX/dt + x_t, dX/dt + x_t\rangle \\ &\quad = \langle dX/dt, dX/dt\rangle + 2\langle x_t, dX/dt\rangle + \langle x_t, x_t\rangle \\ &\quad = 2 + 2(-1) + 1 = 1\,, \end{aligned}$$

by virtue of $\langle x_t, dX/dt\rangle = -\langle dx/dt, X_t\rangle = -1$.

We thus set $Y_t = dX/dt + x_t$, namely, $\nabla'_t X_t = Y_t$. Since $\langle Y_t, X_t\rangle = 0$, we have

$$\begin{aligned} \nabla'_t Y_t &= dY/dt = d^2X/dt^2 + X_t \\ &= \sqrt{2}\begin{bmatrix} \sin(\sqrt{2}\,t) & -\cos(\sqrt{2}\,t) \\ -\cos(\sqrt{2}\,t) & -\sin(\sqrt{2}\,t) \end{bmatrix} \\ &\quad + (1/\sqrt{2})\begin{bmatrix} -\sin(\sqrt{2}\,t) & \cos(\sqrt{2}\,t) \\ \cos(\sqrt{2}\,t) & \sin(\sqrt{2}\,t) \end{bmatrix} \\ &= (1/\sqrt{2})\begin{bmatrix} \sin(\sqrt{2}\,t) & -\cos(\sqrt{2}\,t) \\ -\cos(\sqrt{2}\,t) & -\sin(\sqrt{2}\,t) \end{bmatrix} = -X_t\,. \end{aligned}$$

Thus we have shown that $x_t$ is a circle of curvature $k = 1$.

## 3. Proof of Theorem B.

We now finish the proof of Theorem B. Let $f$ be a Kaehlerian immersion of a complete Kaehler manifold $M^n$ into $P^m(\boldsymbol{C})$ with the property that for each geodesic $\tau$ in $M^n$ the image $f(\tau)$ is a circle in $P^m(\boldsymbol{C})$. We shall first show that

(i) the second fundamental form $\alpha$ is parallel;

(ii) $f$ is isotropic, that is, $\|\alpha(X, X)\|$ is equal to a constant for all unit tangent vectors $X$ to $M^n$ at each point;

(iii) $M^n$ has constant holomorphic curvature.

Let $x_t$ be a geodesic on $M^n$ with tangent vectors $X_t$ of length 1. Denoting by $\tilde{\nabla}$ and $\nabla$ the Kaehlerian connections of $P^m(C)$ and $M^n$, respectively, we have

$$\tilde{\nabla}_t X_t = \nabla_t X_t + \alpha(X_t, X_t) = \alpha(X_t, X_t) ,$$

where $\alpha$ is the second fundamental form. We obtain

$$(1) \qquad \tilde{\nabla}_t^2 X_t = -A_{\alpha(X_t, X_t)} X_t + \nabla_t^{\perp} \alpha(X_t, X_t) ,$$

where $A$ is the second fundamental tensor and $\nabla^{\perp}$ the normal connection. On the other hand, since $f(x_t)$ is a circle by assumption, there exists a field of unit tangent vectors $Y_t$ along $x_t$ and $k > 0$ such that

$$\tilde{\nabla}_t X_t = k Y_t \quad \text{and} \quad \tilde{\nabla}_t Y_t = -k X_t ,$$

thus

$$(2) \qquad \tilde{\nabla}_t^2 X_t = -k^2 X_t .$$

From (1) and (2) we obtain

$$(3) \qquad A_{\alpha(X_t, X_t)} X_t = k^2 X_t$$

and

$$(4) \qquad \nabla_t^{\perp} \alpha(X_t, X_t) = 0 .$$

Since $x_t$ is a geodesic in $M^n$, the covariant derivative

$$(\nabla_t^* \alpha)(X_t, X_t) = \nabla_t^{\perp} \alpha(X_t, X_t) - \alpha(\nabla_t X_t, X_t) - \alpha(X_t, \nabla_t X_t)$$

is equal to 0 by virtue of (4). Evaluating this at $t = 0$ and observing that $X_0$ can be an arbitrary unit tangent vector at an arbitrary point of $M^n$, we have

$$(5) \qquad (\nabla_X^* \alpha)(X, X) = 0 \qquad \text{for all tangent vectors } X \text{ to } M^n.$$

Since $(\nabla_X^* \alpha)(Y, Z)$ is symmetric in $X, Y$ and $Z$, we conclude that $\nabla^* \alpha = 0$, that is, $\alpha$ is parallel.

From (3) it follows that for any unit tangent vector $X$ to $M^n$ there exists a certain constant $k > 0$ such that

$$A_{\alpha(X, X)} X = k^2 X .$$

If $Y$ is a tangent vector perpendicular to $X$, then

$$\langle A_{\alpha(X,X)}X, Y\rangle = 0$$

so that

$$\langle \alpha(X,X), \alpha(X,Y)\rangle = 0 \quad \text{whenever} \quad \langle X, Y\rangle = 0 . \tag{6}$$

This condition implies that $f$ is isotropic, that is, $\|\alpha(X,X)\|$ is equal to a constant for all unit tangent vectors $X$ at each point (see [6], Lemma 1). It also follows that $M^n$ has constant holomorphic sectional curvature (see [6], Lemma 6).

We now wish to prove that $f$ is essentially the Veronese imbedding. Since $\alpha$ is parallel, the first normal spaces (spanned by the range of $\alpha$ at each point) are obviously parallel relative to the normal connection. The (complex) dimension of the normal spaces, say, $p$, is at most $n(n+1)/2$. It is known [1], Proposition 9, that there is a totally geodesic $P^{n+p}(C)$ in $P^m(C)$ such that $f(M^n) \subset P^{n+p}(C)$. We shall see that this immersion $f_0$ of $M^n$ into $P^{n+p}(C)$ is the Veronese imbedding (and indeed $p = n(n+1)/2$).

If $p < n(n+1)/2$, Theorem 2 of [6] says that $f_0$ is totally geodesic. This will mean that the image of a geodesic in $M^n$ is a geodesic in $P^{n+p}(C)$ and hence a geodesic in $P^m(C)$, contrary to the assumption that it is a circle in $P^m(C)$. Hence we must have $p = n(n+1)/2$. We already know that $M^n$ has constant holomorphic sectional curvature. Since the second fundamental form is parallel, it follows from [5], Theorem 4.4, that this constant is half the constant holomorphic sectional curvature of $P^{n+p}(C)$. Moreover, such an immersion $f_0$ is rigid. Thus $M^n$ is $P^n(C)$ with holomorphic sectional curvature, say, 2, if we assume that $P^m(C)$ and hence $P^{n+p}(C)$ has holomorphic sectional curvature 4. Now the Veronese imbedding $v$ is a Kaehlerian imbedding of $P^n(C)$ into $P^{n+p}(C)$. By rigidity, $f_0$ is congruent to $v$ by a holomorphic isometry of $P^{n+p}(C)$. Since this holomorphic isometry can be extended to a holomorphic isometry of $P^m(C)$, we can now conclude that $f: M^n \to P^m(C)$ is in fact congruent to $i \circ v$, where $v$ is the Veronese imbedding of $P^n(C)$ into $P^{n+p}(C)$, $p = n(n+1)/2$, and $i$ is a totally geodesic inclusion of $P^{n+p}(C)$ into $P^m(C)$. We have thus completed the proof of Theorem B.

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