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ERROR ESTIMATES FOR DOMINICI'S HERMITE FUNCTION ASYMPTOTIC FORMULA AND SOME APPLICATIONS

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Abstract

The aim of this paper is to find a concrete bound for the error involved when approximating the nth Hermite function (in the oscillating range) by an asymptotic formula due to D. Dominici. This bound is then used to study the accuracy of certain approximations to Hermite expansions and to Fourier transforms. A way of estimating an unknown probability density is proposed.

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1. Introduction

Let

$$h_n(x) = (-1)^n \gamma_n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2}, \quad \gamma_n = \pi^{-1/4} 2^{-n/2} (n!)^{-1/2}, \tag{1.1}$$

be the nth Hermite function, $n = 0, 1, \ldots$. Dominici [2] obtained an asymptotic formula for the Hermite polynomial, H_n, which yields for h_n in the oscillatory range $|x| < \sqrt{2n}$ (or, writing $x = \sqrt{2n}\sin\theta$, $|\theta| < \pi/2$),

$$h_n(x) = h_n(\sqrt{2n}\sin\theta) \sim \left(\frac{2n}{e}\right)^{n/2} \gamma_n \sqrt{\frac{2}{\cos\theta}} \cos\left(\frac{n}{2}\sin\theta + \left(n + \frac{1}{2}\right)\theta - \frac{n\pi}{2}\right). \tag{1.2}$$

As will be seen in Examples 1, 2 and 5 below, this asymptotic formula leads to remarkably accurate results.

Our aim in this paper is to estimate the error incurred when $h_n(x)$ is replaced by the right side of (1.2). We go on to apply this new information to the approximation of Hermite expansions and Fourier transforms and then to density estimation.

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In fact, we work with a slightly modified version of the asymptotic function. Our principal result is given in Theorem 1.1.

THEOREM 1.1. *Let h_n be the nth Hermite function, $n = 0, 1, \ldots$ and fix $A > 0$. Then, with $x = \sqrt{2n}\sin\theta$, one has, for $|x| \le A$,*

$$h_n(x) = d_n(x) + e_n(x),$$

where

$$d_n(x) = d_n(\sqrt{2n}\sin\theta) = c_n\sqrt{\frac{2}{\cos\theta}}\cos\left(\frac{n}{2}\sin\theta + \left(n+\frac{1}{2}\right)\theta - \frac{n\pi}{2}\right), \tag{1.3}$$

$$c_n = \begin{cases} \dfrac{n!}{\sqrt{2}\Gamma\left((n/2)+1\right)}\pi^{-1/4}2^{-n/2}(n!)^{-1/2}, & n \text{ even}, \\[2ex] \dfrac{2(n!)}{\left(\sqrt{2}n + 1/(\sqrt{2}n)\right)\Gamma\left(n+1/2\right)}\pi^{-1/4}2^{-n/2}(n!)^{-1/2}, & n \text{ odd}, \end{cases} \tag{1.4}$$

and

$$e_n(x) = \frac{r_n(x)}{n^{7/4}} + O\left(\frac{1}{n^{9/4}}\right), \tag{1.5}$$

in which $|r_n(x)| \le 2A$, $n \ge 5$.

The constant c_n in (1.4) has been chosen to guarantee $d_n(0) = h_n(0)$, and $d_n'(0) = h_n'(0)$, $n = 0, 1, \ldots$. Now, h_n satisfies the differential equation

$$L_n y = 0, \quad L_n := \frac{d^2}{dx^2} + (-x^2 + 2n + 1).$$

Defining $f_n = -L_n d_n$, we get that $e_n = h_n - d_n$ is the unique solution of the initial value problem

$$L_n y = f_n, \quad y(0) = y'(0) = 0.$$

Thus,

$$e_n(x) = \int_0^x G_n(x, y) f_n(y)\,dy, \quad x \in \mathfrak{R}, \tag{1.6}$$

G_n being the Green's function of the differential operator L_n.

In Sections 2–4 we successively estimate $|f_n(x)|$, $G_n(x, y)$ and, finally, using (1.5), $|e_n(x)|$. Section 5 has an application of the latter estimate to Hermite expansions and Section 6 has one to the Fourier transform. In Section 7 we briefly discuss density estimation.

2. A uniform bound for f_n on $[-\sqrt{n}, \sqrt{n}]$

Setting $\theta = \sin^{-1}(x/\sqrt{2n})$ in formula (1.3) for $d_n(\sqrt{2n}\sin\theta)$ gives

$$d_n(x) = \frac{c_n\cos(\rho(x))}{\left(4 - (2x^2/n)\right)^{1/4}},$$

in which

$$\rho(x) = \frac{1}{4}\sqrt{2n}x\sqrt{4 - \frac{2x^2}{n}} + \left(n + \frac{1}{2}\right)\sin^{-1}\left(\frac{x}{\sqrt{2n}}\right) - \frac{n\pi}{2}.$$

Next,

$$(L_n d_n)(x) = \frac{2^{-1/4} n^{1/4} c_n}{\left(2n - x^2\right)^{7/4}}\sqrt{(2n - x^2)(-4x^4 + 2(4n+1)x^2 + n)},$$

so

$$\begin{aligned} |f_n(x)| &= \left|(L_n d_n)(\sqrt{2n}\sin\theta)\right| \\ &\le \frac{c_n}{8n\cos^{11/2}\theta}\sqrt{16\sin^6\theta - 40\sin^4\theta + 47\sin^2\theta + 2} \\ &\le 2^{-3/4}\sqrt{35}\frac{c_n}{n}, \end{aligned} \tag{2.1}$$

when $|\theta| < \pi/2$ or $|x| < \sqrt{n}$.

3. The Green's function of L_n

As observed in Arfken [1, Pages 637–638], two linearly independent solutions of $L_n y = 0$ are

$$\phi_{1n}(x) = {}_1F_1\left(-\frac{n}{2}; \frac{1}{2}; x^2\right)e^{-x^2/2}$$

and

$$\phi_{2n}(x) = {}_1F_1\left(-\frac{(n-1)}{2}; \frac{3}{2}; x^2\right)xe^{-x^2/2},$$

where the confluent hypergeometric function of the first kind

$${}_1F_1(a; b; x) := \sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}\frac{x^k}{k!}, \qquad a, b, x \in \mathfrak{R},\ b \neq 0, -1, \ldots$$

and

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)}.$$

Indeed, $\phi_{1n}(0) = 1$, $\phi'_{1n}(0) = 0$ and $\phi_{2n}(0) = 1$, $\phi'_{2n}(0) = 1$, which means the Green's function of L_n is

$$G_n(x, y) = \phi_{1n}(y)\phi_{2n}(x) - \phi_{1n}(x)\phi_{2n}(y).$$

Slater [5, Page 68] gives the following asymptotic formula of Tricomi for the confluent hypergeometric function:

$${}_1F_1(a; b; x) = \Gamma(b)(kx)^{(1-b)/2}e^{x/2}J_{b-1}\left(2\sqrt{kx}\right)\left[1 + O\left(\frac{1}{\sqrt{k}}\right)\right],$$

in which $a \in \mathfrak{R}$, $b > 0$, $x \ge 0$ and $k = b/2 - a$. As usual, J_ν denotes the νth order Bessel function of the first kind.

As

$$J_{-1/2}(y) = \left(\frac{2}{\pi y}\right)^{1/2} \cos y \quad \text{and} \quad J_{1/2}(y) = \left(\frac{2}{\pi y}\right)^{1/2} \sin y,$$

the principal term in $G_n(x, y)$ will be

$$\cos\left(\sqrt{2n+1}y\right)\frac{\sin\left(\sqrt{2n+1}x\right)}{\sqrt{2n+1}} - \cos\left(\sqrt{2n+1}x\right)\frac{\sin\left(\sqrt{2n+1}y\right)}{\sqrt{2n+1}} = \frac{\sin\left(\sqrt{2n+1}(x-y)\right)}{\sqrt{2n+1}};$$

more precisely, for $|x|, |y| \le A$, $A > 0$ being fixed,

$$G_n(x, y) = \frac{\sin\left(\sqrt{2n+1}(x-y)\right)}{\sqrt{2n+1}}\left[1 + O\left(\frac{1}{\sqrt{n}}\right)\right]. \tag{3.1}$$

4. The proof of Theorem 1.1

According to (1.6) and (3.1), for $|x| \le A$, $A > 0$ being fixed,

$$e_n(x) = \frac{r_n(x)}{n^{7/4}}\left[1 + O\left(\frac{1}{\sqrt{n}}\right)\right],$$

where

$$r_n(x) := n^{7/4}\int_0^x \frac{\sin\left(\sqrt{2n+1}(x-y)\right)}{\sqrt{2n+1}} f_n(y)\, dy.$$

In view of (2.1), again, if $|x| \le A$,

$$|r_n(x)| \le \frac{n^{7/4}}{\sqrt{2n+1}} 2^{-3/4}\sqrt{35}\frac{c_n}{n} A < 2^{-5/4}\sqrt{35} n^{1/4} c_n A.$$

However, Stirling's formula in the form

$$\Gamma(x) = x^{x-1/2}e^{-x-1}\sqrt{2\pi}\exp\left(\frac{\theta}{12x}\right), \quad 0 < \theta < 1,$$

(see Whittaker and Watson [7, Page 253]) yields

$$c_n \le \sqrt{\frac{2}{\pi}}\exp\left(\frac{1}{24(n+1)}\right)\frac{1}{n^{1/4}}.$$

Hence, for $n \ge 5$,

$$|e_n(x)| \le \frac{2^{-3/4}}{n^{7/4}}\sqrt{\frac{35}{\pi}}\exp\left(\frac{1}{144}\right) A\left[1 + O\left(\frac{1}{\sqrt{n}}\right)\right] \le \frac{2A}{n^{7/4}}\left[1 + O\left(\frac{1}{\sqrt{n}}\right)\right]. \qquad \square$$

5. Hermite series

As is well known, any square-integrable function f on $\mathfrak{R}$ can be represented in terms of its Hermite series. That is,

$$f = \sum_{n=0}^{\infty} \langle f, h_n \rangle \, h_n \quad \text{with} \quad \langle f, h_n \rangle = \int_{-\infty}^{\infty} f(x) h_n(x) \, dx, \tag{5.1}$$

the convergence in (5.1) being both almost everywhere and in the mean of order two.

For the purpose of computation we will replace the square-integrable function f by $f_A := f \chi_{(-A,A)}$, $A > 0$. The assumption is that f has its essentially compact support contained in $(-A, A)$. This can be gauged by how small $\int_{|x|\geq A} |f(x)|^2 \, dx$ is.

Next, we observe that one can compute h_n and $\langle f, h_n \rangle$ quickly and accurately, using, say, the formula (1.1) for h_n, only when $n \leq N$, with N somewhat less than 100. Fortunately, when $n > N$, d_n and $\langle f, d_n \rangle$ are readily calculated and they approximate h_n and $\langle f, h_n \rangle$ extremely well.

We will use the root-mean-square norm

$$M_2(g; A) = \left[\frac{1}{2A} \int_{-A}^{A} |g(x)|^2 \, dx \right]^{1/2}$$

to measure how well

$$\sum_{n=0}^{N} \langle f_A, h_n \rangle \, h_n + \sum_{n=N+1}^{\infty} \langle f_A, d_n \rangle \, d_n$$

approximates f. Applying Theorem 1.1 we obtain the following theorem.

THEOREM 5.1. *Suppose f is square-integrable on $\mathfrak{R}$ and set $f_A := f \chi_{(-A,A)}$ for a chosen $A > 0$. Given $N \in Z_+$, $N^2 \gg A$, consider the approximation*

$$D_N f := \sum_{n=0}^{N} \langle f_A, h_n \rangle \, h_n + \sum_{n=N+1}^{\infty} \langle f_A, d_n \rangle \, d_n \tag{5.2}$$

to the Hermite series of f_A. Then, for $|x| \leq A$,

$$M_2\left(\sum_{n=0}^{\infty} \langle f_A, h_n \rangle \, h_n - D_N f; \, A \right) \leq \left[\frac{c_1(A)}{N^{5/4}} + \frac{c_2(A)}{N^{5/2}} \right] \left[1 + O\left(\frac{1}{\sqrt{N}} \right) \right], \tag{5.3}$$

where

$$c_1(A) = \frac{2}{\sqrt{5}} A^{1/2} \, \|f_A\|_1 + 2\sqrt{\frac{2}{5}} A \left(\|f_A\|_2^2 - \sum_{n=0}^{N} |\langle f_A, h_n \rangle|^2 \right)^{1/2},$$

$$c_2(A) = \frac{8}{5} A^2 \, \|f_A\|_1 \, ,$$

$$\|f_A\|_2 = \left[\int_{-A}^{A} |f(x)|^2 \, dx \right]^{1/2} \quad \text{and} \quad \|f_A\|_1 = \int_{-A}^{A} |f(x)| \, dx.$$

PROOF. One readily shows

$$\sum_{n=0}^{\infty} \langle f_A, h_n\rangle\, h_n - D_N f = \sum_{n=N+1}^{\infty} \langle f_A, e_n\rangle\, h_n + \sum_{n=N+1}^{\infty} \langle f_A, h_n\rangle\, e_n - \sum_{n=N+1}^{\infty} \langle f_A, e_n\rangle\, e_n.$$

Now,

$$\begin{aligned}
\frac{1}{2A}\int_{-A}^{A} \left|\sum_{n=N+1}^{\infty} \langle f_A, e_n\rangle\, h_n(x)\right|^2 dx &\le \frac{1}{2A}\int_{-\infty}^{\infty} \left|\sum_{n=N+1}^{\infty} \langle f_A, e_n\rangle\, h_n(x)\right|^2 dx \\
&\le \frac{1}{2A}\sum_{n=N+1}^{\infty} |\langle f_A, e_n\rangle|^2 \quad \text{(by Parseval's theorem)} \\
&\le \frac{1}{2A}\sum_{n=N+1}^{\infty} \frac{4A^2\,\|f_A\|_1^2}{n^{7/2}}\left[1+O\left(\frac{1}{\sqrt{n}}\right)\right] \\
&\le \frac{4A}{5}\,\|f_A\|_1^2\,\frac{1}{N^{5/2}}\left[1+O\left(\frac{1}{\sqrt{N}}\right)\right].
\end{aligned}$$

Again,

$$\begin{aligned}
&\frac{1}{2A}\int_{-A}^{A} \left|\sum_{n=N+1}^{\infty} \langle f_A, h_n\rangle\, e_n(x)\right|^2 dx \\
&\quad\le \frac{1}{2A}\int_{-A}^{A} \left(\sum_{n=N+1}^{\infty} |\langle f_A, h_n\rangle|^2\right)\left(\sum_{n=N+1}^{\infty} |e_n(x)|^2\right) dx \\
&\quad\le \left(\|f_A\|_2^2 - \sum_{n=0}^{N} |\langle f_A, h_n\rangle|^2\right)^{1/2} \left(\frac{1}{2A}\int_{-A}^{A}\left(\sum_{n=N+1}^{\infty} |e_n(x)|^2\right)^2 dx\right)^{1/2} \\
&\quad\le \left(\|f_A\|_2^2 - \sum_{n=0}^{N} |\langle f_A, h_n\rangle|^2\right)^{1/2} \sum_{n=N+1}^{\infty} \frac{4A^2}{n^{7/2}}\left[1+O\left(\frac{1}{\sqrt{n}}\right)\right] \\
&\quad\le \frac{8}{5}A^2\left(\|f_A\|_2^2 - \sum_{n=0}^{N} |\langle f_A, h_n\rangle|^2\right)^{1/2} \frac{1}{N^{5/2}}\left[1+O\left(\frac{1}{\sqrt{N}}\right)\right].
\end{aligned}$$

Finally,

$$\begin{aligned}
&\frac{1}{2A}\int_{-A}^{A} \left|\sum_{n=N+1}^{\infty} \langle f_A, e_n\rangle\, e_n(x)\right|^2 dx \\
&\quad\le \frac{1}{2A}\int_{-A}^{A} \left[\sum_{n=N+1}^{\infty} |\langle f_A, e_n\rangle|\, |e_n(x)|\right]^2 dx
\end{aligned}$$

$$\le \frac{1}{2A}\int_{-A}^{A}\left[\sum_{n=N+1}^{\infty}\frac{2A\,\|f_A\|_1}{n^{7/4}}\frac{2A}{n^{7/4}}\right]^2 dx\left[1+O\left(\frac{1}{\sqrt{N}}\right)\right]$$
$$\le 16A^4\,\|f_A\|_1^2\left[\sum_{n=N+1}^{\infty}\frac{1}{n^{7/4}}\right]^2\left[1+O\left(\frac{1}{\sqrt{N}}\right)\right]$$
$$\le 16A^4\,\|f_A\|_1^2\left[\frac{2}{5}\frac{1}{N^{5/2}}\right]^2\left[1+O\left(\frac{1}{\sqrt{N}}\right)\right].$$

We conclude from Minkowski's inequality that

$$M_2\left(\int_{-A}^{A}\sum_{n=0}^{\infty}\langle f_A, h_n\rangle\, h_n - D_N f;\ A\right)\le\left[\frac{c_1(A)}{N^{5/4}}+\frac{c_2(A)}{N^{5/2}}\right]\left[1+O\left(\frac{1}{\sqrt{N}}\right)\right],$$

as asserted. □

EXAMPLE 1. We study the Hermite series approximation of the trimodal density function treated in Härdle *et al.* [3, Pages 176–181], namely,

$$f(x)=0.5\phi(x)+3\phi(10(x-0.8))+2\phi(10(x-1.2)), \tag{5.4}$$

in which

$$\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2},\qquad -\infty<x<\infty,$$

is the standard normal density. As is seen from its graph in Figure 1, f is essentially supported in $[-3, 3]$. Now, the graphs of f and $\sum_{n=0}^{40}\langle f, h_n\rangle h_n$ in Figure 2(a) show that many more than 40 terms of its Hermite series are required to accurately represent f. Accordingly, we take $N = 40$ in (5.2) and sum the second series there from $n = 41$ to $n = 500$ to get the asymptotic Hermite approximation to f, shown in Figure 2(b) to be almost indistinguishable from the trimodal density function given in Figure 1.

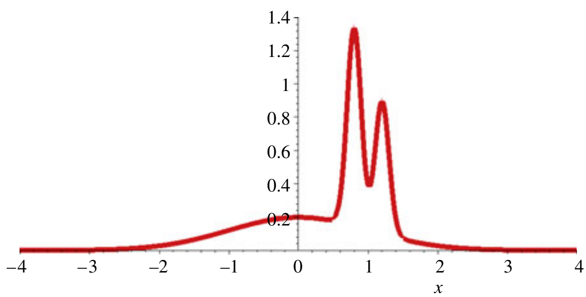

FIGURE 1. Trimodal density function.

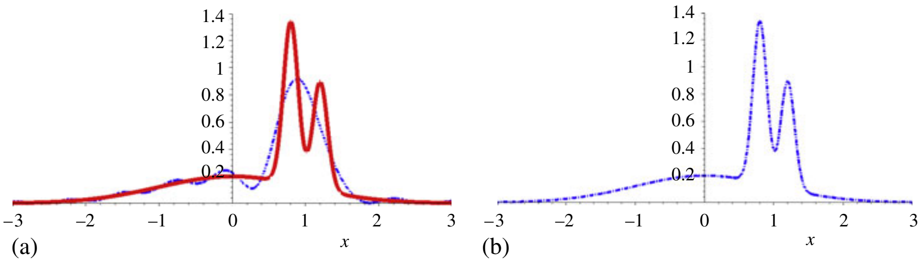

FIGURE 2. Approximations to the trimodal density function. (a) Hermite series approximation with 40 terms, (b) an approximation using a Hermite series for the first 40 terms and the Dominici approximation in the next 460 terms. The solid line shows the trimodal density function, the dashed line shows the approximation. In (b) the trimodal density function is not shown as it is visually indistinguishable from the approximation.

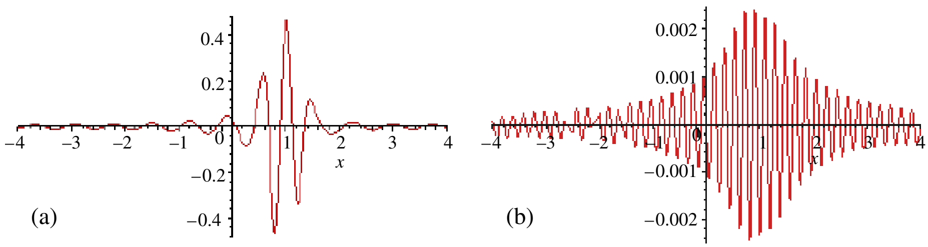

FIGURE 3. Error function of (a) Hermite series approximation with 40 terms and (b) the Hermite series approximation up to 40 terms with the Dominici approximation in the next 460 terms.

Figures 3(a) and (b) display the error function involved when approximating the trimodal density by using Hermite series up to 40 terms and the above-described asymptotic Hermite series, respectively. In connection with Figure 3(b), we observe that the error estimate of (5.3), when $A = 3$ and $N = 40$, is 0.02689, while the absolute maximum error is about 0.0025.

6. Fourier transforms

Let f be both absolutely integrable and square-integrable on $\mathfrak{R}$. Following Wiener [8], we define $\widehat{f}$, the Fourier transform of f, by

$$\widehat{f}(\lambda) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\lambda x}\, dx, \qquad \lambda \in \mathfrak{R}.$$

Recall that with this definition, $\widehat{h_n}(\lambda) = (-i)^n h_n(\lambda)$, and, hence,

$$\widehat{f}(\lambda) = \sum_{n=0}^{\infty} (-i)^n \langle f, h_n \rangle\, h_n(\lambda), \qquad \lambda \in \mathfrak{R}.$$

This suggests approximating $\widehat{f}$ by an asymptotic Hermite series.

We choose $A > 0$ in Theorem 6.1 below so that f is essentially supported in $(-A, A)$. Indeed, we replace f by $f_A := f\chi_{(-A,A)}$, provided that

$$\int_{-\infty}^{\infty} \left|\widehat{f}(\lambda) - \widehat{f_A}(\lambda)\right|^2 d\lambda = \int_{-\infty}^{\infty} |f(x) - f_A(x)|^2\, dx = \int_{|x|\geq A} |f(x)|^2\, dx$$

is as small as deemed necessary. The constant $B > 0$ in the theorem is selected after the approximation to $\widehat{f_A}$ has been computed.

THEOREM 6.1. *Suppose f is square-integrable on $\mathfrak{R}$ and set $f_A := f\chi_{(-A,A)}$ for a chosen $A > 0$. Given $N \in Z_+$, $N \gg A^2$, consider the approximation to $\widehat{f_A}(\lambda)$, $\lambda \in \mathfrak{R}$,*

$$\widehat{F}_{A,N}(\lambda) := \sum_{n=0}^{N} (-i)^n \langle f_A, h_n\rangle\, h_n(\lambda) + \sum_{n=N+1}^{\infty} (-i)^n \langle f_A, d_n\rangle\, d_n(\lambda).$$

Then,

$$M_2\left(\widehat{f_A} - \widehat{F}_{A,N};\ B\right) \leq \left[\frac{c_1(A, B)}{N^{5/4}} + \frac{c_2(A, B)}{N^{5/2}}\right]\left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right],$$

where

$$c_1(A, B) := \frac{2A}{\sqrt{5B}}\, \|f_A\|_1 + 2\sqrt{\frac{2}{5}}B\sqrt{\|f_A\|_2^2 - \sum_{n=0}^{N} |\langle f_A, h_n\rangle|^2} \quad \textit{and}$$

$$c_2(A, B) := \frac{8}{5} AB\, \|f_A\|_1\, .$$

PROOF. The proof is similar to that of Theorem 5.1 and is omitted. □

EXAMPLE 2. Let f be the trimodal density function from Example 1. Figures 4(a)–(c) display the graphs of Re $\widehat{F}_{A,N}$, Im $\widehat{F}_{A,N}$ and $|\widehat{F}_{A,N}|$, respectively, for $A = 3$, $N = 40$. They indicate that $\widehat{f_A}$ is essentially supported in $(-8, 8)$, for which interval

$$M_2\left(\widehat{f_A} - \widehat{F}_{A,N}; 8\right) \leq 0.0401.$$

The graph of $|\widehat{f_A}(\lambda) - \widehat{F}_{A,N}(\lambda)|$ in Figure 5 reveals the actual maximum absolute error is approximately 0.0003.

7. Hermite density estimation

One method of estimating an unknown density function f involves the use of orthogonal expansions. In particular, Hermite series were used in this connection by Schwartz [4] and Walter [6].

The idea is to suppose the density function f is square-integrable on $\mathfrak{R}$, with Hermite expansion $f(x) = \sum_{n=0}^{\infty} \langle f, h_n\rangle\, h_n(x)$. One then takes a sequence

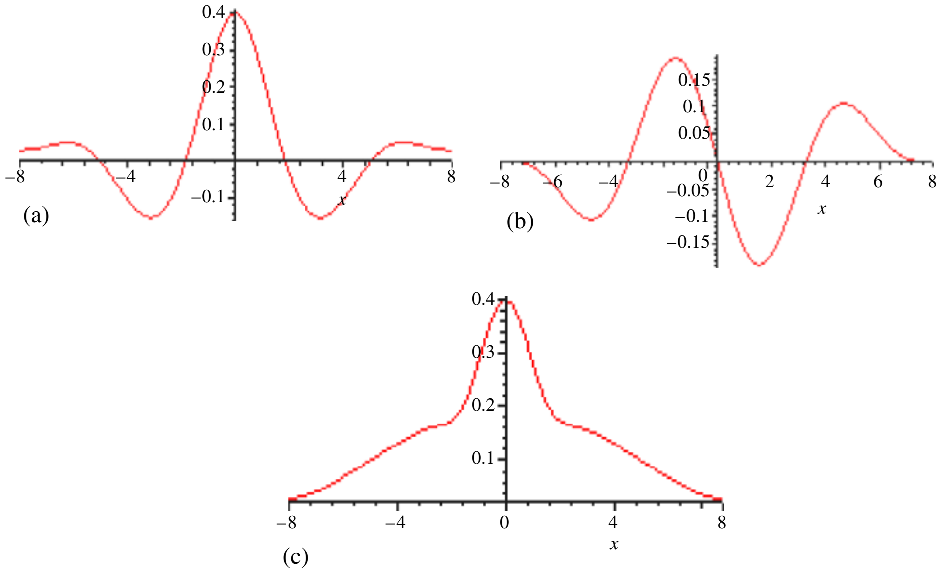

FIGURE 4. The (a) real part, (b) imaginary part and (c) absolute value of the approximate Fourier transform of the trimodal density.

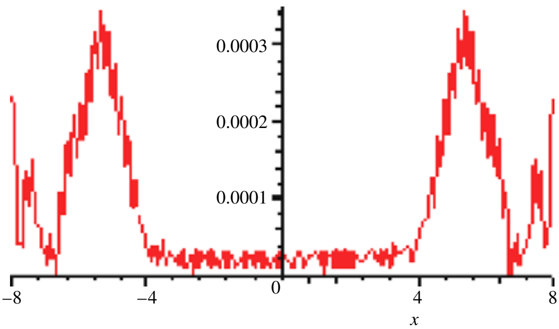

FIGURE 5. Absolute value of the error of the approximate Fourier transform of the trimodal density.

of m independent identically distributed random samples $X_1, X_2, \ldots, X_m$ from the population random variable X with density f and computes the sums

$$E_{n,m} := \frac{1}{m} \sum_{i=1}^{m} h_n(X_i).$$

Now, the law of large numbers ensures that, almost surely,

$$\lim_{m \to \infty} E_{n,m} = \text{Expected value of } h_n(X) \equiv \int_{-\infty}^{\infty} h_n(x) f(x)\, dx = \langle f, h_n \rangle.$$

This leads to the following definition.

DEFINITION 3. The Hermite series density estimate obtained with a sequence of random samples from a population with square-integrable density f is given by

$$f_{\mathrm{HS}}(x; m) := \sum_{n=1}^{q(m)} \left(\frac{1}{m} \sum_{i=1}^{m} h_n(X_i) \right) h_n(x), \quad m = 1, 2, \ldots . \tag{7.1}$$

In (7.1), $\{q(m)\}$ is an increasing sequence of positive integers satisfying $q(m)/m \to 0$ as $m \to \infty$.

To avoid the numerical problems associated with computing high-degree Hermite polynomials, we make a further definition.

DEFINITION 4. Suppose the vast majority of the sample values of Definition 3 lie in $(-A, A)$, $A > 0$, and let $N \in Z^+$ be such that $A^2 \ll N < q(m)$. The asymptotic Hermite series estimate of the density f is defined to be

$$\begin{aligned} f_{\mathrm{AHS}}(x; m) :&= \sum_{n=1}^{N} \left(\frac{1}{m} \sum_{i=1}^{m} h_n(X_i) \right) h_n(x) \\ &\quad + \sum_{n=N+1}^{q(m)} \left(\frac{1}{m} \sum_{i=1}^{m} d_n(X_i) \right) d_n(x), \quad m = 1, 2, \ldots . \end{aligned} \tag{7.2}$$

EXAMPLE 5. We illustrate our method with the trimodal density function again. A Monte Carlo simulation (with MAPLE11 software) of the distribution defined by (5.4) was used to generate $m = 1000$ random samples. The graph of f_{HS} in (7.1), when $m = 1000$, $q(m) = 64$ is shown, together with the density, in Figure 6(a).

Taking $N = 40$ in (7.2) and summing the second series from $n = 41$ to 500 yields for f_{AHS} in Figure 6(b) a graph that, except for the tails, fits the true density f well.

The error functions involved in the two approximations appear in Figure 7(a) and (b). In the first case the maximum absolute error is about 0.4, while in the second case it is about 0.15.

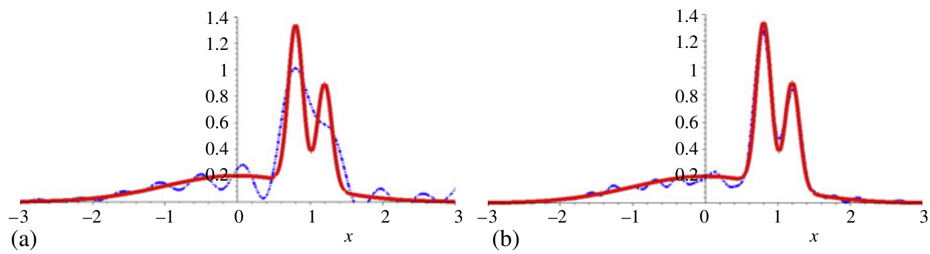

FIGURE 6. Sample density estimation using (a) a Hermite series of 64 terms and (b) the Dominici approximation on the next 460 terms. In each case the solid line shows the trimodal density function, the dashed line shows the approximation.

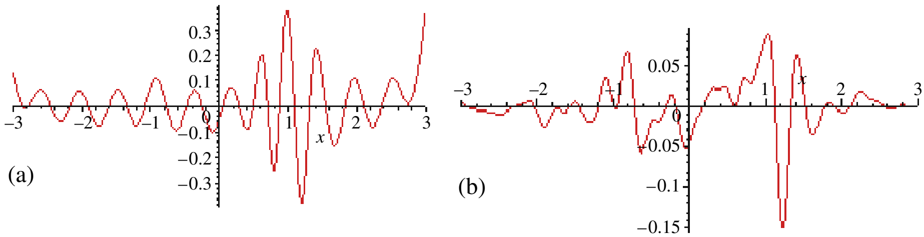

FIGURE 7. Error function of the density estimate using (a) Hermite series and (b) the Dominici approximation.

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