# LIPSCHITZ SPACES AND BOUNDED MEAN OSCILLATION OF HARMONIC MAPPINGS

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## Abstract

We first study the bounded mean oscillation of planar harmonic mappings. Then we establish a relationship between Lipschitz-type spaces and equivalent modulus of real harmonic mappings. Finally, we obtain sharp estimates on the Lipschitz number of planar harmonic mappings in terms of the bounded mean oscillation norm, which shows that the harmonic Bloch space is isomorphic to $BMO_2$ as a Banach space.

## 1. Introduction and main results

Let $\mathbb{C}$ denote the complex plane. For $a \in \mathbb{C}$, let $\mathbb{D}(a, r) = \{z : |z - a| < r\}$. In particular, we use $\mathbb{D}_r$ to denote the disc $\mathbb{D}(0, r)$ and $\mathbb{D}$ the unit disc $\mathbb{D}_1$. A complex-valued function $f$ defined on $\mathbb{D}$ is called *harmonic* in $\mathbb{D}$ if and only if both the real and imaginary parts of $f$ are real harmonic in $\mathbb{D}$. It is known that every harmonic mapping $f$ defined in $\mathbb{D}$ admits a decomposition $f = h + \overline{g}$, where $h$ and $g$ are analytic in $\mathbb{D}$. We refer to [10, 12, 13, 19, 34] for the theory of planar harmonic mappings. For harmonic mappings $f$ defined on $\mathbb{D}$, we use the following standard notation:

$$\Lambda_f(z) = \max_{0 \le \theta \le 2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = |f_z(z)| + |f_{\bar{z}}(z)|$$

and

$$\lambda_f(z) = \min_{0 \le \theta \le 2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = ||f_z(z)| - |f_{\bar{z}}(z)||.$$

A continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ is called a *majorant* if $\omega(t)/t$ is nonincreasing for $t > 0$ (see [14, 28]). Given a subset $\Omega$ of $\mathbb{C}$, a

function $f : \Omega \to \mathbb{C}$ is said to belong to the *Lipschitz space $L_\omega(\Omega)$* if there is a positive constant $M$ such that

$$|f(z) - f(w)| \le M\omega(|z - w|) \quad \text{for all } z, w \in \Omega. \tag{1.1}$$

For $\delta_0 > 0$ and $0 < \delta < \delta_0$, we consider the following conditions on a majorant $\omega$:

$$\int_0^\delta \frac{\omega(t)}{t} \, dt \le M\omega(\delta) \tag{1.2}$$

and

$$\delta \int_\delta^{+\infty} \frac{\omega(t)}{t^2} \, dt \le M\omega(\delta), \tag{1.3}$$

where $M$ denotes a positive constant. A majorant $\omega$ is said to be *regular* if it satisfies (1.2) and (1.3) (see [14, 28]).

Dyakonov [14] discussed the relationship between Lipschitz space and bounded mean oscillation on holomorphic functions in $\mathbb{D}$, and obtained the following result. In order to state Theorem A, we first introduce some notation. Let $G$ be a domain of $\mathbb{C}$. We use $d_G(z)$ to denote the Euclidean distance from $z$ to the boundary $\partial G$ of $G$. In particular, we always use $d(z)$ to denote the Euclidean distance from $z$ to the boundary of $\mathbb{D}$.

THEOREM A [14, Theorem 1]. *Suppose that $f$ is a holomorphic function in $\mathbb{D}$ which is continuous up to the boundary of $\mathbb{D}$. If $\omega$ and $\omega^2$ are regular majorants, then*

$$f \in L_\omega(\mathbb{D}) \iff P_{|f|^2}(z) - |f(z)|^2 \le M\omega^2(d(z)),$$

*where*

$$P_{|f|^2}(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} |f(e^{i\theta})|^2 \, d\theta.$$

The following result is an analogue of Theorem A for planar harmonic mappings.

THEOREM 1.1. *Suppose that $\omega$ is a majorant and that $f$ is a harmonic mapping in $\mathbb{D}$. Then $\Lambda_f(z) \le M\omega(1/d(z))$ in $\mathbb{D}$ if and only if, for every $r \in (0, 1 - |z|]$,*

$$\frac{1}{|\mathbb{D}(z, r)|} \int_{\mathbb{D}(z,r)} |f(\zeta) - f(z)| \, dA(\zeta) \le Mr\omega\left(\frac{1}{r}\right),$$

*where $dA$ denotes the area measure in $\mathbb{D}$.*

DEFINITION 1.2. Let $f$ be harmonic in $\mathbb{D}$. For $p \in [1, \infty)$, we say that $f \in BMO_p$ if

$$\|f\|_{BMO_p} = \sup_{\mathbb{D}(z,r) \subseteq \mathbb{D}} \left( \frac{1}{|\mathbb{D}(z, r)|} \int_{\mathbb{D}(z,r)} \left| f(\zeta) - \frac{1}{|\mathbb{D}(z, r)|} \int_{\mathbb{D}(z,r)} f(\xi) \, dA(\xi) \right|^p dA(\zeta) \right)^{1/p}$$

is bounded, where $r \in (0, 1 - |z|]$.

In particular, by taking $\omega(t) = t$ in Theorem 1.1, we get the following result.

COROLLARY 1.3. *Let $f$ be a harmonic mapping in $\mathbb{D}$. Then $f \in BMO_1$ if and only if $\Lambda_f(z) \le M/d(z)$ holds in $\mathbb{D}$.*

In [14], Dyakonov also investigated the property of equivalent modulus for holomorphic functions in $\mathbb{D}$ and obtained the following theorem.

THEOREM B [14, Theorem 2]. *Let $\omega$ be a regular majorant and $f$ be a holomorphic function in $\mathbb{D}$ and continuous up to the boundary $\partial\mathbb{D}$. Then*

$$f \in L_\omega(\mathbb{D}) \Longleftrightarrow |f| \in L_\omega(\mathbb{D}) \Longleftrightarrow |f| \in L_\omega(\mathbb{D}, \partial\mathbb{D}),$$

*where $L_\omega(\mathbb{D}, \partial\mathbb{D})$ denotes the class of continuous functions $F$ on $\mathbb{D} \cup \partial\mathbb{D}$ which satisfy (1.1) with some positive constant $C$, whenever $z \in \mathbb{D}$ and $w \in \partial\mathbb{D}$.*

Later, in [28, Theorems A], Pavlović came up with a relatively simple proof of the results of Dyakonov. Recently, many authors have considered this topic and generalised Dyakonov's results to quasiconformal mappings and real harmonic functions in several variables for some special majorant $\omega(t) = t^\alpha$, where $\alpha > 0$ (see [1, 15, 23, 26, 27, 29–31]). For the general majorant $\omega$ to holomorphic mappings and pluriharmonic mappings in the unit ball, see [7, 15, 33].

We will prove the analogue of Theorem B for real harmonic functions in the following form.

THEOREM 1.4. *Suppose that $\omega$ is a majorant satisfying (1.2), and that $G$ is an $L_\omega$-extension domain. If $f$ is a real harmonic function in $G$ and continuous up to the boundary $\partial G$, then*

$$f \in L_\omega(G) \Longleftrightarrow |f| \in L_\omega(G) \Longleftrightarrow |f| \in L_\omega(G, \partial G),$$

*where $L_\omega(G, \partial G)$ denotes the class of continuous functions $F$ on $G \cup \partial G$ which satisfy (1.1) with some positive constant $C$, whenever $z \in G$ and $w \in \partial G$.*

Here a proper subdomain $G$ of $\mathbb{C}$ or $\mathbb{R}^2$ is said to be an $L_\omega$-extension if $L_\omega(G) = \mathrm{loc}L_\omega(G)$, where $\mathrm{loc}L_\omega(G)$ denotes the set of all functions $f : G \to \mathbb{C}$ satisfying (1.1) with a fixed positive constant $M$, whenever $z \in G$ and $w \in G$ such that $|z - w| < \frac{1}{2}d_G(z)$. Obviously, the unit disc $\mathbb{D}$ is an $L_\omega$-extension domain.

In [25], the author proved that $G$ is an $L_\omega$-extension domain if and only if each pair of points $z, w \in G$ can be joined by a rectifiable curve $\gamma \subset G$ satisfying

$$\int_\gamma \frac{\omega(d_G(z))}{d_G(z)} \, ds(z) \le M\omega(|z - w|) \tag{1.4}$$

with some fixed positive constant $M = M(G, \omega)$, where $ds$ stands for the arc length measure on $\gamma$. See [17, 25] for more details on $L_\omega$-extension domains.

We remark that in Theorem 1.4, we replace 'the unit disc $\mathbb{D}$' and 'the regular majorant' in Theorem B by 'an $L_\omega$-extension domain' and 'a majorant satisfying (1.2), but not necessarily (1.3)', respectively. In fact, by using [30, Lemma A, Theorem 4,

and Corollary 2] and the similar proof method of Theorem 1.4, we can prove that Theorem 1.4 also holds for real harmonic functions in the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$.

For planar harmonic mappings, we obtain the following result which is a generalisation of Theorem B.

THEOREM 1.5. *Let $\omega$ be a majorant satisfying (1.2) and $G$ be an $L_\omega$-extension domain. Let $f = h + \overline{g}$ be a harmonic mapping in $G$, where $g$ and $h$ are analytic functions in $G$. Then*

$$f \in L_\omega(G) \Longleftrightarrow g, h \in L_\omega(G) \Longleftrightarrow |g|, |h| \in L_\omega(G).$$

DEFINITION 1.6. A planar harmonic mapping $f$ in $\mathbb{D}$ is called a *harmonic Bloch mapping* if

$$\beta_f = \sup_{z,w \in \mathbb{D}, z \neq w} \frac{|f(z) - f(w)|}{\rho(z, w)} < \infty.$$

Here $\beta_f$ is called the *Lipschitz number* of $f$ and

$$\rho(z, w) = \frac{1}{2} \log\left(\frac{1 + |\frac{z-w}{1-\overline{z}w}|}{1 - |\frac{z-w}{1-\overline{z}w}|}\right) = \operatorname{arctanh}\left|\frac{z - w}{1 - \overline{z}w}\right|$$

denotes the hyperbolic distance between $z$ and $w$ in $\mathbb{D}$.

It is known that

$$\beta_f = \sup_{z \in \mathbb{D}}\{(1 - |z|^2)\Lambda_f(z)\}.$$

Clearly, a harmonic Bloch mapping $f$ is uniformly continuous as a map between metric spaces

$$f : (\mathbb{D}, \rho) \rightarrow (\mathbb{C}, |\cdot|)$$

and for all $z, w \in \mathbb{D}$ we have the Lipschitz inequality

$$|f(z) - f(w)| \leq \beta_f \, \rho(z, w).$$

The reader is referred to [12, Theorem 2] (or [2, 3, 8]) for a proof. Then the set of all harmonic Bloch mappings in $\mathbb{D}$ forms a harmonic Bloch space which is denoted by $\mathcal{B}_h$. Uniform continuity with respect to a hyperbolic metric is a central theme in [35, 36].

In [9, 20, 32], the authors provided several characterisations of $BMO_2$ on holomorphic functions. For extensive discussions on $BMO_2$, see [11, 16, 18, 21, 24]. In this paper, we will use the $BMO_2$ norm to obtain a sharp estimate on harmonic Bloch mappings, which shows that $\mathcal{B}_h$ is isomorphic to $BMO_2$ as a Banach space. Our result is given below.

THEOREM 1.7. *If $f$ is harmonic in $\mathbb{D}$, then*

$$\|f\|_{BMO_2} \leq \beta_f \leq 2\|f\|_{BMO_2}. \tag{1.5}$$

*Moreover, the estimates of (1.5) are sharp. The extreme harmonic mappings of the first inequality are constant functions, and the extreme harmonic mappings of the second inequality are the mappings with the form $f(z) = C(z + \overline{z})$, where $C$ denotes a constant.*

The proofs of Theorems 1.1 and 1.4 will be presented in Section 2, and the proof of Theorem 1.7 will be given in Section 3.

## 2. Bounded mean oscillation and equivalent modulus

The following lemma easily follows from a simple computation (as in [6]).

LEMMA 2.1. *Let f be a complex-valued continuously differentiable function defined on* $\mathbb{D}$ *and* $f = u + iv$, *where u and v are real-valued functions. Then for* $z = x + iy \in \mathbb{D}$,

$$\Lambda_f(z) \leq |\nabla u(x, y)| + |\nabla v(x, y)|,$$

*where* $\nabla u = (u_x, u_y)$ *and* $\nabla v = (v_x, v_y)$.

Then we have the following lemma.

LEMMA 2.2. *Suppose that f is a harmonic mapping in* $\overline{\mathbb{D}}(a, r)$, *where r is a positive constant. Then*

$$\Lambda_f(a) \leq \frac{2}{\pi r} \int_0^{2\pi} |f(a) - f(a + re^{i\theta})| \, d\theta.$$

PROOF. Let $f = u + iv$ be a harmonic mapping in $\overline{\mathbb{D}}(a, r)$, where $u$ and $v$ are real harmonic functions. Without loss of generality, we may assume that $a = 0$ and $f(0) = 0$. By Poisson's formula,

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} u(re^{i\theta}) \, d\theta, \quad |z| < r.$$

By calculations, we get ($z = x = iy$)

$$u_x(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{-2x|z - re^{i\theta}|^2 - 2(r^2 - |z|^2)(x - r\cos\theta)}{|z - re^{i\theta}|^4} u(re^{i\theta}) \, d\theta$$

and similarly

$$u_y(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{-2y|z - re^{i\theta}|^2 - 2(r^2 - |z|^2)(y - r\sin\theta)}{|z - re^{i\theta}|^4} u(re^{i\theta}) \, d\theta,$$

which imply that

$$
\begin{aligned}
|\nabla u(0)| &= \left( \left| \frac{1}{r\pi} \int_0^{2\pi} u(re^{i\theta}) \cos\theta \, d\theta \right|^2 + \left| \frac{1}{r\pi} \int_0^{2\pi} u(re^{i\theta}) \sin\theta \, d\theta \right|^2 \right)^{1/2} \\
&\leq \frac{1}{r\pi} \int_0^{2\pi} (|\cos\theta| + |\sin\theta|) |u(re^{i\theta})| \, d\theta \\
&\leq \frac{\sqrt{2}}{r\pi} \int_0^{2\pi} |u(re^{i\theta})| \, d\theta.
\end{aligned}
\tag{2.1}
$$

A similar argument shows that

$$|\nabla v(0)| \leq \frac{\sqrt{2}}{r\pi} \int_0^{2\pi} |v(re^{i\theta})| \, d\theta. \tag{2.2}$$

By (2.1), (2.2) and Lemma 2.1,

$$\Lambda_f(0) \leq |\nabla u(0)| + |\nabla v(0)|$$

$$\leq \frac{\sqrt{2}}{r\pi} \int_0^{2\pi} (|u(re^{i\theta})| + |v(re^{i\theta})|) \, d\theta$$

$$\leq \frac{2}{r\pi} \int_0^{2\pi} |f(re^{i\theta})| \, d\theta.$$

Finally, the desired conclusion follows if we apply the last inequality to the function $F(z) = f(a) - f(z + a)$.                                                                    □

**2.1. Proof of Theorem 1.1.**    First, we show the 'if' part. By Lemma 2.2,

$$\Lambda_f(z) \leq \frac{2}{\pi\rho} \int_0^{2\pi} |f(z) - f(z + \rho e^{i\theta})| \, d\theta,$$

where $\rho \in (0, d(z)]$, which gives

$$\int_0^r \Lambda_f(z)\rho^2 \, d\rho \leq \frac{2}{\pi} \int_0^r \left( \rho \int_0^{2\pi} |f(z) - f(z + \rho e^{i\theta})| \, d\theta \right) d\rho,$$

whence

$$\Lambda_f(z) \leq \frac{6}{\pi r^3} \int_{\mathbb{D}(z,r)} |f(z) - f(\zeta)| \, dA(\zeta)$$

$$= \frac{6}{r|\mathbb{D}(z, r)|} \int_{\mathbb{D}(z,r)} |f(z) - f(\zeta)| \, dA(\zeta)$$

$$\leq \frac{6Mk(r)}{r} = 6M\omega\left(\frac{1}{d(z)}\right),$$

where $r = d(z)$.

Next, we prove the 'only if' part. For $z, w \in \mathbb{D}$ and $t \in (0, 1)$,

$$d(z + t(w - z)) = 1 - |z + t(w - z)| \geq d(z) - t|w - z|.$$

If $d(z) - t|w - z| > 0$, then

$$|f(z) - f(w)| \leq \left| \int_0^1 \frac{df}{dt}(z + t(w - z)) \, dt \right|$$

$$\leq |w - z| \int_0^1 \Lambda_f(z + t(w - z)) \, dt$$

$$\leq M|w - z| \int_0^1 \omega\left(\frac{1}{d(z) - t|w - z|}\right) dt$$

$$= M \int_0^{|w-z|} \omega\left(\frac{1}{d(z) - t}\right) dt.$$

Hence

$$\frac{1}{|\mathbb{D}(z,r)|} \int_{\mathbb{D}(z,r)} |f(\zeta) - f(z)| \, dA(\zeta) \le \frac{M}{|\mathbb{D}_r|} \int_{\mathbb{D}_r} \left( \int_0^{|\xi|} \omega\left(\frac{1}{d(z)-t}\right) dt \right) dA(\xi)$$

$$= \frac{2M}{r^2} \int_0^r \rho\left( \int_0^\rho \omega\left(\frac{1}{d(z)-t}\right) dt \right) d\rho$$

$$\le \frac{2M}{r^2} \int_0^r \left( \int_t^r \rho \, d\rho \right) \omega\left(\frac{1}{r-t}\right) dt$$

$$\le \frac{2M}{r} \int_0^r (r-t) \omega\left(\frac{1}{r-t}\right) dt$$

$$\le \frac{2M}{r} r\omega\left(\frac{1}{r}\right) \int_0^r dt$$

$$= 2Mr\omega\left(\frac{1}{r}\right).$$

The proof of this theorem is complete.                                                          □

   The following result from [22] is needed in the proof of Theorem 1.4.

LEMMA C  [22, Theorem 1]. *Let u be a real harmonic function of $\mathbb{D}$ into $(-1, 1)$. Then for $z \in \mathbb{D}$, the following sharp inequality holds:*

$$|\nabla u(z)| \le \frac{4}{\pi} \frac{1 - u^2(z)}{1 - |z|^2}.$$

**2.2. Proof of Theorem 1.4.**  Without loss of generality, we assume that $f$ is not constant. The implication $f \in L_\omega(G) \Rightarrow |f| \in L_\omega(G) \Rightarrow |f| \in L_\omega(G, \partial G)$ is obvious, and so we only need to prove the implication $|f| \in L_\omega(G) \Rightarrow f \in L_\omega(G)$. For a fixed $z \in G$, let

$$M_z = \sup\{|f(\zeta)| : |\zeta - z| < d_G(z)\}$$

and for $\xi \in \mathbb{D}$,

$$T_f(\xi) = \frac{f(z + d_G(z)\xi)}{M_z}.$$

Obviously, $|T_f(\xi)| < 1$ and thus Lemma C implies that

$$|\nabla T_f(\xi)| \le \frac{4}{\pi}\left(\frac{1 - T_f^2(\xi)}{1 - |\xi|^2}\right),$$

which gives

$$\frac{d_G(z)|\nabla f(z)|}{M_z} = |\nabla T_f(0)| \le \frac{4}{\pi}\left(1 - \frac{f^2(z)}{M_z^2}\right) \le \frac{8}{\pi}\left(1 - \frac{|f(z)|}{M_z}\right),$$

that is,

$$d_G(z)|\nabla f(z)| \le \frac{8}{\pi}(M_z - |f(z)|). \tag{2.3}$$

For a fixed $\varepsilon_0 > 0$, there exists a $\zeta \in \partial G$ such that $|\zeta - z| < (1 + \varepsilon_0)d_G(z)$. Then, for $w \in \mathbb{D}(z, d_G(z))$,

$$|f(w)| - |f(z)| \leq ||f(w)| - |f(\zeta)|| + ||f(\zeta)| - |f(z)||$$
$$\leq M\omega((2 + \varepsilon_0)d_G(z)) + M\omega((1 + \varepsilon_0)d_G(z)).$$

Now we take $\varepsilon_0 = 1$. It follows that

$$\sup_{w \in \mathbb{D}(z,d_G(z))} (|f(w)| - |f(z)|) \leq M(\omega(3d_G(z)) + \omega(2d_G(z))) \leq 5M\omega(d_G(z))$$

whence

$$M_z - |f(z)| \leq 5M\omega(d_G(z)). \tag{2.4}$$

By (2.3) and (2.4), we conclude that

$$|\nabla f(z)| \leq \frac{40M}{\pi} \frac{\omega(d_G(z))}{d_G(z)}. \tag{2.5}$$

Finally, for $z_1, z_2 \in G$, by [25], there must exist a rectifiable curve $\gamma$ in $G$ which joins $z_1$ and $z_2$ and satisfies (1.4). Integrating (2.5) along $\gamma$,

$$|f(z_1) - f(z_2)| \leq \int_\gamma |\nabla f(\zeta)| \, ds(z) \leq \frac{40M}{\pi} \int_\gamma \frac{\omega(d_G(z))}{d_G(z)} \, ds(z) \leq C\omega(|z_1 - z_2|),$$

where $C$ is a constant. The proof of this theorem is complete. $\qquad\square$

PROOF OF THEOREM 1.5. The implication $g, h \in L_\omega(G) \Longleftrightarrow |g|, |h| \in L_\omega(G)$ follows from Theorem B. We only need to prove that $f \in L_\omega(G) \Longrightarrow g, h \in L_\omega(G)$, because the implication $g, h \in L_\omega(G) \Longrightarrow f \in L_\omega(G)$ is obvious. Let $f = h + \overline{g}$ in $G$, where $h$ and $g$ are holomorphic in $G$. It is easy to see that $f \in L_\omega(G) \Longrightarrow \overline{f} \in L_\omega(G)$. This implies that $u = \operatorname{Re} f_1 \in L_\omega(G)$ and $v = \operatorname{Im} f_2 \in L_\omega(G)$, where $f_1 = h + g$ and $f_2 = h - g$.

We claim that $f_1, f_2 \in L_\omega(G)$. We now prove this claim. For a fixed $z \in G$, let

$$M_z = \sup\{|u(\zeta)| : |\zeta - z| < d(z)\} \quad \text{and} \quad T_u(\xi) = \frac{u(z + d(z)\xi)}{M_z}, \quad \xi \in \mathbb{D}.$$

Then for any $\xi \in \mathbb{D}$, $|T_u(\xi)| < 1$ and by Lemma C,

$$|\nabla T_u(\xi)| \leq \frac{4}{\pi}\left(\frac{1 - T_u^2(\xi)}{1 - |\xi|^2}\right).$$

This gives

$$\frac{d(z)|\nabla u(z)|}{M_z} = |\nabla T_u(0)| \leq \frac{4}{\pi}\left(1 - \frac{u^2(z)}{M_z^2}\right) \leq \frac{8}{\pi}\left(1 - \frac{|u(z)|}{M_z}\right),$$

which yields

$$d(z)|f_1'(z)| = d(z)|\nabla u(z)| \leq \frac{8}{\pi}(M_z - |u(z)|). \tag{2.6}$$

For a fixed $\varepsilon_0 > 0$, there exists a $\zeta \in \partial G$ such that $|\zeta - z| < (1 + \varepsilon_0)d_G(z)$. Then, for $w \in \mathbb{D}(z, d_G(z))$,

$$
\begin{aligned}
|u(w)| - |u(z)| &\leq \|u(w)| - |u(\zeta)\| + \|u(\zeta)| - |u(z)\| \\
&\leq M\omega((2 + \varepsilon_0)d_G(z)) + M\omega((1 + \varepsilon_0)d_G(z)).
\end{aligned}
$$

Now we take $\varepsilon_0 = 1$. It follows that

$$
\sup_{w \in \mathbb{D}(z, d_G(z))} (|u(w)| - |u(z)|) \leq M(\omega(3d_G(z)) + \omega(2d_G(z))) \leq 5M\omega(d_G(z))
$$

whence

$$
M_z - |u(z)| \leq 5M\omega(d_G(z)). \tag{2.7}
$$

By (2.6) and (2.7), we conclude that

$$
|f_1'(z)| \leq \frac{40M}{\pi} \frac{\omega(d_G(z))}{d_G(z)}. \tag{2.8}
$$

Finally, for $z_1, z_2 \in G$, by [25], there must exist a rectifiable curve $\gamma$ in $G$ which joins $z_1$ and $z_2$, and satisfies (1.4). Integrating (2.8) along $\gamma$, we obtain that

$$
|f_1(z_1) - f_1(z_2)| \leq \int_\gamma |f_1'(\zeta)| \, ds(z) \leq \frac{40M}{\pi} \int_\gamma \frac{\omega(d_G(z))}{d_G(z)} \, ds(z) \leq C\omega(|z_1 - z_2|),
$$

where $C$ is a constant. This gives $f_1 \in L_\omega(G)$. By similar arguments, we know that $f_2 \in L_\omega(G)$. Hence $(f_1 + f_2) \in L_\omega(G)$ and $(f_1 - f_2) \in L_\omega(G)$. Therefore,

$$
h = \frac{f_1 + f_2}{2} \in L_\omega(G) \quad \text{and} \quad g = \frac{f_1 - f_2}{2} \in L_\omega(G).
$$

The proof of this theorem is complete. $\square$

## 3. Estimates on $BMO_2$

Green's theorem (see [4, 5]) states that if $g \in C^2(\mathbb{D})$, that is, is twice continuously differentiable in $\mathbb{D}$, then

$$
\frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}) \, d\theta = g(0) + \frac{1}{2} \int_{\mathbb{D}_r} \Delta g(z) \log \frac{r}{|z|} \, d\sigma(z) \tag{3.1}
$$

for $r \in (0, 1)$, where $d\sigma$ denotes the normalised area measure in $\mathbb{D}$.

LEMMA 3.1. *For $r \in (0, 1)$, let*

$$
M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta,
$$

*where $f$ is a harmonic mapping in $\mathbb{D}$. Then for $p \in [2, \infty)$, $M_p^p(r, f)$ is an increasing function on $r$ in $(0, 1)$ and*

$$r\frac{d}{dr}M_p^p(r, f) = p \int_{\mathbb{D}_r} \left( \left(\frac{p}{2} - 1\right)|f(z)|^{p-4}|f_z(z)\overline{f(z)} + f(z)\overline{f_{\bar{z}}(z)}|^2 \right.$$
$$\left. + |f(z)|^{p-2}|\widehat{\nabla f}(z)|^2 \right) d\sigma(z),$$

*where $|\widehat{\nabla f}| = (|f_z|^2 + |f_{\bar{z}}|^2)^{1/2}$.*

PROOF. Since $|f|^p$ is subharmonic in $\mathbb{D}$, we see that $M_p^p(r, f)$ is an increasing function on $r$ in $(0, 1)$, where $p \in [2, \infty)$. On the other hand, by (3.1),

$$r\frac{d}{dr}M_p^p(r, f) = \frac{1}{2}\int_{\mathbb{D}_r} \Delta(|f(z)|^p)\, d\sigma(z)$$
$$= p \int_{\mathbb{D}_r} \left( \left(\frac{p}{2} - 1\right)|f(z)|^{p-4}|f_z(z)\overline{f(z)} + f(z)\overline{f_{\bar{z}}(z)}|^2 \right.$$
$$\left. + |f(z)|^{p-2}|\widehat{\nabla f}(z)|^2 \right) d\sigma(z).$$

The proof of this lemma is complete. □

LEMMA 3.2. *For $r \in (0, 1)$ and $p \in [2, \infty)$, let*

$$I_p(r, f) = \left( \frac{1}{|\mathbb{D}_r|} \int_{\mathbb{D}_r} |f(z)|^p\, dA(z) \right)^{1/p},$$

*where $f$ is harmonic in $\mathbb{D}$. Then the function $I_p(r, f)$ is increasing on $r$ in $(0, 1)$.*

PROOF. Since

$$\int_{\mathbb{D}_r} |f(z)|^p dA(z) = 2\pi \int_0^r \rho M_p^p(\rho, f)\, d\rho, \tag{3.2}$$

we see that

$$\frac{d}{dr} \int_{\mathbb{D}_r} |f(z)|^p\, dA(z) = 2\pi r M_p^p(r, f). \tag{3.3}$$

By (3.2), (3.3) and Lemma 3.1,

$$M_p^p(r, f) - I_p^p(r, f) = \frac{1}{|\mathbb{D}_r|} \int_0^r \frac{d}{dt}M_p^p(t, f)|\mathbb{D}_t|\, dt \geq 0. \tag{3.4}$$

By (3.2), (3.4) and elementary computations,

$$\frac{d}{dr}I_p^p(r, f) = \frac{|\mathbb{D}_r|\frac{d}{dr}\int_{\mathbb{D}_r} |f(z)|^p\, dA(z) - \int_{\mathbb{D}_r} |f(z)|^p\, dA(z)\frac{d}{dr}|\mathbb{D}_r|}{|\mathbb{D}_r|^2}$$
$$= \frac{2\pi r\left[|\mathbb{D}_r|M_p^p(r, f) - \int_{\mathbb{D}_r} |f(z)|^p\, dA(z)\right]}{|\mathbb{D}_r|^2}$$
$$\geq 0.$$

Hence the function $I_p(r, f)$ is increasing on $r$ in $(0, 1)$. The proof of this lemma is complete. □

LEMMA 3.3. *For fixed $a \in \mathbb{D}$, let $\phi_a(z) = a + (1 - |a|)z$ in $\mathbb{D}$. Then for $p \in [2, \infty)$,*

$$\|f\|_{BMO_p} = \sup_{a \in \mathbb{D}} \left( \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |f(\phi_a(z)) - f(\phi_a(0))|^p \, dA(z) \right)^{1/p}, \tag{3.5}$$

*where $f$ is harmonic in $\mathbb{D}$.*

PROOF. It is not difficult to see that

$$\sup_{a \in \mathbb{D}} \left( \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |f(\phi_a(z)) - f(\phi_a(0))|^p \, dA(z) \right)^{1/p} \leq \|f\|_{BMO_p}. \tag{3.6}$$

On the other hand, by elementary calculations and Lemma 3.2,

$$\left( \frac{1}{|\mathbb{D}(a, r)|} \int_{\mathbb{D}(a,r)} |f(\zeta) - f(a)|^p dA(\zeta) \right)^{1/p}$$
$$\leq \left( \frac{1}{|\mathbb{D}(a, 1 - |a|)|} \int_{\mathbb{D}(a,1-|a|)} |f(\zeta) - f(a)|^p dA(\zeta) \right)^{1/p}$$
$$= \left( \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |f(\phi_a(\zeta)) - f(\phi_a(0))|^p \, dA(\zeta) \right)^{1/p},$$

where $r \in (0, 1 - |a|]$. Then

$$\|f\|_{BMO_p} \leq \sup_{a \in \mathbb{D}} \left( \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |f(\phi_a(z)) - f(\phi_a(0))|^p \, dA(z) \right)^{1/p}. \tag{3.7}$$

Obviously, (3.5) follows from (3.6) and (3.7). □

LEMMA 3.4. *For each fixed $a \in \mathbb{D}$, let $\phi_a(z) = a + (1 - |a|)z$ in $\mathbb{D}$. Then*

$$|\phi_a'(z)| \leq \frac{1 - |\phi_a(z)|^2}{1 - |z|^2}. \tag{3.8}$$

PROOF. It is easy to see that $f$ is analytic and, for all $z \in \mathbb{D}$, $|\phi_a(z)| \leq 1$. Then (3.8) follows from the Schwarz–Pick lemma. □

**3.1. Proof of Theorem 1.7.** We first prove that $\|f\|_{BMO_2} \leq \beta_f$. For a fixed $a \in \mathbb{D}$, let

$$F_a(\zeta) = f(\phi_a(\zeta))$$

in $\mathbb{D}$, where $\phi_a(\zeta) = a + (1 - |a|)\zeta$. By Lemma 3.4,

$$\sup_{\zeta \in \mathbb{D}} ((1 - |\zeta|^2) \Lambda_{F_a}(\zeta)) = \sup_{\zeta \in \mathbb{D}} ((1 - |\zeta|^2) \Lambda_f(\phi_a(\zeta)) |\phi_a'(\zeta)|)$$
$$\leq \sup_{\zeta \in \mathbb{D}} ((1 - |\phi_a(\zeta)|^2) \Lambda_f(\phi_a(\zeta)))$$
$$\leq \beta_f.$$

Then Lemma 3.1 leads to

$$\frac{d}{dr} M_2^2(r, F_a(re^{i\theta}) - F_a(0)) = \frac{2}{r\pi} \int_{\mathbb{D}_r} |\widehat{\nabla F_a}(\zeta)|^2 \, dA(\zeta)$$

$$\leq \frac{2}{r\pi} \int_{\mathbb{D}_r} \Lambda_{F_a}^2(\zeta) \, dA(\zeta)$$

$$\leq \frac{2\beta_f^2}{r\pi} \int_{\mathbb{D}_r} \frac{dA(\zeta)}{(1 - |\zeta|^2)^2}$$

$$= \frac{4\beta_f^2}{r} \int_0^r \frac{\rho}{(1 - \rho^2)^2} \, d\rho$$

$$= 2\beta_f^2 \sum_{n=1}^{\infty} r^{2n-1},$$

which gives

$$M_2^2(r, F_a(re^{i\theta}) - F_a(0)) \leq \beta_f^2 \sum_{n=1}^{\infty} \frac{r^{2n}}{n}.$$

Since

$$\int_0^1 2r M_2^2(r, F_a(re^{i\theta}) - F_a(0)) \, dr = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} r |F_a(re^{i\theta}) - F_a(0)|^2 \, d\theta \, dr$$

$$= \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |F_a(\zeta) - F_a(0)|^2 \, dA(\zeta),$$

we see that

$$\frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |F_a(\zeta) - F_a(0)|^2 \, dA(\zeta) \leq \int_0^1 2\beta_f^2 \sum_{n=1}^{\infty} \frac{r^{2n+1}}{n} \, dr = \beta_f^2 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \beta_f^2,$$

whence

$$\|f\|_{BMO_2} \leq \beta_f.$$

Next, we prove that $\beta_f \leq 2\|f\|_{BMO_2}$. By Lemma 3.1 and the subharmonicity of $|\widehat{\nabla F_a}|^2$,

$$\frac{2}{r} \int_0^r \rho |\widehat{\nabla F_a}(0)|^2 \, d\rho \leq \frac{2}{r} \int_0^r \rho \left( \frac{1}{2\pi} \int_0^{2\pi} |\widehat{\nabla F_a}(\rho e^{i\theta})|^2 \, d\theta \right) d\rho$$

$$= \frac{1}{r\pi} \int_{\mathbb{D}_r} |\widehat{\nabla F_a}(\zeta)|^2 \, dA(\zeta)$$

$$= \frac{1}{2} \frac{d}{dr} M_2^2(r, F_a(re^{i\theta}) - F_a(0)),$$

which implies that

$$|\widehat{\nabla F_a}(0)|^2 r^2 \leq M_2^2(r, F_a(re^{i\theta}) - F_a(0)).$$

It follows that

$$\frac{|\widehat{\nabla F_a}(0)|^2}{4} = \int_0^1 |\widehat{\nabla F_a}(0)|^2 r^3 \, dr \le \frac{1}{2\pi} \int_{\mathbb{D}} |F_a(\zeta) - F_a(0)|^2 \, dA(\zeta),$$

whence

$$\frac{\Lambda_{F_a}^2(0)}{4} \le \frac{|\widehat{\nabla F_a}(0)|^2}{2} \le \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |F_a(\zeta) - F_a(0)|^2 \, dA(\zeta). \tag{3.9}$$

On the other hand,

$$\beta_f \le \sup_{a \in \mathbb{D}} \Lambda_{F_a}(0). \tag{3.10}$$

By (3.9) and (3.10),

$$\beta_f \le 2\|f\|_{BMO_2}.$$

It remains to prove the sharpness in the inequalities. Obviously, the equality sign in the first inequality of (1.5) occurs when $f$ is constant. For the sharpness part of the second inequality of (1.5), we let

$$f(z) = C(z + \overline{z}),$$

where $C$ is a constant. Then

$$\beta_f = \sup_{z \in \mathbb{D}} \{(1 - |z|^2)\Lambda_f(z)\} = 2|C|$$

and

$$\begin{aligned}
\|f\|_{BMO_2} &= \sup_{a \in \mathbb{D}} \left( \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |F_a(z) - F_a(0)|^2 \, dA(z) \right)^{1/2} \\
&= |C| \sup_{a \in \mathbb{D}} \left( \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} (1 - |a|)^2 |z + \overline{z}|^2 \, dA(z) \right)^{1/2} \\
&= |C| \sup_{a \in \mathbb{D}} \left( \frac{4(1 - |a|)^2}{\pi} \int_0^1 \int_0^{2\pi} r^3 \cos^2 \theta \, d\theta \, dr \right)^{1/2} \\
&= |C| \sup_{a \in \mathbb{D}} (1 - |a|) \\
&= |C|,
\end{aligned}$$

whence

$$\beta_f = 2\|f\|_{BMO_2}.$$

The proof of this theorem is complete.                                                              □

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