This is a ``preproof'' accepted article for *Canadian Mathematical Bulletin*
This version may be subject to change during the production process.
DOI: 10.4153/S0008439524000389

ON IRREDUCIBLE REPRESENTATIONS OF FUCHSIAN GROUPS

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Abstract. Let $\mathcal{R} \subset \mathbb{P}^1_{\mathbb{C}}$ be a finite subset of markings. Let G be an almost simple, simply–connected algebraic group over $\mathbb{C}$. Let K_G denote the compact real form of G. Suppose for each lasso l around the marked point a conjugacy class C_l in K_G is prescribed. The aim of this paper is to give verifiable criteria for the existence of an *irreducible* homomorphism of $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \mathcal{R})$ into K_G such that the image of l lies in C_l.

1. Introduction

Let G be an almost simple simply–connected algebraic group over $\mathbb{C}$. Let K_G denote a maximal compact subgroup. Let $\mathcal{R} \subset \mathbb{P}^1_{\mathbb{C}}$ be a finite subset of distinct closed points or markings. Recall that the fundamental group $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \mathcal{R})$ is a free group on $s = \|\mathcal{R}\|$-number of generators $\gamma_1, \dots, \gamma_s$ such that $\gamma_1 \dots \gamma_s = 1$. Recall that a subset $H \subset K_G$ is called *irreducible* if the $\{Y \in \mathrm{Lie}(G) \mid \mathrm{ad}h(Y) = Y, \forall h \in H\} =$ centre of $\mathrm{Lie}(G) = 0$ and a homomorphism $\rho : \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \mathcal{R}) \to K_G$ is called *irreducible* if the image $\rho(\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \mathcal{R})) \subset K_G$ is irreducible (see Ramanathan [16]). For $i \geq 3$, let $\{C_i | 1 \leq i \leq s\}$ denote a prescribed set of conjugacy classes in K_G. The aim of this paper is to give verifiable criteria for the existence of an *irreducible* homomorphism $\rho : \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \mathcal{R}) \to K_G$, such that the conjugacy class of $\rho(\gamma_j)$ lies in C_j.

The multiplicative Horn problem asks whether there exists a set of lifts $\{c_i \in C_i\}$ satisfying $\prod c_i = 1$. S.Bauer [4] was the first to relate the representation question to standard algebro-geometric objects. For $G = SU_n$ such a criteria was obtained independently by Agnihotri-Woodward [1] and P.Belkale [5]. Teleman and Woodward [19] gave numerical criteria for this problem for arbitrary G. We note that the additive Horn problem was solved in the late nineties independently by A.A. Klyachko [9] and Knutson and Tao [10].

By Balaji-Seshadri [2, Thm 8.1.7, Cor 8.1.8] for finite-order conjugacy classes and by [3, Cor 10.6] for arbitrary ones, the existence of an irreducible set of lifts is equivalent to the existence of a stable torsor under a suitable parahoric Bruhat-Tits group scheme on $\mathbb{P}^1_{\mathbb{C}}$. When $K_G = U_n$, by Mehta-Seshadri [12] this problem is equivalent to the existence of a stable parabolic vector bundle on $\mathbb{P}^1_{\mathbb{C}}$.

2000 *Mathematics Subject Classification.* 14L15,14D23,14D20.

Key words and phrases. Bruhat-Tits group scheme, parahoric group, Moduli stack, Gromov-Witten, stability.

The support of Science and Engineering Research Board under Mathematical Research Impact Centric Support File number: MTR/2017/000229 is gratefully acknowledged. We thank ICTS, Bangalore for excellent hospitality where this work was finally accomplished.

From the perspective of the root system of G, we recall that conjugacy classes in K_G are parametrized precisely by the points of the Weyl alcove $\mathbf{a}_0$ (see [14, Page 151]). In this setting, our aim is to give a numerical verifiable criteria for the existence of such an irreducible set of lifts in terms of the points of the Weyl alcoves determined by the $\{C_i\}$. More precisely, let $\mathbf{a}_0$ denote the points of the *closed* Weyl alcove. Let Δ^s denote the set of points $\boldsymbol{\theta} = \{\theta_x\}_{x \in \mathcal{R}} \in Maps(\mathcal{R}, \mathbf{a}_0)$ such that there exists a stable parahoric torsor on $\mathbb{P}^1$ with weight $\boldsymbol{\theta}$. In this note we want to describe the stable polytope $\Delta^s \subset Maps(\mathcal{R}, \mathbf{a}_0)$. One defines the semistability polytope Δ^{ss} similarly.

Such criteria were obtained by I.Biswas [6] for U_2 and later [7] for U_n, by P.Belkale [5] for SU_n and by Y.Pandey [15] for the maximal compact subgroups of $SO_n(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$.

Returning to the setting of [2], let $\mathcal{M}_{\mathbb{P}^1}(\mathcal{G})$ denote the moduli stack of $\mathcal{G}$-torsors on $\mathbb{P}^1_{\mathbb{C}}$ where $\mathcal{G}$ is a Bruhat-Tits group scheme on $\mathbb{P}^1_{\mathbb{C}}$ at a fixed set of marked points on $\mathbb{P}^1_{\mathbb{C}}$ given by a choice of weights $\boldsymbol{\theta}$. Let $\mathcal{M}_{\mathbb{P}^1}(\mathcal{G}^{\mathtt{I}})$ be the moduli stack of torsors with Iwahori structures at these marked points. Recall that the points of $\mathcal{M}_{\mathbb{P}^1}(\mathcal{G}^{\mathtt{I}})$ can be viewed as principal G-bundles with parabolic structures given by the Borel B at points $x \in \mathcal{R}$, analogous to vector bundles with full-flag parabolic structures. Under this identification, the set consisting of the trivial G-bundle with varying B-structures, gives a subset of the points of $\mathcal{M}_{\mathbb{P}^1}(\mathcal{G}^{\mathtt{I}})$.

Denoting the moduli stack of principal G-bundles by $\mathcal{M}_X(G)$ for an arbitrary curve X, it can be seen that these fit in together in the following Hecke-modification diagram

$$\mathcal{M}_X(\mathcal{G}) \xleftarrow{\;p\;} \mathcal{M}_X(\mathcal{G}^{\mathtt{I}}) \xrightarrow{\;q\;} \mathcal{M}_X(G) \qquad (1.0.1)$$

where the morphism q is simply the one which forgets the flag structures (see the discussion after (2.1.3) for details).

In §5, we explain the main construction of this paper generalizing the completing flag construction of [5, Appendix]. For each representation ρ of G, this construction *extends weights* $\boldsymbol{\theta}$ on $\mathcal{G}$-torsors to $\mathcal{G}^{\mathtt{I}}$-torsors. Recall that if V_* and W_* are two parabolic vector bundles, then the quasi-parabolic structure underlying their tensor product depends on the weights and is not determined by the quasi-parabolic structures underlying V_* and W_*. Keeping this feature in mind, as it happens in [3, §4.2], in order to carry the construction out for each point $x \in \mathcal{R}$ we need to make the choice of a smaller facet $\mathbf{a}_\rho^{\circ,x}$ (see Def 5.2.1). Then in §6, given a choice of weights $\boldsymbol{\theta}$ for $\mathcal{G}$, we show how one can derive an extended set of weights $(\boldsymbol{\theta}, \{\mathbf{a}_\rho^{\circ,x}\}_{x\in\mathcal{R}})$ for $\mathcal{G}^{\mathtt{I}}$-torsors $\mathcal{E}^{\mathtt{I}}$ such that the stability condition for a $\mathcal{G}$ torsor $\mathcal{E}$ with weights $\boldsymbol{\theta}$ becomes equivalent to an intrinsic stability condition for all the $\mathcal{G}^{\mathtt{I}}$ torsors $\mathcal{E}^{\mathtt{I}}$ with weights $(\boldsymbol{\theta}, \{\mathbf{a}_\rho^{\circ,x}\}_{x\in\mathcal{R}})$ sitting above $\mathcal{E}$ under the map p (see Prop 6.5.1). Thus for varying choices of $\{\mathbf{a}_\rho^{\circ,x}\}_{x\in\mathcal{R}}$, the $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}_\rho^{\circ,x}\}_{x\in\mathcal{R}})$ are (semi)stable or not simultaneously. We turn this observation into a definition (see 6.5.2). The equivalence of stability of parahoric torsors to that of usual parabolic G-bundles but for *extended weights* is the main point of §5 and §6 and is the technical heart of this work.

https://doi.org/10.4153/S0008439524000389 Published online by Cambridge University Press

As we noted above in (1.0.1), through q the stability condition can alternatively be seen as an extended weight $\boldsymbol{\theta}$ stability condition on the underlying parabolic G-bundle. Denoting by Ad the adjoint representation, we prove

Theorem 1.0.1. *Let $X = \mathbb{P}^1$. The open sub-stack $\mathcal{M}_{\mathbb{P}^1}(\mathcal{G})^s$ of $\mathcal{M}_{\mathbb{P}^1}(\mathcal{G})$ consisting of stable torsors is non-empty if and only if the trivial G-bundle with generic B-structures and extended weight $\boldsymbol{\theta}$ (see 6.5.2) is stable as a point of $\mathcal{M}_{\mathbb{P}^1}(\mathcal{G}^{\mathtt{I}})$.*

The setting is as in [19], and by Proposition 6.5.4 the sought-after criterion now gets translated into one in terms of Gromov-Witten numbers. We describe the stable polytope Δ^s in Corollary 3.1.1 and Corollary 3.1.2 shows that the difference between Δ^{ss} and Δ^s is at the boundary of $Maps(\mathcal{R}, \mathbf{a}_0)$.

Let $\mathcal{E}^{\mathtt{I}}$ be the *trivial G-bundle* with parabolic structures of the full-flag type, i.e.. B-structures at the marked points $\mathcal{R}$. For a parabolic subgroup $P \subset G$, let $\mathcal{E}^{\mathtt{I}}_{P}$ be a reduction of structure group to P. We then have an inclusion of Lie algebra bundles $\mathcal{E}^{\mathtt{I}}_{P}(\mathfrak{p}) \subset \mathcal{E}^{\mathtt{I}}(\mathfrak{g})$. Observe that the associated Lie algebra bundle $\mathcal{E}^{\mathtt{I}}(\mathfrak{g})$ gets canonical parabolic structures at the marked points (these will not be full-flag types though). We denote this Lie algebra bundle with parabolic structures by $\mathcal{E}^{\mathtt{I}}(\mathfrak{g})_*$. The sub-bundle $\mathcal{E}^{\mathtt{I}}_{P}(\mathfrak{p})$ gets the canonical induced parabolic structures and we have similarly $\mathcal{E}^{\mathtt{I}}_{P}(\mathfrak{p})_* \subset \mathcal{E}^{\mathtt{I}}(\mathfrak{g})_*$.

Say a P-reduction $\mathcal{E}^{\mathtt{I}}_{P} \subset \mathcal{E}^{\mathtt{I}}$ is of the minus 1 type if the parabolic degree of the quotient $\mathcal{E}^{\mathtt{I}}(\mathfrak{g})_*/\mathcal{E}^{\mathtt{I}}_{P}(\mathfrak{p})_*$ is 0 and further, the degree of the vector bundle underlying the quotient $\mathcal{E}^{\mathtt{I}}(\mathfrak{g})_*/\mathcal{E}^{\mathtt{I}}_{P}(\mathfrak{p})_*$ is -1. We prove

Theorem 1.0.2. *A point $\boldsymbol{\theta} \in \Delta^{ss}$ lies in Δ^s if and only if the trivial G-bundle $\mathcal{E}^{\mathtt{I}}$ with generic B-structures and extended weight $(\boldsymbol{\theta}, \{\mathbf{a}^{\circ,x}_{Ad}\}_{x\in\mathcal{R}})$ does not have any P-reduction $\mathcal{E}^{\mathtt{I}}_{P}$ of the minus 1 type.*

Proposition 7.0.1 reduces the condition in the above theorem to Gromov-Witten numbers. This gives new verifiable criteria for points in Δ^{ss} to lie in Δ^s.

1.1. **Comparison with [19].** The inequalities in Corollary 3.1.1 determining Δ^s are strict versions of those in [19, Prop 4.4] determining Δ^{ss}. A point to be noted is that, whereas [19] had an underlying principal G-bundle to work with, in general we do not have such a bundle. In fact, the extending weight construction and Theorem 1.0.1 fill the void left by the non-existence of an underlying principal G-bundle. The bundle-theoretic description of difference between points in Δ^{ss} and Δ^s sheds some light on the role played by them.

1.2. **Some remarks on the far wall and stability.** We conclude by giving a few words of justification on why the far wall cannot be avoided for the stability question (although it can be avoided for semistability *in the very special case of $G = \mathrm{GL}(n)$* because of the presence of an underlying $\mathrm{GL}(n)$-bundle). Let us mention some difficulties. Firstly, in [15, §7] some examples of stable parahoric symplectic and (special orthogonal) torsors are shown to lie on the product of far walls. Secondly, Belkale has shown that $(\Delta^{ss})^\circ \subset \Delta^s$ (see [15, Prop 7.0.5]). So they have the same closures in $Maps(\mathcal{R}, \mathbf{a}_0)$. Also the origin in $Maps(\mathcal{R}, \mathbf{a}_0)$ lies in $\Delta^{ss} \setminus \Delta^s$ because it corresponds to the case of principal G-bundles with trivial parabolic structures. Lastly, Meinrenken and Woodward (see [13, Cor 4.13]) have shown that Δ^{ss} is a closed convex polytope of maximal dimension inside $Maps(\mathcal{R}, \mathbf{a}_0)$ and to the best of our knowledge, for stability no such argument is known. Thus, it does not seem

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possible to reduce the problem of determining Δ^s to the case of generic weights i.e. the interior of $Maps(\mathcal{R}, \mathbf{a}_0)$. We are forced to consider $Maps(\mathcal{R}, \mathbf{a}_0)$ fully and directly. Now weight tuples, one of whose factor lies on the far wall of $\mathbf{a}_0$, correspond to strictly parahoric (non-parabolic) torsors under parahoric group schemes. These need to be reckoned with in the sense there is no underlying principal G-bundle. Consequently, one cannot deform to the trivial bundle without going through Hecke-modifications. In Corollary 3.1.2 we find that the difference between Δ^{ss} and Δ^s is at the boundary of $Maps(\mathcal{R}, \mathbf{a}_0)$.

1.3. **Some remarks on non-existence of underlying bundle and Hecke-modification.** We carry out the technique of Hecke-modification in this paper for the following reason. In [5] or [15] one finds that for Δ^s the corresponding hyperplanes are not merely the strict versions of the hyperplanes for Δ^{ss}. Indeed, this is in fact expected, because for stability, unlike for semistability, one cannot restrict oneself only to the case when the Gromov-Witten invariant is one. But, even if one allows for higher Gromov-Witten numbers, as we do in this paper, the formulation of the slope inequalities for stability in [5] could be done because in the case $G = SL_n$, by implicitly extending structure group to GL_n, there was an underlying parabolic vector bundle to work with and the stability of the parahoric torsor, when $G = SL_n$, is equivalent to that of the parabolic vector bundle. However in general, for an arbitrary G taking an associated vector bundle under a faithful representation *does not preserve stability.*

The hyperplanes defining the semistablity polytope (see [19] and [5] (from which even the stability polytope may be deduced)) arise by a translation of semistability conditions on the *trivial principal G-bundle with generic parabolic structures.* Further, the reduction to this special case relies on the existence of an underlying G-bundle to work with. For these reasons, for stability, strict versions of slope inequalities, seem inadequate to address the situation for weight tuples with at least one coordinate in the far wall.

Lastly, the reduction to Gromov-Witten inequalities works when one has a trivial principal G-bundle but in the parahoric situation, *with weights constrained to lie on the far wall*, there is no reduction to anything analogous to the trivial bundle.

It might seem that parahoric weights from the far wall can still be handled by going to an equivariant bundle as in [2]. But again, in the equivariant setup there is no analogue of Ramanathan's theorem [17] on deformations of principal G-bundles on $\mathbb{P}^1$. In this context, we wish to clarify that in [15, §4.2, Page 109], for the case of classical groups G of $\mathtt{B_n}$,$\mathtt{C_n}$ and $\mathtt{D_n}$ type, the second-named author found a Hecke-modification allowing to pass from parahoric torsors to usual parabolic G-bundles preserving (semi)stability. Such a route seems more elusive for the case of exceptional G. In this note, we give a unifying argument in the general case. We also wish to point out that whereas in [5] and [15], the weights do change under Hecke-modifications, they do not in this note. In terms of line bundles, we are merely pulling them back under p of (1.0.1) and then viewing them through q.

1.4. **Layout.** We develop notions over a general smooth projective curve X over an algebraically closed field k of arbitrary characteristic throughout the paper and specialize to the case $X = \mathbb{P}^1$ and $k = \mathbb{C}$ only to prove the main theorems in §3 and §7. In §2, we explain our basic set-up. Then after recalling the main consequences of our construction, we prove our first main theorem in §3 and derive the stable

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polytope in Corollary 3.1.1. In §4 after recalling [5, Appendix], we recast it in our setup of alcoves, weights, facets and Hecke-modification diagram. The introduction of §5 explains the main constructions of the paper. It also has a few examples serving to highlight the key issues. In §6 we show the equivalence of (semi)stability of usual parahoric torsors $(\mathcal{E}, \boldsymbol{\theta})$ with that of extended weight parahoric torsors $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}_\rho^{\circ,x}\}_{x\in\mathcal{R}})$. In §7, we prove the second main theorem.

1.5. **Acknowledgement.** We wish to sincerely thank the anonymous referee for a careful reading of the manuscript and for suggesting improvements. In fact, Corollary 3.1.2 was formulated in response to a question raised by the referee.

2. The moduli stack $\mathcal{M}_X(\mathcal{G})$

2.1. **Local group theoretical data of parahoric group schemes.** We will write k instead of $\mathbb{C}$ whenever the results we use holds for an algebraically closed field of arbitrary characteristic. Let $A := k[[t]]$ and $K := k((t)) = k[[t]][t^{-1}]$, where t denotes a uniformizing parameter. Let G be an *almost simple simply connected affine algebraic group* defined over k. We now want to consider the group $G(K)$.

We shall fix a maximal torus $T \subset G$. Let $R = R(T, G)$ denote the root system of G (see. [18, p. 125]). Let $Y(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ denote the group of all one–parameter subgroups of T. The *standard affine apartment* $\mathcal{A}_T$ is an affine space under $Y(T) \otimes_{\mathbb{Z}} \mathbb{R}$. We may identify $\mathcal{A}_T$ with $Y(T) \otimes_{\mathbb{Z}} \mathbb{R}$ (see [2, § 2]) by choosing a point $v_0 \in \mathcal{A}_T$. This v_0 is also called an origin. For a root r of G and an integer $n \in \mathbb{Z}$, we get an affine functional

$$\alpha = r + n : \mathcal{A}_T \to \mathbb{R}, x \mapsto r(x - v_0) + n. \tag{2.1.1}$$

These are called the *affine roots* of G. For any point $x \in \mathcal{A}_T$, let Y_x denote the set of affine roots vanishing on x. For an integer $n \geq 0$, define

$$\mathcal{H}_n = \{x \in \mathcal{A}_T \,||Y_x| = n\}. \tag{2.1.2}$$

A facet σ of $\mathcal{A}_T$ is defined to be a connected component of $\mathcal{H}_n$ for some n. The dimension of a facet is its dimension as a real manifold. We refer the reader to [3, §3.2] for the definition of the parahoric subgroup of $G(K)$ corresponding to a facet.

The subgroup $G(A) \subset G(K)$ is an example of a maximal parahoric subgroup. We have a natural evaluation map $ev : G(A) \to G(k)$ and the inverse image $\mathtt{I} := ev^{-1}(B)$ of the standard Borel subgroup $B \subset G$ is called the standard Iwahori subgroup. Any parahoric subgroup contains a $G(K)$-conjugate of the standard Iwahori subgroup $\mathtt{I}$. *In this paper* Θ *will always be either a* facet *or a point of* $\mathcal{A}_T$. By [8, Section 1.7] we have an affine flat smooth group scheme $\mathcal{G}_\Theta \longrightarrow \mathrm{Spec}(A)$ called the *parahoric group scheme* associated to Θ. It is uniquely determined by its A–valued points which equals the parahoric group corresponding to Θ. In particular, the group scheme whose A–valued points is $\mathtt{I}$ is called the standard Iwahori group scheme. For a facet $\sigma \subset \mathcal{A}_T$, let $\mathcal{G}_\sigma \to \mathrm{Spec}(A)$ be the parahoric group scheme defined by σ.

2.1.1. *Alcove.* We choose a Borel B in G/k containing T. This determines a choice of positive roots. Let $\mathbf{a}_0$ denote the unique closed alcove in $\mathcal{A}_T$ whose closure contains v_0 and is contained in the finite Weyl chamber determined by positive simple roots. We will denote its interior by $\mathbf{a}_0^\circ$. It is the facet corresponding to the standard Iwahori subgroup. The affine walls defining $\mathbf{a}_0$ determine a set $\mathbf{S}$ of simple *affine roots*. We will denote these simple roots by the symbols $\{\alpha_i\}$.

2.1.2. *The parahoric Bruhat-Tits group scheme.* For an arbitrary closed point $y \in \mathcal{R}$ let $\mathbb{D}_y := \mathrm{Spec}(\hat{\mathcal{O}}_y)$, let K_y be the quotient field of $\hat{\mathcal{O}}_y$. Let $\mathcal{R} \subset X$ be a non-empty finite set of closed points. For each $x \in \mathcal{R}$, we choose a facet $\sigma_x \subset \mathcal{A}_T$. Let $\mathcal{G}_{\sigma_x} \to \mathbb{D}_x$ be the parahoric group scheme corresponding to σ_x. Let $X^\circ = X \setminus \mathcal{R}$. *In this paper, by a Bruhat-Tits group scheme $\mathcal{G} \to X$ we shall mean that $\mathcal{G}$ restricted to X° is isomorphic to the trivial constant group scheme $X^\circ \times G$ on X°, and for any closed point $x \in X$, $\mathcal{G}$ restricted to $\mathbb{D}_x$ is a parahoric group scheme $\mathcal{G}_{\sigma_x}$ such that the gluing functions take values in $Mor(\mathbb{D}_x^\circ, G) = G(K_x)$.* This is also the set-up of [2, Defn 5.2.1]. Thus the $\hat{\mathcal{O}}_y$-points of the restriction of $\mathcal{G}$ to this disc give parahoric subgroups of $G(K)$.

We also suppose that the facets $\{\sigma_x\}_{x\in\mathcal{R}}$ lie in $\mathbf{a}_0$ because it can easily be seen that for results and constructions in this note the general case of arbitrary facets reduces to this one.

Remark 2.1.1. The group scheme $\mathcal{G}$ depends on the gluing data. But if $\mathcal{G}$ and $\mathcal{G}'$ are two parahoric group schemes on X which differ only in their gluing data, then it is straightforward to check that the stacks $\mathcal{M}_X(\mathcal{G})$ and $\mathcal{M}_X(\mathcal{G}')$ are isomorphic. For this reason, we fix one gluing data to get $\mathcal{G}$ and work with this.

We use the notation $X \times G$ to denote the trivial group scheme on X. Let $\mathcal{G}^{\mathtt{I}} \to X$ (resp. $\mathcal{G}^{\mathtt{I}}_{tr} \to X$) be the group scheme obtained by gluing $X^\circ \times G$ with the standard Iwahori group scheme $\mathcal{G}_{\mathbf{a}_0^\circ}$ at each parabolic point $x \in \mathcal{R}$ using the same gluing functions as $\mathcal{G}$ (resp. the identity in $G(K_x)$ as a gluing function). For each $x \in \mathcal{R}$, the inclusions $\mathtt{I} \subset \mathcal{G}_{\mathbb{D}_x}(\hat{\mathcal{O}}_x)$ (resp. $\mathtt{I} \subset G(\hat{\mathcal{O}}_x)$) induce morphisms of group schemes $\mathcal{G}^{\mathtt{I}} \to \mathcal{G}$ (resp. $\mathcal{G}^{\mathtt{I}}_{tr} \to X \times G$) over the whole of X.

2.1.3. *Parahoric torsors.* Let $\mathcal{G} \to X$ be a group scheme as in §2.1.2. A *quasi-parahoric* torsor $\mathcal{E}$ is a $\mathcal{G}$–torsor on X. This means that $\mathcal{E} \times_X \mathcal{E} \simeq \mathcal{E} \times_X \mathcal{G}$ and there is an action map $a : \mathcal{E} \times_X \mathcal{G} \to \mathcal{E}$ which satisfies the usual axioms for principal G-bundles. A *parahoric torsor* is a pair $(\mathcal{E}, \boldsymbol{\theta})$ consisting of the pair of a quasi-parahoric torsor and weights $\boldsymbol{\theta} = \{\theta_x | x \in \mathcal{R}\} \in (Y(T) \otimes \mathbb{R})^m$ such that θ_x lies in the facet σ_x (see §2.1.2) and $m = |\mathcal{R}|$. Let $\mathcal{M}_X(\mathcal{G})$ denote the moduli stack of $\mathcal{G}$-torsors on X. The natural morphisms of group schemes seen above induces the following morphisms of stacks:

$$\mathcal{M}_X(\mathcal{G}) \leftarrow \mathcal{M}_X(\mathcal{G}^{\mathtt{I}}) \overset{2.1.1}{\simeq} \mathcal{M}_X(\mathcal{G}^{\mathtt{I}}_{tr}) \to \mathcal{M}_X(G). \tag{2.1.3}$$

In particular, the morphism $\mathcal{M}_X(\mathcal{G}^{\mathtt{I}}) \xrightarrow{q} \mathcal{M}_X(G)$ induced by the morphism $\mathcal{G}^{\mathtt{I}} \to X \times G$ can be viewed as follows. The points of the stack $\mathcal{M}_X(\mathcal{G}^{\mathtt{I}})$ are G-bundles on X with B-structures at the marked points $\mathcal{R}$. The morphism $\mathcal{M}_X(\mathcal{G}^{\mathtt{I}}) \to \mathcal{M}_X(G)$ forgets the B-structures. Thus, $\mathcal{M}_X(\mathcal{G}^{\mathtt{I}})$ seen from the standpoint of $\mathcal{M}_X(G)$ is the analogue of the moduli stack of vector bundles with full-flag structures at the marked points. *We will call morphisms in the diagram (1.0.1) as Hecke-modification.*

Although in the literature a sequence of flip-flop is called a Hecke-modification, we wish to emphasize that often only a *single* morphism as above will be required for the proofs in this paper. For the usual case of parabolic vector bundles these one-step modification morphisms correspond to usual Hecke-modifications. For torsors, we will call them both by the same name following Balaji-Seshadri [2].

3. Main Theorem

In this section we suppose that $X = \mathbb{P}^1$. We wish to show that after general results proved in later sections, our main result follows by standard arguments well-known to experts. Let us begin by the steps of this reduction:

Step 1: Let $(\mathcal{E}, \boldsymbol{\theta})$ be a parahoric torsor. Under $p : \mathcal{M}_{\mathbb{P}^1}(\mathcal{G}^{\mathtt{I}}) \to \mathcal{M}_{\mathbb{P}^1}(\mathcal{G})$, let $\mathcal{E}^{\mathtt{I}}$ be an arbitrary $\mathcal{G}^{\mathtt{I}}$-torsor that maps to $\mathcal{E}$ and consider the Hecke-modification diagram (1.0.1). Given a parabolic vector bundle with possibly partial flags, a construction called *completing flags* is described in [5, Appendix]. In an analogous fashion (with a somewhat involved "parahoric" adaptation), in §5 we explain how after choosing any finer facet $\mathbf{a}_{Ad}^{\circ,x} \subset \mathbf{a}^{\circ}$, in whose closure θ_x lies, $\boldsymbol{\theta}$ may be extended as a weight $(\boldsymbol{\theta}, \{\mathbf{a}_{Ad}^{\circ;x}\}_{x \in \mathcal{R}})$ (see (5.2.1)) on $\mathcal{E}^{\mathtt{I}}$.

Step 2: We extend the definition of (semi)stablity for such objects and call this construction *extending weights* (see (5.2.1)). Then in Proposition 6.5.1 we show that $(\mathcal{E}, \boldsymbol{\theta})$ is stable if and only if $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}_{Ad}^{\circ;x}\}_{x \in \mathcal{R}})$ is stable as a extended weight parahoric torsor. This result justifies Definition 6.5.2 where we say that $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta})$ is stable if for some (and therefore all) choice of $\{\mathbf{a}_0^{\circ,x}\}_{x \in \mathcal{R}}$, $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}_{Ad}^{\circ,x}\}_{x \in \mathcal{R}})$ is stable. Now we view $\mathcal{E}^{\mathtt{I}}$ as a parabolic G-bundle with additional parabolic structures at $\mathcal{R}$.

Proof of Theorem 1.0.1. Since G is simply-connected and $X = \mathbb{P}^1$, so by [17, Thm 7.4] the principal G-bundle E underlying $\mathcal{E}^{\mathtt{I}}$ may be put in a family $\mathbf{E} \to \mathbb{P}^1 \times T$ where the generic bundle is the trivial G-bundle and T is affine . Consider $T_1 := T \times_{\mathcal{M}_{\mathbb{P}^1}(G)} \mathcal{M}_{\mathbb{P}^1}(\mathcal{G}^{\mathtt{I}})$ corresponding to the classifying map $T \to \mathcal{M}_{\mathbb{P}^1}(G)$ of $\mathbf{E}$. The family $\mathbf{E} \to \mathbb{P}^1 \times T$ can be used to make a T_1-family of parabolic principal G-bundles $(\mathbf{E}^{\mathtt{I}}, \boldsymbol{\theta}) \to \mathbb{P}^1 \times T_1$ with extended weights. Thus we have a degeneration to $\mathcal{E}^{\mathtt{I}}$ where the underlying bundle of the generic object is trivial. Under the morphism $\mathcal{M}_{\mathbb{P}^1}(\mathcal{G}^{\mathtt{I}}) \to \mathcal{M}_{\mathbb{P}^1}(\mathcal{G})$ let $\mathbf{E}^{\mathtt{I}} \to \mathbb{P}^1 \times T_1$ give the family $\mathbf{E_1} \to \mathbb{P}^1 \times T_1$ of $\mathcal{G}$-torsors. It degenerates to $\mathcal{E}$ and we view it as a family with weight $\boldsymbol{\theta}$. Further, by [15, Prop 6.1.2] stability is an open property of parahoric torsors. Therefore by Proposition 6.5.1 for generic $t \in T_1$, $(\mathbf{E}^{\mathtt{I}}, \boldsymbol{\theta})_t \to \mathbb{P}^1$ is stable. So by q (see (1.0.1)) the trivial G-bundle with extended weight $\boldsymbol{\theta}$ and generic quasi-parabolic structure is stable. □

3.1. Gromov-Witten numbers and the stability polytope.

By Proposition 6.5.4, the above result on parahoric torsors can be interpreted back in in the setting of parabolic G-bundles. With notations as in [19, Page 716], let us denote the Gromov-Witten number as

$$n_d(w_x | x \in \mathcal{R}). \tag{3.1.1}$$

It counts the number of regular maps $\phi : X \to G/P$ of degree d such that for $x \in \mathcal{R}$, $\phi(x)$ lies in a generic translate of the Schubert variety $Y_{w_x} \subset G/P$ corresponding to $w_x \in W$. By Remark 6.5.3, for computing the slope inequality of extended weights parahoric torsors we may switch from our definition to that in [19] *including the far wall.* Then by repeating the arguments exactly as in [19, Page 741, (13)], we get the following checkable corollary which agrees with [5] and [15].

Corollary 3.1.1. *The polytope Δ^s is the set of points $\boldsymbol{\theta}$ satisfying the inequality*

$$\sum_{x \in \mathcal{R}} (w_x \omega_P, \theta_x) < d \tag{3.1.2}$$

for all maximal parabolic subgroups $P \subset G$ and non-negative integers d such that the Gromov-Witten invariant (see (3.1.1)) $n_d(w_x | x \in \mathcal{R}) \neq 0$.

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Corollary 3.1.2. *The interior of Δ^{ss} is contained in Δ^s. Its complement is the intersection of Δ^s with the boundary of $Maps(\mathcal{R}, \mathbf{a}_0)$.*

Proof. Let $\boldsymbol{\theta} \in (\Delta^{ss})^\circ$. Consider a Euclidean ball B in $(\Delta^{ss})^\circ$ about $\boldsymbol{\theta}$. Thus B is also a ball in the interior of $Maps(\mathcal{R}, \mathbf{a}_0)$ which equals $Maps(\mathcal{R}, \mathbf{a}_0^\circ)$. Observe if one of the inequalities in (3.1.2) is non-strict for $\boldsymbol{\theta}$, then points in one of the open hemisphere of B cannot satisfy (3.1.2). Thus all inequalities must be strict for $\boldsymbol{\theta}$ i.e. $\boldsymbol{\theta}$ lies in Δ^s. We now prove the second statement. The intersection of the boundary of $Maps(\mathcal{R}, \mathbf{a}_0)$ with $(\Delta^{ss})^\circ$ is ofcourse empty. We now show the inclusion of $\Delta^s \setminus (\Delta^{ss})^\circ$ in the boundary of $Maps(\mathcal{R}, \mathbf{a}_0)$. Since Δ^s is an open polytope in the polytope $Maps(\mathcal{R}, \mathbf{a}_0)$, we have $\Delta^s \cap Maps(\mathcal{R}, \mathbf{a}_0)^\circ = (\Delta^s)^\circ \subset (\Delta^{ss})^\circ$. Therefore $\Delta^s \setminus (\Delta^{ss})^\circ$ is contained in the boundary of $Maps(\mathcal{R}, \mathbf{a}_0)$. □

4. Filtrations associated to Parabolic vector bundles

In section §4-§6, the results are of a general nature and so X will be an *arbitrary smooth projective curve of genus $g \geq 0$.*

4.1. **Completing flags** [5, Appendix]**.** For simplicity, we will first assume that we are working with one parabolic point x. Recall a quasi-parabolic structure is giving a possibly partial filtration

$$\cdots \supset E = F_0(E) \supset F_1(E) \supset \cdots \supset F_{l-1}(E) \supset F_l(E) = E(-\mathcal{R}) \supset \cdots \tag{4.1.1}$$

by subsheaves, which can be continued infinitely in both directions. Here l is called the length of the filtration. It can at most be the rank of E. It is called a parabolic sheaf if it has a system of weights $\alpha_0, \cdots, \alpha_{l-1}$ such that

$$0 \leq \alpha_0 < \alpha_1 \cdots < \alpha_{l-1} < 1. \tag{4.1.2}$$

The weight α_i is called the weight of the subsheaf $F_i(E)$. A given filtration (4.1.1) need not be full. By choosing any complete flag for a given parabolic bundle, in [5] the notion of $\mathbb{R}$-filtration of [11] is extended to complete flag parabolic vector bundles with extended weights in [5, Appendix]. This process is carried out as follows. One considers a filtration

$$\cdots \supset E_n \supset E_{n+1} \supset \cdots \tag{4.1.3}$$

of sheaves with strict inclusions as in (4.1.1) parametrized by $\mathbb{Z}$ together with weights $\{\alpha_n\}$ in $\mathbb{R}$, which are allowed to coincide now. Thus, (4.1.2) has become non-strict and is extended by the law:

$$\alpha_{k+ml} = \alpha_k + m. \tag{4.1.4}$$

The constructions in [5] extend in an obvious way to mutliple parabolic points as well as to non-complete flag parabolic vector bundles, just that the indices are harder to write because there may be jumps because of partial flags. Note that, as subsheaves become smaller, their weights become larger.

4.2. **Construction inverse to §4.1, sliding weights.** In 4.1.3 if we *forget sheaves E_j for which there exists a bigger sheaf with the same weight*, then the reduced subset of $\{E_n\}$, together with the corresponding weights, which are now distinct, correspond to (4.1.1). On any term E_m of the filtration (4.1.3) we can induce the structure of a parabolic vector bundle by using the l successive subsheaves in (4.1.1) to get the flags; their corresponding weights may lie outside of $[0, 1)$, but, after subtracting α_m from each of them the weight of E_m becomes zero and the

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remaining will lie in $[0,1)$. This will be denoted E_{m*}. Conversely, E_{m*} gives (4.1.1) upto shifting indices, and the same weights upto a constant. Let us call the process of adding an arbitrary constant $a \in \mathbb{R}$ to all the weights at a point $x \in \mathcal{R}$ as **sliding weights**. Sliding weights adds a to the parabolic slope of the bundle or any of its parabolic sub-bundle. Thus, it leaves (semi)stability invariant. Therefore, upto sliding weights, the formation of E_{m*} is the *inverse construction* to making the filtration 4.1.3 with weights.

4.3. **Key takeaway on degree computation of sub-bundles from [5].** For simplicity, we first work in the setting of [5] which involves one parabolic point. Now weights may now lie outside $[0,1)$. We will denote this as E_{m*}. The weights are defined by (4.1.3) and (4.1.4) in such a way that the parabolic degree of E_{m*} becomes independent of $m \in \mathbb{Z}$ (see [5, page 83 last para]). Further, on [5, page 84] for any $m_1, m_2 \in \mathbb{Z}$ a natural procedure is explained to go from sub-bundles of E_{m_1*} to E_{m_2*}. By [5, Lemma 8] this procedure preserves parabolic degree of sub-bundles too. Thus E_{m_1*} is (semi)stable if and only if E_{m_2*} is (semi)stable. These results generalize to multiple parabolic points also.

4.4. **Interpretation of passage from E_{m_1*} to E_{m_2*} in our set-up of alcoves, weights, facets and Diagram 1.0.1.** To enable us to adapt aspects of this process in the setting of parahoric torsors, we need to interpret it in the language of alcoves.

Let us consider the case of $SL(n)$. Let us label the vertices of $\mathbf{a}_0$ by integers $\{0, \cdots, n-1\}$. Any facet σ in $\mathcal{A}_T$ of dimension d determines a set of $d+1$ vertices. Let us call **the far wall of** σ as the codimension one facet determined by forgetting the smallest vertex. Define alcove $\mathbf{a_{k+1}}$ inductively by reflecting the alcove $\mathbf{a_k}$ along the far wall and label the new vertex by $n+k$. Let us view the weights of E_{k*} (4.1.3) as a point in $\mathbf{a_k}$ by taking barycentric coordinates $\{\alpha_{E_{k+1}} - \alpha_{E_k}, \cdots, \alpha_{E_{k+n}} - \alpha_{E_{k+n-1}}\}$. When we pass from E_{m*} to E_{m+1*}, it follows from (4.1.4) that the weights of E_{m+1*} are obtained by reflecting the weights of E_{m*} along the far wall of $\mathbf{a}_m$. Thus, in terms of barycentric coordinates, as a set they remain the same, just that their indexing is shifted by $-1\,(mod\, n)$ respectively.

Let σ be a facet in the closure of $\mathbf{a}_0$ of codimension one where only the affine root α_d vanishes. Then the morphism $\mathcal{M}_X(\mathcal{G}^{\mathrm{I}}) \to \mathcal{M}_X(\mathcal{G}^{\sigma})$ corresponds to forgeting subsheaves in the $\mathbb{Z}$-filtration whose index is $d \pmod n$. In terms of complete flag parabolic vector bundles, this corresponds to forgetting exactly one term for $d \pmod n \neq 0$ and a Hecke-modification by E_0/E_1 for $d = 0$ which gives a shift by one. These facets are of course much more general than those of $\mathbf{a}_0$ and its facets. They hold for any pair of facet σ_1 and its codimension one subfacet σ. Going to far wall of σ_1 corresponds to a Hecke-modification by a sky-scraper sheaf while forgetting other vertices corresponds merely to forgetting flags in $\mathcal{M}_X(\mathcal{G}^{\sigma_1})$. The above process also suitably generalizes to the graph of the hyperplane structure. More precisely, for any two facets (σ_1, σ) where σ lies in the closure of σ_1, the path we take to come from σ_1 to σ is not important, i.e. the processes of forgetting terms in flags and Hecke-modifications by sky-scraper sheaves commute.

5. Extending Weights on $\mathcal{G}^{\mathrm{I}}$ torsors

The curve X is arbitrary. For simplicity of notation, we further assume that only one parabolic point $x \in X$ is fixed. It will become clear to the reader that, for

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all that is done, the processes can be carried out independently at several points. However, for the main application the final conclusions will be made in terms of multiple points on X. In this section completing flags with induced weights construction of [5] is generalized to *extending weights for* $\mathcal{G}^{\mathtt{I}}$. Recall that similar to the definition of Mehta-Seshadri, in [2] weights have been defined for parahoric torsors to be points lying in the facets (see §2.1.3). Consider the morphism of stacks $p : \mathcal{M}_X(\mathcal{G}^{\mathtt{I}}) \to \mathcal{M}_X(\mathcal{G})$. As in §2.1.3, suppose that we are given weights $\boldsymbol{\theta} = \{\theta_x | x \in \mathcal{R}\} \in (Y(T) \otimes \mathbb{R})^m$ such that θ_x lies in the facet σ_x and $\sigma_x \subset \mathbf{a}_0$ for all $x \in \mathcal{R}$.

In [3, §5], given a representation $\rho : G \to SL(V)$ and a parahoric Bruhat-Tits torsors $\mathcal{E}$, the parabolic vector bundle $(\mathcal{E}(V)_*, \boldsymbol{\theta}_V)$ has been constructed. A priori, a naive approach would be to take the parabolic bundle $(\mathcal{E}(V)_*, \boldsymbol{\theta}_V)$ and carry out a process as in [5], of deforming the underlying bundle after possibly some Hecke-modifications, to get one with a full-flag and suitable schema of weights. But the serious obstruction is that, the new parabolic vector bundle obtained by the deformation need not come as an extension of structure group from any parahoric torsor via the representation ρ.

5.1. **Two problems that arise when we try to extend the weight $\boldsymbol{\theta}$ for $\mathcal{M}_X(\mathcal{G}^{\mathtt{I}})$.** Firstly, the weights θ_x only belong to $\mathbf{a}_0$ and not to its interior $\mathbf{a}_0^\circ$. So the set-up of [3] does not apply. Secondly, in order to apply the associated construction of [3], even if we take a weight $\boldsymbol{\theta}_V$ in $\mathbf{a}_0$ arbitrarily close to $\boldsymbol{\theta}$, already the quasi-parabolic structure of $(\mathcal{E}^{\mathtt{I}}(V)_*, \boldsymbol{\theta}_V)$ is sensitive to the choice of $\boldsymbol{\theta}_V$ (see Example 5.2.6 which shows that even the underlying vector bundle depends on the choice of weight). On weights $\boldsymbol{\theta}_V$, we are forced to consider the equivalence relation that arises from the quasi-parabolic structure.

We address these problems *by choosing for each $x \in \mathcal{R}$ a finer ρ-facet $\mathbf{a}_\rho^{\circ,x}$ (see §5.2) in whose closure θ_x lies. These are defined by the requirement that all points in $\mathbf{a}_\rho^{\circ,x}$ under $\rho : \mathcal{A}_T \to \mathcal{A}_{T_{SL(V)}}$ go to a fixed open facet of $SL(V)$, of dimension at most the dimension of T, in whose closure $\rho(\theta_x)$ lies.* Now the flag structure on $(\mathcal{E}(V)_*, \boldsymbol{\theta}_V)$ of [3, §5.1] or $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}_\rho^{\circ,x}\})$ (see 5.2.1) may not be full since it will have at most $\dim(T)$ many distinct flags (or weights). More importantly, unlike [5] the vector bundle underlying it may only be related to the one underlying $(\mathcal{E}(V)_*, \boldsymbol{\theta}_V)$ by a Hecke-modification. So instead of completing flags we call this construction *extending weights for parahoric torsors*. In the set-up of [5] we have $\rho = Id$. This reflects the facts that a parabolic vector bundle determines a choice of an alcove whict itself is a finer ρ-facet. The adjoint representation being in general the most canonical choice, *in the applications of the constructions carried out here, we will mostly have to take $\rho = Ad$ and so $V = \mathfrak{g}$.*

5.2. **Extending weights construction for $\mathcal{G}^{\mathtt{I}}$-torsors with respect to a representation ρ.** Let $\mathcal{E}$ be a $\mathcal{G}$-torsor and let $\mathcal{E}^{\mathtt{I}}$ be a $\mathcal{G}^{\mathtt{I}}$-torsor lying in the fiber of $\mathcal{M}_X(\mathcal{G}^{\mathtt{I}}) \to \mathcal{M}_X(\mathcal{G})$. To lighten the notation, it suffices to treat the case of one parabolic point i.e.. $\mathcal{R} = \{x\}$. For a representation $\rho : G \to SL(V)$ we choose tori $T_G \subset G$ and $T_{SL} \subset SL(V)$ such that ρ maps T_G to T_{SL}. Thus we get a linear map

$$\rho : \mathcal{A}_T \to \mathcal{A}_{T_{SL}} \tag{5.2.1}$$

between the apartments. In [3, BBP], the usual definition of facet is generalized to *facets associated to a homomorphism* ρ as follows. By a generalized affine functional on $\mathcal{A}_T$ we mean affine functionals for G together with those of $\mathcal{A}_{T_{SL}}$ viewed as functionals on $\mathcal{A}_T$. For any point $x \in \mathcal{A}_T$, let Y_x^g denote the set of generalized affine functionals vanishing on x. For an integer $n \geq 0$, define

$$\mathcal{H}_n^g = \{x \in \mathcal{A}_T || Y_x^g| = n\}. \tag{5.2.2}$$

A ρ-facet σ of $\mathcal{A}_T$ corresponding to a representation ρ is defined to be a connected component of $\mathcal{H}_n^g$ for some n. The dimension of a ρ-facet is its dimension as a real manifold. The finer facets satisfy the property that, for any two weights belonging to it, the parabolic vector bundles associated to ρ have the same quasi-parabolic structure.

For each $x \in \mathcal{R}$ we choose a ρ-facet $\mathbf{a}_\rho^{\circ,x} \subset \mathbf{a}_0$ whose closure contains θ_x. Then, given a weight θ_x, we choose a sequence of rational weights $\theta_{x,n}$ lying in our chosen alcove $\mathbf{a}_\rho^{\circ,x}$ and converging to θ_x. Thus, the quasi-parabolic structure of $(\mathcal{E}^{\mathtt{I}}(V)_*, \rho(\theta_{x,n}))$ is independent of n and is also independent of the choice of the limiting sequence $\{\theta_{x,n}\}$. Keeping this quasi-parabolic structure fixed, the weight $\rho(\theta_{x,n}) \in \mathcal{A}_{T_{SL}}$ equips the vector space G_x^j of the flag at x with a real number $\alpha_{x,n}^j$. We set

$$\alpha_x^j = lim\ \alpha_{x,n}^j. \tag{5.2.3}$$

By linearity of (5.2.1), this process is independent of the choice of weights $\{\theta_{x,n}\}$ and depends only on $\mathbf{a}_\rho^{\circ,x}$ and θ_x.

Definition 5.2.1. *Let $(\mathcal{E}, \boldsymbol{\theta})$ be a parahoric torsor. With notations as above, extending weights for a $\mathcal{G}^{\mathtt{I}}$-torsor $\mathcal{E}^{\mathtt{I}}$ with respect to a given representation $\rho : G \to SL(V)$ is giving the triple $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}_\rho^{\circ,x}\}_{x\in\mathcal{R}})$. The quasi-parabolic vector bundle $(\mathcal{E}^{\mathtt{I}}(V)_*, \rho(\boldsymbol{\theta_n}))$ associated to $\boldsymbol{\theta_n} = \{\theta_{x,n}\}_{x\in\mathcal{R}}$ endowed with weights $\{\alpha_x^j\}$ is a PVB which we will also, by an abuse of notation, denote as the triple $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}_\rho^{\circ,x}\}_{x\in\mathcal{R}})$.*

Remark 5.2.2. Say $\mathcal{R}$ is a single point. A generic point θ in the interior of the Weyl alcove $\mathbf{a}^x$, lies in a single ρ-facet $\mathbf{a}_\rho^{\circ,x}$. In this case, $\boldsymbol{\theta}$ determines $\{\mathbf{a}_\rho^{\circ,x}\}_{x\in\mathcal{R}}$. Thus $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}_\rho^{\circ,x}\})$ is equivalent to the associated construction $(\mathcal{E}^{\mathtt{I}}(V)_*, \boldsymbol{\theta})$ of [3]. Belkale's completion of flag in [5] is precisely extending weights for the case θ lies in $\mathbf{a}_0 \setminus \{\text{far wall}\}$. This is enough to determine the semistability polytope Δ^{ss}.

Example 5.2.3. In [5], we have $G = \mathrm{SL}_n$, $\rho = \mathrm{Id}$ and so $\mathbf{a}_\rho^{\circ,x}$ is the alcove $\mathbf{a}_0^\circ$ of SL_n. For simplicity let $\mathcal{R} = \{x\}$. Let θ_d be the vertex of the alcove where only the affine root α_d does not vanish for $0 \leq d < n$. For $G = \mathrm{SL}_2, \rho = \mathrm{Id}$ and only one parabolic point x consider a weight $\theta \in [0, 1)$ and the parabolic vector bundle V_* given by $\mathcal{O}_X^{\oplus 2}$ with one flag at x of weight θ. Doing the extending weight construction for the pair $(\theta_1, \mathbf{a}_\rho^{\circ,x} := \mathbf{a}_0^\circ)$, taking limit as θ tends to θ_1, the flag acquires weight one. For $G = \mathrm{SL}_n, \rho = \mathrm{Id}$ weights $(b_0, b_1 \cdots, b_n)$ in barycentric coordinates correspond to weights $(0, b_1, b_1 + b_2, \cdots, b_1 + \cdots + b_n)$. In particular, for $\mathbf{a}_\rho^{\circ,x} = \mathbf{a}_0^\circ$, θ_d corresponds to vector bundle of degree $-d$. Further, for $d \geq 1$, $\rho(\theta_d)$ does lie on the far wall.

Let us contrast with the case when $\rho(\theta_d)$ does **not** lie on the far wall of the chosen $\mathbf{a}_\rho^{\circ,x}$. For instance suppose $\rho = Id$, $G = SL_2$ and $\theta_d = 1$ but we choose $\mathbf{a}_\rho^{\circ,x} = (1, 2)$ instead of the standard $(0, 1)$. In this case, the extended weight torsor $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}_\rho^{\circ,x}\})$ corresponds to a vector bundle $V \to X$ of determinant $-d$, and full-flag $0 \subset F_x^1 \subset F_x^2 \subset \cdots F_x^n = V_x$ at V_x with extended weights d/n and $\rho(\theta_d)$ is **not** on the far wall of $\mathbf{a}_\rho^{\circ,x}$. This happens because the flags are of V which is not a principal SL_n-bundle.

Remark 5.2.4. We illustrate sliding of weights (see §4.2) for the situation when $\mathcal{R} = \{x\}$, $G = SL_3$, $\rho = Id$. Let $\theta \in \mathbf{a}_0^\circ$ tend to a point θ_1 on the far wall with barycentric coordinates $(0, b_1, b_2)$. Hence $b_2 = 1 - b_1$. Doing the extending weight construction for the pair $(\theta_1, \mathbf{a}_\rho^{\circ,x} := \mathbf{a}_0^\circ)$, we see that rank three vector bundle with full flags acquire weights $(0, b_1, b_1 + b_2)$. On the other hand, θ_1 corresponds to rank three vector bundles with a single flag of dimension two with weight $1 - b_1$. This corresponds to the fact that $\theta_1 = (b_1, b_2)$ in the barycentric coordinates of the far wall. The general case of sliding of weights is only notationally harder to write.

Remark 5.2.5. We acquire weight 1 exactly when $\rho(\theta_x)$ lies on the far wall of the facet corresponding to $\mathbf{a}_\rho^{\circ,x}$.

Example 5.2.6. Let V_* be as in Example 5.2.3. Now $Ad : \mathrm{SL}_2 \to \mathrm{SL}_3$ corresponds to $V_* \mapsto \mathrm{Sym}^2(V_*)$. Now $\mathrm{Sym}^2(V_*)$ is the PVB with underlying bundle $\mathcal{O}_X^{\oplus 3}$ with weights $\{\theta, 2\theta\}$ if $\theta \in [0, 1/2)$ or $\mathcal{O}_X^{\oplus 2} \oplus \mathcal{O}_X(x)$ with weights $\{2\theta - 1, \theta\}$ if $\theta \in [1/2, 1)$. When $\theta \in [0, 1/2)$, then $\mathrm{Sym}^2(V_*)$ corresponds to $\mathbf{a}_0 \setminus \{\text{far wall}\}$ and thus does not have a map forgetting the flags to torsors on the far wall. When $\theta \in [1/2, 1)$, then the underlying degree of $\mathrm{Sym}^2(V_*)$ is not congruent to zero modulo three, and it corresponds to a parahoric SL_3 torsor which maps to torsors on the far wall. Let $W = \mathcal{O}_X \oplus \mathcal{O}_X(x)$. It corresponds to the far wall of SL_2 but $\mathrm{Sym}^2(W) = \mathcal{O}_X \oplus \mathcal{O}_X(x) \oplus \mathcal{O}_X(2x)$ corresponds to an affine Weyl group translate of the origin of SL_3 i.e. $\mathrm{Sym}^2(W) \otimes \mathcal{O}_X(-x)$ is a principal SL_3-bundle. We see that $\mathrm{Sym}^2(W)$ and the bundle underlying $\mathrm{Sym}^2(V_*)$ are related by a Hecke-modification when $\theta \in [1/2, 1)$. In this sense as θ tends to 1, V_* tends to W. This observation is formalized in the proposition below.

Proposition 5.2.7. *The vector bundles underlying $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}_\rho^{\circ,x}\})$ and $(\mathcal{E}(V), \rho(\boldsymbol{\theta}))$ are related by a Hecke-modification and are comparable under inclusion.*

Proof. By construction, the quasi-parabolic structure of $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}_\rho^{\circ,x}\})$ is the same as that of $(\mathcal{E}^{\mathtt{I}}(V), \rho(\boldsymbol{\theta_n}))$. Notice that for each $x \in \mathcal{R}$ the weights $\{\rho(\theta_{x,n})\}$ lie in a fixed facet σ_{SL}^x of $SL(V)$ in whose closure $\rho(\theta_x)$ lies. So considering the quasi-parabolic structures associated to these weights, we are in the situation of a *single Hecke-modification morphism* between the stacks associated to $\{\sigma_{SL}^x\}_{x \in \mathcal{R}}$ and $\boldsymbol{\theta}$ as in Diagram 1.0.1. Equivalently we have a morphism of stacks of quasi-parabolic vector bundles

$$QPVB(\rho(\theta_{x,n}), x \in \mathcal{R}) \to QPVB(\rho(\theta_x), x \in \mathcal{R}). \tag{5.2.4}$$

Under this morphism, the underlying vector bundles are related by a Hecke modification (or are the same if the morphism (5.2.4) corresponds merely to forgetting flags) and are comparable under inclusion. □

Remark 5.2.8. In the following proposition we show that the filtration (4.1.3) of $(\mathcal{E}(V)_*, \rho(\boldsymbol{\theta}))$ is refined by that of $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}_\rho^{\circ,x}\}_{x \in \mathcal{R}})$ upto (possibly) shifting of indices, the weights are also preserved upto sliding by a real number (see §4.2) and it forgets only those sheaves which do not matter for parabolic degree computations of these bundles as well as their sub-bundles as it happens in (4.3). We assume that $\mathcal{R} = \{x\}$ because the following argument can be applied point by point in the general case.

Proposition 5.2.9. *Assume $\mathcal{R} = \{x\}$. Let U (resp $U^{\mathtt{I}}$) be the vector bundles underlying $(\mathcal{E}(V)_*, \rho(\boldsymbol{\theta}))$ (resp. $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}_\rho^{\circ,x}\})$). Consider the filtration (4.1.3) of*

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$(\mathcal{E}^{\mathrm{I}}, \boldsymbol{\theta}, \{\mathbf{a}_\rho^{\circ,x}\})$ *at* x. *Then* U *is the* m*-term for* $0 \leq m \leq$ *the dimension of the facet of* $\mathrm{SL}(V)$ *containing* $\rho(\mathbf{a}_\rho^{\circ,x})$.

Proof. The underlying bundle is the 0-th term of (4.1.1). Further by Proposition 5.2.7, U is a term in the infinite filtration (4.1.3) of $(\mathcal{E}^{\mathrm{I}}, \boldsymbol{\theta}, \{\mathbf{a}_\rho^{\circ,x}\}_{x\in\mathcal{R}})$ because it is related to U^{I} by a Hecke-modification while being comparable to it under inclusion. It is the 0-term if and only if U is obtained from U^{I} *only by forgetting flags*, but if U is obtained by a non-trivial Hecke-modification of vector bundles, then U will be the m-th term for $m \geq 1$ as described above because U may correspond to any flag at x and the flag length l_x is given by the dimension of the facet of $\mathrm{SL}(V)$ containing $\rho(\mathbf{a}_\rho^{\circ,x})$. □

Proposition 5.2.10. *The notations are as in in §4.2. We carry out the following three steps. (i) We slide weights so that the bundle* U *of* (5.2.9) *has weight zero. (ii) Then, we discard sheaves for which there is a bigger sheaf with the same weight, and (iii) we shift indices so that* U *index zero. Then, we recover the filtration (4.1.1) for* $(\mathcal{E}(V)_*, \rho(\boldsymbol{\theta}))$ *at* x.

Proof. We slide the weights (see (4.2)) to make U have weight zero. Under the morphism $\mathcal{M}_X(\mathcal{G}^{\mathrm{I}}) \to \mathcal{M}_X(\mathcal{G})$ for each $x \in \mathcal{R}$ depending on the facet in $\mathcal{A}_{T_{SL(V)}}$ containing $\rho(\theta_x)$ we forget the corresponding terms in the filtration. These are exactly the sheaves in the infinite filtration (4.1.3) for which there is a bigger sheaf with the same weight. In particular, we forget all subsheaves of U containing $U(-x)$ which are different from these and which have weight zero or one. Now U_* has weights lying in $[0,1)$ because the formation of U_* ignores $U(-x)$. Thus U_* is the parabolic vector bundle corresponding to $(\mathcal{E}(V)_*, \rho(\boldsymbol{\theta}))$. Further, we get the filtration corresponding to $(\mathcal{E}(V)_*, \rho(\boldsymbol{\theta}))$ if we shift indices so that the index of U becomes zero. □

6. (Semi)-stability of extended weight torsors

The curve X is arbitrary. The aim of this section is to formulate a notion of extended weight $\boldsymbol{\theta}$ (semi)stability for $\mathcal{E}^I$ i.e the case when $\boldsymbol{\theta}$ does not lie in $Maps(\mathcal{R}, \mathbf{a}_0^\circ)$ but only in its closure. We do this by showing the equivalence of (semi)stability of $(\mathcal{E}, \boldsymbol{\theta})$ with that of $(\mathcal{E}^{\mathrm{I}}, \boldsymbol{\theta}, \{\mathbf{a}_\rho^{\circ,x}\}_{x\in\mathcal{R}})$ as defined in (5.2.1). In this section, we work with $\rho = Ad : G \to \mathrm{SL}(\mathfrak{g})$, the "adjoint" representation of G.

6.1. **Invariant direct image functor.** We briefly summarise [3, §3.1]. Let $p : W \to U$ be a finite flat surjective morphism of normal integral Noetherian schemes that is Galois with Galois group Γ. Let $\mathcal{H}$ be an affine group scheme on W. Assume further that $\mathcal{H}$ is equipped with an action of Γ lifting the one on W, so that "mult" and "inverse" maps are Γ-equivariant.

Definition 6.1.1. *The invariant direct image* $p_*^\Gamma(\mathcal{H})$ *of* $\mathcal{H}$ *is the group scheme on* U *whose valued points for any* U*-scheme* S *are given by* $p_*^\Gamma(\mathcal{H})(S) = \mathcal{H}(S \times_U W)^\Gamma$.

Further, the invariant direct image functor commutes with the Lie-algebra functor in the following sense

$$Lie(p_*^\Gamma(\mathcal{H})) = p_*^\Gamma(Lie(\mathcal{H})). \tag{6.1.1}$$

More generally, the invariant direct image may be defined for any *affine* scheme over W together with a lift of Γ-action.

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6.2. **Rational weight parahoric torsors as $\Gamma - G$ bundles.** We briefly recall some results from [2, §5]. Let $\mathcal{G} \to X$ be a parahoric group scheme as in §2.1.2. Say $\boldsymbol{\theta}$ is a weight such that each θ_x is a rational point of the facet σ_x corresponding to the restriction of $\mathcal{G}$ at the formal disc $\mathbb{D}_x$. By [2, Thm 5.3.1], there exists a finite Galois cover $p : Y \to X$ branched over $\mathcal{R}$ with Galois group Γ and a principal G-bundle F (see [2, Notn. 5.1.0.1]) equipped with a left action of Γ such that if $Isom_Y(F, F)$ denotes the adjoint group scheme of F then taking the invariant direct image sets up a canonical isomorphism of group schemes:

$$p_*^{\Gamma}(Isom_Y(F, F)) = \mathcal{G}. \tag{6.2.1}$$

Let $\mathcal{R}_Y = p^{-1}(\mathcal{R}) \subset Y$ be the ramification points of the covering p. For each $y \in \mathcal{R}_Y$, let $\Gamma_y \subset \Gamma$ denote the isotropy subgroup that fixes y. Let $\tau_y : \Gamma_y \to Aut(F_y)$ be the action on the fiber of y. We denote its conjugacy class as $[\tau_y]$. By the type τ of F one means the set $\tau = \{[\tau_y] | y \in \mathcal{R}_y\}$ of conjugacy classes. Let $\mathcal{M}_Y^{\tau}(\Gamma, G)$ denote the moduli stack of $(\Gamma - G)$ bundles on Y of type τ. We have an isomorphism of algebraic stacks

$$\alpha_F : \mathcal{M}_Y^{\tau}(\Gamma, G) \to \mathcal{M}_X(\mathcal{G}), \tag{6.2.2}$$

given by F as follows: denote by $Isom_Y(E, F)$ the sheaf of local isomorphisms

$$E \mapsto p_*^{\Gamma}(Isom_Y(E, F)). \tag{6.2.3}$$

The inverse map is given by $\mathcal{E} \mapsto p^*(\mathcal{E}) \times_{p^*(\mathcal{G})} F$.

6.3. **Definition of (semi)stability for a parahoric torsor** $(\mathcal{E}, \boldsymbol{\theta})$ *with rational weight* $\boldsymbol{\theta}$**.** Keeping all the notations and the setup as in §6.2, we make [3, §6] more precise using the notion of *parabolically associated constructions.* Let $\mathcal{E}(\mathfrak{g})_* = (\mathcal{E}(\mathfrak{g}), \boldsymbol{\theta})$ denote the associated parabolic vector bundle ([3, §5]). Let $\mathcal{E}_{\boldsymbol{\theta}}^{par}(\mathcal{G})$ denote the *parabolicaly associated adjoint group scheme of* $\mathcal{E}$ defined as follows: let F and E be be a principal $\Gamma - G$ bundles on Y such that

$$p_*^{\Gamma}(Isom_Y(F, F)) = \mathcal{G} \quad \text{and} \quad \alpha_F(E) = \mathcal{E}.$$

Let $E(G) = Isom_Y(E, E)$ denote the adjoint group scheme of E. We define

$$\mathcal{E}_{\boldsymbol{\theta}}^{par}(\mathcal{G}) := p_*^{\Gamma}(E(G)). \tag{6.3.1}$$

Let $E(\mathfrak{g})$ denote the Lie-algebra bundle of E defined by associated constructions as follows: $E \times_{G,Ad} \mathfrak{g}$. By (6.1.1), we define the *parabolically associated Lie-algebra bundle* $\mathrm{Lie}(\mathcal{E}_{\boldsymbol{\theta}}^{par}(\mathcal{G}))$ of $(\mathcal{E}, \boldsymbol{\theta})$ as:

$$\mathrm{Lie}(\mathcal{E}_{\boldsymbol{\theta}}^{par}(\mathcal{G})) = \mathcal{E}_{\boldsymbol{\theta}}^{par}(\mathfrak{g}) := p_*^{\Gamma}(Lie(E(G))) = p_*^{\Gamma}(E(\mathfrak{g})). \tag{6.3.2}$$

The Lie algebra bundle $\mathcal{E}_{\boldsymbol{\theta}}^{par}(\mathfrak{g})$ is given the structure of a parabolic Lie algebra bundle by identifying it with the vector bundle underlying $\mathcal{E}(\mathfrak{g})_*$ (see also [3, §5,§6]).

Let η be the generic point of the curve X. Let $\mathcal{E}_{\boldsymbol{\theta}}^{par}(\mathcal{G})_\eta$ denote the restriction of $\mathcal{E}_{\boldsymbol{\theta}}^{par}(\mathcal{G})$ to η. Let $\mathcal{P}_\eta \subset \mathcal{E}_{\boldsymbol{\theta}}^{par}(\mathcal{G})_\eta$ be a parabolic subgroup scheme. Taking the flat closure of $\mathcal{P}_\eta$ in $\mathcal{E}_{\boldsymbol{\theta}}^{par}(\mathcal{G})$ we get the subgroup scheme $\mathcal{P}_{\boldsymbol{\theta}} \subset \mathcal{E}_{\boldsymbol{\theta}}^{par}(\mathcal{G})$. The Lie algebra bundle $\mathrm{Lie}(\mathcal{P}_{\boldsymbol{\theta}})$ is a sub-bundle of $\mathrm{Lie}(\mathcal{E}_{\boldsymbol{\theta}}^{par}(\mathcal{G}))$, and we give it the *canonical induced parabolic structure* to get a parabolic Lie subbundle $\mathrm{Lie}(\mathcal{P}_{\boldsymbol{\theta}})_*$ of $\mathcal{E}(\mathfrak{g})_*$.

Definition 6.3.1. *(*[3, Defn 6.1]*) One calls a parahoric torsor* $(\mathcal{E}, \boldsymbol{\theta})$ *(semi)stable if for every generic parabolic subgroup scheme* $\mathcal{P}_\eta \subset \mathcal{E}_{\boldsymbol{\theta}}^{par}(\mathcal{G})_\eta$ *as above, we have*

$$pardeg(Lie(\mathcal{P})_*) < 0 \ (\textit{respectively } pardeg(Lie(\mathcal{P})_*) \leq 0).$$

In Remark 6.4.3 we show how to extend this definition to real weights.

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6.4. **Definition of (semi)stability for extended real weights.** Let $(\mathcal{E}, \boldsymbol{\theta})$ be a parahoric torsor where $\boldsymbol{\theta} = \{\theta_x | x \in \mathcal{R}\}$ and θ_x are real weights. Under $p : \mathcal{M}_X(\mathcal{G}^{\mathtt{I}}) \to \mathcal{M}_X(\mathcal{G})$, let $\mathcal{E}^{\mathtt{I}}$ map to $\mathcal{E}$. For each $x \in \mathcal{R}$, we will fix an Ad-facet $\mathbf{a}^{\circ,x}_{Ad}$ in whose closure θ_x lies. In each $\mathbf{a}^{\circ,x}_{Ad}$, we pick a sequence of *rational points* $\theta_{x,n}$ *converging to* θ_x. We denote by $\boldsymbol{\theta}_n$ the rational weight $\{\theta_{x,n} | x \in \mathcal{R}\}$. Let $\mathcal{E}^{\mathtt{I},par}_{\boldsymbol{\theta}_n}(\mathcal{G}^{\mathtt{I}})$ be the parabolically associated adjoint group scheme (see 6.3.1) of $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}_n)$.

Proposition 6.4.1. *The vector bundle underlying* $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}^{\circ,x}_{Ad}\})$ *identifies naturally with the Lie algebra bundle* $Lie(\mathcal{E}^{\mathtt{I},par}_{\boldsymbol{\theta}_n}(\mathcal{G}^{\mathtt{I}}))$ *of* $\mathcal{E}^{\mathtt{I},par}_{\boldsymbol{\theta}_n}(\mathcal{G}^{\mathtt{I}})$.

Proof. For simplicity, we may suppose that $\mathcal{R} = \{x\}$. By the construction in Definition 5.2.1 , the quasi-parabolic structure of $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}^{\circ,x}_{Ad}\})$ and $(\mathcal{E}^{\mathtt{I}}(\mathfrak{g})_*, Ad(\theta_{x,n}))$ are the same. Since for varying n the $\theta_{x,n}$ all lie in the same Ad-facet $\mathbf{a}^{\circ,x}_{Ad}$, the vector bundle underlying $(\mathcal{E}^{\mathtt{I}}(\mathfrak{g})_*, Ad(\theta_{x,n}))$ is independent of n. Further, just as $\mathcal{E}^{par}_{\boldsymbol{\theta}}(\mathfrak{g})$ identifies with the vector bundle underlying $\mathcal{E}(\mathfrak{g})_*$ by (6.1.1), $\mathrm{Lie}(\mathcal{E}^{\mathtt{I},par}_{\boldsymbol{\theta}_n}(\mathcal{G}^{\mathtt{I}}))$ identifies with the vector bundle underlying $(\mathcal{E}^{\mathtt{I}}(\mathfrak{g})_*, Ad(\theta_{x,n}))$. □

We induce the parabolic structure on $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}^{\circ,x}_{Ad}\})$ to $\mathrm{Lie}(\mathcal{E}^{\mathtt{I},par}_{\boldsymbol{\theta}_n}(\mathcal{G}^{\mathtt{I}}))$ and, by an abuse of notation forgetting the smaller facets, denote it by $\mathrm{Lie}(\mathcal{E}^{\mathtt{I}}_{\boldsymbol{\theta}}(\mathcal{G}^{\mathtt{I}}))_*$. We observe the independence with respect to n of the following:

$$\mathrm{Lie}(\mathcal{E}^{\mathtt{I},par}_{\boldsymbol{\theta}_n}(\mathcal{G}^{\mathtt{I}})) \quad \text{and} \quad \mathcal{E}^{\mathtt{I},par}_{\boldsymbol{\theta}_n}(\mathcal{G}^{\mathtt{I}})_\eta \tag{6.4.1}$$

The latter equals $\mathcal{E}^{par}_{\boldsymbol{\theta}}(\mathcal{G})_\eta$. Let $\mathcal{P}_\eta \subset \mathcal{E}^{par}_{\boldsymbol{\theta}}(\mathcal{G})_\eta$ be a parabolic subgroup scheme. We denote by $\mathrm{Lie}(\mathcal{P}^{\mathtt{I}})$ the sheaf that is the closure of $\mathrm{Lie}(\mathcal{P}_\eta)$ in $\mathrm{Lie}(\mathcal{E}^{\mathtt{I},par}_{\boldsymbol{\theta}_n}(\mathcal{G}^{\mathtt{I}}))$ with torsion-free quotient. Thus $\mathrm{Lie}(\mathcal{P}^{\mathtt{I}})$ is a sub-bundle of $\mathrm{Lie}(\mathcal{E}^{\mathtt{I},par}_{\boldsymbol{\theta}_n}(\mathcal{G}^{\mathtt{I}}))$. We endow $\mathrm{Lie}(\mathcal{P}^{\mathtt{I}})$ with the *canonical induced parabolic structure* from $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}^{\circ,x}_{Ad}\}_{x\in\mathcal{R}})$ and forgetting the smaller facets denote the associated PVB by

$$\mathrm{Lie}(\mathcal{P}^{\mathtt{I}}_{\boldsymbol{\theta}})_*. \tag{6.4.2}$$

Definition 6.4.2. *Let* $(\mathcal{E}, \boldsymbol{\theta})$ *be a parahoric torsor with real weights. Under* $p : \mathcal{M}_X(\mathcal{G}^{\mathtt{I}}) \to \mathcal{M}_X(\mathcal{G})$*, let* $\mathcal{E}^{\mathtt{I}}$ *map to* $\mathcal{E}$*. We say that the parahoric torsor with extended weights* $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}^{\circ,x}_{Ad}\})$ *(see 5.2.1) is (semi)stable if for every generic parabolic subgroup scheme* $\mathcal{P}_\eta \subset \mathcal{E}^{\mathtt{I},par}_{\boldsymbol{\theta}_n}(\mathcal{G}^{\mathtt{I}})_\eta$ *we have*

$$pardeg(Lie(\mathcal{P}^{\mathtt{I}}_{\boldsymbol{\theta}})_*) < 0 \quad (respectively \quad pardeg(Lie(\mathcal{P}^{\mathtt{I}}_{\boldsymbol{\theta}})_*) \le 0). \tag{6.4.3}$$

Remark 6.4.3. If for each $x \in \mathcal{R}$ instead of the Ad-facets $\mathbf{a}^{\circ,x}_{Ad}$, we had chosen simply the unique Ad-facet in which θ_x lies, and instead of $\mathcal{E}^{\mathtt{I}}$ we worked with the quasi-parahoric torsor $\mathcal{E}$, then the construction in this subsection recovers the extension [3, Defn 6.1] of Definition 6.3.1 to real weights.

6.5. **Equivalence of (semi)stability.**

Proposition 6.5.1. *The parahoric torsor* $(\mathcal{E}, \boldsymbol{\theta})$ *is (semi)stable (see 6.4.3) if and only if the extended weight parahoric torsor* $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}^{\circ,x}_{Ad}\}_{x\in\mathcal{R}})$ *is (semi)stable . The (semi)stability of* $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}^{\circ,x}_{Ad}\}_{x\in\mathcal{R}})$ *is independent of the choices of* $\{\mathbf{a}^{\circ,x}_{Ad}\}_{x\in\mathcal{R}}$.

Proof. By Proposition 5.2.7, the Lie algebra bundles $\mathrm{Lie}(\mathcal{E}^{\mathtt{I},par}_{\boldsymbol{\theta}_n}(\mathcal{G}^{\mathtt{I}}))$ and $\mathrm{Lie}(\mathcal{E}^{par}_{\boldsymbol{\theta}}(\mathcal{G}))$, being the vector bundles underlying $(\mathcal{E}^{\mathtt{I}}(\mathfrak{g}), \boldsymbol{\theta})$ and $(\mathcal{E}(\mathfrak{g}), \boldsymbol{\theta})$ respectively, are related by a Hecke-modification and they are comparable under inclusion. In Proposition 5.2.9 we denoted them by U^I and U respectively. Let a denote the difference between the weights attached to U and U^I in (4.1.3). Sliding of weights

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(see §4.2) by a real number a only changes the parabolic slope by a. So by Remark 5.2.8 if we slide weights a then we have

$$par\mu(\mathrm{Lie}(\mathcal{E}^{\mathtt{I},par}_{\boldsymbol{\theta}_n}(\mathcal{G}^{\mathtt{I}}))_*) - par\mu(\mathrm{Lie}(\mathcal{E}^{par}_{\boldsymbol{\theta}}(\mathcal{G}))_*) = a.$$

Let us consider a generic parabolic subgroup scheme $\mathcal{P}_\eta \subset \mathcal{E}^{\mathtt{I},\mathtt{par}}_{\theta_n}(\mathcal{G}^{\mathtt{I}})_\eta = \mathcal{E}^{par}_{\boldsymbol{\theta}}(\mathcal{G})$.

We have the natural identification $\mathrm{Lie}(\mathcal{P}^{\mathtt{I}}_{\boldsymbol{\theta}})_\eta = \mathrm{Lie}(\mathcal{P}^{\mathtt{I}}_\eta) = \mathrm{Lie}(\mathcal{P}_\eta) = \mathrm{Lie}(\mathcal{P}_{\boldsymbol{\theta}})_\eta$. The parabolic structure on $\mathrm{Lie}(\mathcal{P}^{\mathtt{I}}_{\boldsymbol{\theta}})_*$ is induced from $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}^{\circ,x}_{Ad}\}_{x\in\mathcal{R}})$. So from the infinite filtration (4.1.3) of $\mathrm{Lie}(\mathcal{P}^{\mathtt{I}}_{\boldsymbol{\theta}})_*$ we can extract, like in Remark 5.2.8, the infinite filtration of $\mathrm{Lie}(\mathcal{P}_{\boldsymbol{\theta}})_*$ by forgetting some sheaves which turn out to be only those that do not matter critically for parabolic degree computations. Thus

$$par\mu(\mathrm{Lie}(\mathcal{P}^{\mathtt{I}}_{\boldsymbol{\theta}})_*) - par\mu(\mathrm{Lie}(\mathcal{P}_{\boldsymbol{\theta}})_*) = a.$$

This shows the first assertion. Now the second statement follows. □

We turn the independence observed above into a definition.

Definition 6.5.2. *We say that a $\mathcal{G}^{\mathtt{I}}$-torsor $\mathcal{E}^{\mathtt{I}}$ is (semi)stable with extended weight $\boldsymbol{\theta}$ if for some choice of facets $\{\mathbf{a}^{\circ,x}_{Ad}\}_{x\in\mathcal{R}}$ (and therefore any by (6.5.1)) the torsor $(\mathcal{E}^{\mathtt{I}}, \boldsymbol{\theta}, \{\mathbf{a}^{\circ,x}_{Ad}\}_{x\in\mathcal{R}})$ is (semi)stable in the sense of (6.4.2).*

Remark 6.5.3. The (semi)stability Definition [19, Defn 2.2] for parabolic principal G-bundle is for the product of alcoves without their far walls (see [19, Defn 2.1]). By continuity, it may be naturally extended to the product of closed Weyl alcoves.

Via the map $q : \mathcal{M}(\mathcal{G}^{\mathtt{I}}) \to \mathcal{M}(G)$ of (1.0.1), a $\mathcal{G}^{\mathtt{I}}$-torsor $\mathcal{E}^{\mathtt{I}}$ with extended weight $\boldsymbol{\theta}$ may be viewed as a parabolic G-bundle but with *extended weights*, especially in the sense that some weights may become equal and may lie on the boundary of $Maps(\mathcal{R}, \mathbf{a}_0)$. The following proposition shows that [19, Defn 2.2] agrees with 6.5.2 even for extended weights.

Proposition 6.5.4. *A $\mathcal{G}^{\mathtt{I}}$-torsor $\mathcal{E}^{\mathtt{I}}$ with extended weights $\boldsymbol{\theta}$ is (semi)stable (see 6.5.2) if and only if it is (semi)stable as a parabolic G-bundle (see 6.5.3).*

Proof. In the case of parahoric $\mathcal{G}^{\mathtt{I}}$ torsors, the Definition 6.5.2 reduces to Definition 6.3.1. Viewing parahoric $\mathcal{G}^{\mathtt{I}}$ torsors as parabolic G-bundles, (6.3.1) in terms of slopes of the adjoint PVB with respect to reductions of structure group of G to parabolic subgroups is *equivalent* to the standard definition (see [19, Defn 2.2]). When we take limits of weights (5.2.3), this equivalence extends between $\mathcal{G}^{\mathtt{I}}$ torsors equipped with extended weights and their associated PVBs in the sense of (6.5.2) and parabolic G-bundles with weights lying in the boundary of the space of weights, which is a product of alcoves. □

7. Some deformation theory and the Proof of Theorem 1.0.2

We are now back in the case $X = \mathbb{P}^1$. Let $(\mathcal{E}, \boldsymbol{\theta})$ be a parahoric $\mathcal{G}$-torsor on $\mathbb{P}^1$ with parahoric structure on the marked points $\mathcal{R}$. Recall that if $\mathcal{E}(\mathfrak{g})$ denotes the Lie-algebra bundle underlying the parabolic bundle $\mathcal{E}(\mathfrak{g})_*$, the first order deformations of $(\mathcal{E}, \boldsymbol{\theta})$ are controlled by the cohomology space $H^1(\mathbb{P}^1, \mathcal{E}(\mathfrak{g}))$.

Let $\mathcal{E}^{\mathtt{I}}$ be a trivial bundle with generic B-structures (B being a fixed Borel subgroup of G) and *extended weight* $\boldsymbol{\theta}$ as in Theorem 1.0.1. In §5.2, after choosing smaller Ad-facets, we have explained the construction of the parabolic vector bundle V_* associated to it by the Ad representation. We further observe that if $\mathcal{E}$ is a

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torsor for the Iwahori group scheme, then there is an underlying principal G-bundle with standard parabolic structures with fibres B at the marked points. Under these conditions, the (semi)stability conditions in §6.3 can be rephrased in terms of the usual (semi)stability of principal G-bundles but carrying along the Iwahori structures. In other words, $\mathcal{E}$ is (semi)stable if and only if for every parabolic subgroup $P \subset G$ and reduction of structure group $\mathcal{E}_P$ to P, we have $pardeg(\mathcal{E}_P(\mathfrak{p})_*) \leq 0 (< 0)$, where $\mathcal{E}_P(\mathfrak{p})_* \subset \mathcal{E}(\mathfrak{g})_*$ gets the canonical induced parabolic structure. This definition coincides with the one in [19].

With these notions in place, we now complete the proof of Theorem 1.0.2.

Proof of Theorem 1.0.2. $\Longrightarrow$ If $\boldsymbol{\theta} \in \Delta^s$, by Theorem 1.0.1, $\mathcal{E}^{\mathtt{I}}$, the trivial bundle with generic B-structure with extended weights $\boldsymbol{\theta}$, is stable. So, owing to stability, it has no sub-bundles whose associated parabolic vector bundle has degree zero.

$\Longleftarrow$ Since $\boldsymbol{\theta} \in \Delta^{ss}$, so by Theorem 1.0.1 the trivial bundle with generic B-structure $\mathcal{E}^{\mathtt{I}}$ is semistable. Now the stack of B-structures on the trivial bundle is algebraic, smooth and irreducible. Therefore, we have a smooth and irreducible versal space $\mathcal{T}$. Let η denote its generic point and let $\mathcal{E}^{\mathtt{I}}$ be a versal torsor on $\mathcal{T}$. If $\mathcal{E}^{\mathtt{I}}$ is only semistable and not stable at η, then in the setting of §6.3, by restricting $\mathcal{T}$ if necessary, we may assume that for all $t \in \mathcal{T}$ the family $\mathcal{E}^{\mathtt{I}}_t(\mathfrak{g})_*$ of torsors admits parabolic reductions $\mathcal{E}^{\mathtt{I}}_P(\mathfrak{p})_{t,*}$ with $\mathrm{pardeg}(\mathcal{E}^{\mathtt{I}}_P(\mathfrak{p})_{t,*}) = 0$.

At a point $t \in \mathcal{T}$ corresponding to $\mathcal{E}^{\mathtt{I}}_t$ with generic B-structures, let $\mathcal{E}^{\mathtt{I}}_{t,P}$ be a parabolic reduction. Let us consider the map $\phi_{\mathcal{E}^{\mathtt{I}}_{t,P}}$ from $\{$*deformations of* $\mathcal{E}^{\mathtt{I}}_{t,P}\}$ to $\{$*deformations of* $\mathcal{E}^{\mathtt{I}}_t\}$. We have the following exact sequence of parabolic bundles:

$$0 \to \mathcal{E}^{\mathtt{I}}_{t,P}(\mathfrak{p})_* \to \mathcal{E}^{\mathtt{I}}_t(\mathfrak{g})_* \to \mathcal{E}^{\mathtt{I}}_t(\mathfrak{g})_*/\mathcal{E}^{\mathtt{I}}_{t,P}(\mathfrak{p})_* \to 0. \tag{7.0.1}$$

By assumption on $\mathcal{E}^{\mathtt{I}}_t$, no P-reduction $\mathcal{E}^{\mathtt{I}}_{t,P}$ is of the *minus 1 type*. So the quotient $\mathcal{E}^{\mathtt{I}}_t(\mathfrak{g})_*/\mathcal{E}^{\mathtt{I}}_{t,P}(\mathfrak{p})_*$ has $\mathrm{pardeg} = 0$ but the degree of the underlying vector bundle is **not** -1. Hence it has a non-trivial H^1. Thus $\phi_{\mathcal{E}^{\mathtt{I}}_{t,P}}$ is *not surjective*. Therefore $\mathcal{E}^{\mathtt{I}}_{t,P}$ does not deform to the generic point of the versal space. And this holds for all parabolic reductions of $\mathcal{E}^{\mathtt{I}}_t$. So $\mathcal{E}^{\mathtt{I}}_\eta$ has no destabilizing $\mathcal{E}^{\mathtt{I}}_{P,\eta}$ arising as a deformation from a closed point of $\mathcal{T}$. However, any $\mathcal{E}^{\mathtt{I}}_{P,\eta}$ of $\mathcal{E}_\eta$ spreads to a Zariski neighbourhood of η in $\mathcal{T}$. So $\mathcal{E}$ must be *stable*. Hence $\boldsymbol{\theta} \in \Delta^s$.

□

Proposition 7.0.1. *The condition "no P-reduction $\mathcal{E}^{\mathtt{I}}_{P}$ is of the minus 1 type" of Theorem 1.0.2 can be formulated in terms of vanishing of Gromov-Witten numbers.*

Proof. For parabolic vector bundles, one considers the trivial bundle with generic quasi-parabolic structures of a fixed type. Then Gromov-Witten input data exactly corresponds to the triples of rank, degree and type of induced parabolic structure on sub-bundles. Recall that this generalizes suitably also for principal G-bundles (see [19]). Consider any P-reduction of structure group $\mathcal{E}^{\mathtt{I}}_{P}$ of $\mathcal{E}^{\mathtt{I}}$. By associated constructions, it leads to a sub-PVB $Lie(\mathcal{P}^{\mathtt{I}}_{\boldsymbol{\theta}})_*$ in (6.4.2). We have also denoted it as $\mathcal{E}^{\mathtt{I}}_P(\mathfrak{p})_*$ above. Notice that for $x \in \mathcal{R}$, the local parabolic degree of $Lie(\mathcal{P}^{\mathtt{I}}_{\boldsymbol{\theta}})_*$ gets determined by $\mathcal{E}^{\mathtt{I}}_{P}$ locally around x of which there are only finitely many possibilities analogous to the induced flag types in the case of PVBs. Further, if $Lie(\mathcal{P}^{\mathtt{I}}_{\boldsymbol{\theta}})_*$ happens to have the same slope as $\mathcal{E}^{\mathtt{I}}(\mathfrak{g})_*$, then its underlying degree, say d, gets determined as well. Thus, the set of input data $\{d\}$ and $\{w_x | x \in \mathcal{R}\}$ for Gromov-Witten numbers of sub-bundles that could potentially violate stabilty gets determined and is finite. We now further observe that these input data also determine the underlying degree of the quotient bundle $\mathcal{E}^{\mathtt{I}}(\mathfrak{g})_*/Lie(\mathcal{P}^{\mathtt{I}}_{\boldsymbol{\theta}})_*$. Therefore, the condition "no P-reduction $\mathcal{E}^{\mathtt{I}}_{P}$ is of the *minus 1 type*" can be formulated in terms of vanishing of Gromov-Witten numbers. □

Thus Theorem 1.0.2 gives verifiable criteria for points in Δ^{ss} to lie in Δ^{s}.

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