MEAN-CONTINUOUS INTEGRALS

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Introduction. Descriptive definitions of Cesàro-Denjoy integrals (CD-integrals) equivalent to the Cesàro-Perron integrals (CP-integrals) introduced by J. C. Burkill [1, 2] have been given by Miss Sargent [6] (see § **2**). The CD-integrals are generalizations of the special Denjoy integral [5, p. 201]. They are somewhat complicated in that modifications of the definitions of continuity, generalized absolute continuity in the restricted sense (ACG_*) [5, p. 231], and of derivatives are required for each order. In the present paper a scale of integrals is obtained which is based on the descriptive definition of the general Denjoy integral [5, p. 241]. The approximate derivative and a slightly modified definition of generalized absolute continuity (ACG) are used for all orders so that the only concept generalized for increasing orders is that of continuity. The resulting r^{th} order integral, $r = 0, 1, 2, \ldots$, called the r^{th} generalized mean integral (GM_r-integral), contains the corresponding C_rD- and C_rP-integrals.

In § **1** the descriptive definition of the GM_r-integral is given and some of the more important properties of the integral, including a theorem on integration by parts, are derived. The relation between the GM_r-integral and the C_rD-, C_rP-integrals is considered in § **2**. In § **3** a constructive definition of the GM_r-integral is given and shown to be equivalent to the descriptive definition. The paper concludes with a proof that the indefinite GM_r-integral takes all values between its upper and lower bounds on any interval over which the integral exists.

1. The descriptive generalized mean integrals. We shall obtain a scale or series of generalized mean integrals, GM_r-integrals, $r = 0, 1, \ldots$ of increasing generality in the sense that each integral will be contained in but not equivalent to all those with higher subscripts. We take as our starting point the general Denjoy integral.

Notation. We number theorems by Rcman numerals, lemmas by Arabic numerals, and the definitions which change with the order by groups of letters. The order concerned in each case is indicated by a subscript, e.g. Definition (M_rC) is the definition of mean continuity of order r. With this notation we refer, for example, to "Theorem II_r" rather than "Theorem II for the GM_r-integral."

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DEFINITION (M_r). *We define the M_r-mean of $F(x)$ on (a, b) as $F(b)$ for $r = 0$ and as*

$$M_r(F, a, b) = \frac{1}{(b-a)^r}\int_a^b (b-t)^{r-1}F(t)dt,$$

for r a positive integer, where the integral in the definition of the M_1-mean is in the general Denjoy sense and the sense in which the integral involved in the M_r-mean is required to exist will be stated below.

DEFINITION (M_rC). *The function $F(x)$ is continuous in the r^{th} mean sense (M_r-continuous) at x_0 if $M_r(F, x_0, x_0 + h) \to F(x_0)$ as $h \to 0$.*

DEFINITION. *A function $F(x)$ is generalized absolutely continuous (ACG) on a set E if E can be expressed as the sum of a finite or denumerable sequence of closed sets $E_1, E_2, \ldots$ such that $F(x)$ is absolutely continuous (AC) on each set E_n.*[1]

DEFINITION (M_rI). *The function $f(x)$ is GM_r-integrable on (a, b) if there exists an M_r-continuous function $F(x)$ that is ACG on (a, b) and is such that the approximate derivative of $F(x)$, $ADF(x)$,* [5, *p.* 220] *exists and is equal to $f(x)$ almost everywhere on (a, b). The function $F(x)$ is then called an indefinite GM_r-integral of $f(x)$ on (a, b). The definite GM_r-integral of f over (a, b) is designated by*

$$GM_r(f, a, b) = (GM_r)\int_a^b f(x)dx = F(b) - F(a).$$

For $r = 0$ this definition is seen to be equivalent to the descriptive definition of the general Denjoy integral.

The integral in the definition of the M_r-mean, $r \geqq 2$, is required to exist in the sense of the GM_{r-1}-integral. In order to define the GM_r-integral we must therefore assume that the GM_{r-1}-integral has been defined. To establish the properties of the GM_r-integral we must assume the following properties for the GM_{r-1}-integral.

PROPERTY I_{r-1}. *If $f_1(x), f_2(x)$ are GM_{r-1}-integrable on (a, b) and $f_1(x) \geqq f_2(x)$ almost everywhere on (a, b) then*

$$GM_{r-1}(f_1, a, b) \geqq GM_{r-1}(f_2, a, b).$$

PROPERTY II_{r-1}. *The GM_{r-1}-integral contains the GM_{r-2}-integral.*

PROPERTY III_{r-1}. *Let $f_n(x)$ be GM_n-integrable on (a, b), let $F_n(x) = GM_n(f_n, a, x)$, $n = r-1, r-2, \ldots 0$, and let*

$$g_n(x) = \int_a^x dt_1 \int_a^{t_1} \ldots dt_{n-1} \int_a^{t_{n-1}} g(t)dt,$$

where $g(x)$ is of bounded variation on (a, b). Then $f_n(x)g_n(x)$ is GM_n-integrable on (a, b) and

$$(GM_n)\int_a^b f_n(x)g_n(x)dx = F_n(b)g_n(b) - (GM_{n-1})\int_a^b F_n(x)g_{n-1}(x)dx.$$

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[1]Saks' definition of ACG [5, p. 222] implies that $F(x)$ is continuous and does not require the sets E_n to be closed. The condition that the sets E_n be closed gives no restriction when $F(x)$ is continuous since the continuity of $F(x)$ is sufficient to ensure that if $F(x)$ is AC on an arbitrary set it is AC on the closure of this set.

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(Property III_{r-1} *implies that, if* $f(x)$ *is* GM_{r-1}*-integrable, then* $f(x)g_{r-1}(x)$ *can be integrated by parts r times.)*

That properties I_{r-1}-III_{r-1} may be presumed is justified by induction. Since the GM_0-integral is the general Denjoy integral it is clear that Property I_0 is true and that Property III_0 is true provided that the GM_{-1}-integral is interpreted as a Stieltjes integral and $g(x)dx$ is replaced by $dg(x)$ [5, p. 246]. If $f(x)$ is GM_0-integrable on (a, b), $F(x) = GM_0(f, a, x)$ is ACG on (a, b) and $ADF(x) = f(x)$ almost everywhere on (a, b). To establish II_1 we need only prove that $F(x)$ is M_1-continuous and, since $F(x)$ is continuous, this is easily done. Our inductive process will be complete if, when we define the GM_r-integral and assume Properties I_{r-1}-III_{r-1}, we can establish Properties I_r-III_r.

LEMMA 1_r. *If there exists an interval* $(x_0, x_0 + h)$, $h > 0$ *and a positive number* d *such that* $F(t) - F(x_0) < d$ *for all* t *except at most a set of measure zero, then* $F(x)$ *cannot be* M_r*-continuous at* x_0.

Let x be any point in $(x_0, x_0 + h)$. Then, using Property I_{r-1},

$$\begin{aligned} M_r(F, x_0, x) &= r(x - x_0)^{-r}(GM_{r-1}) \int_{x_0}^{x} (x - t)^{r-1} F(t)dt \\ &\geqq r(x - x_0)^{-r} \int_{x_0}^{x} (x - t)^{r-1}[F(x_0) + d]\, dt \\ &= F(x_0) + d. \end{aligned}$$

It follows that $F(x)$ cannot be M_r-continuous at x_0. Similar results hold for $h < 0$ and also if $F(t) - F(x_0) > d$ is replaced by $F(t) - F(x_0) < -d$.

THEOREM I_r. *If* $F(x)$ *is monotone and* M_r*-continuous for* $a < x < b$ *then* $F(x)$ *is monotone and continuous for* $a \leqq x \leqq b$.

If we suppose that $F(x)$ is not continuous at some point x, $a \leqq x \leqq b$, Lemma 1_r gives a contradiction.

THEOREM II_r. *If* $F(x)$ *is* M_r*-continuous and* ACG *on* (a, b) *and if* $ADF(x) \geqq 0$ *almost everywhere on* (a, b), *then* $F(x)$ *is non-decreasing on* (a, b).

DEFINITION [4, p. 130]. *A function* $F(x)$ *is lower semi-absolutely continuous* $(\underline{AC})$ *over a set* E *if to a given positive number* ϵ *there corresponds a positive number* δ *such that for any non-overlapping set of intervals* (a_j, a'_j) *with* a_j, a'_j *points of* E, $\Sigma_j \{F(a'_j) - F(a_j)\} > -\epsilon$ *when* $\Sigma_j (a'_j - a_j) < \delta$.

DEFINITION. *A function* $F(x)$ *is generalized lower semi-absolutely continuous* $(\underline{ACG})$ *on* E *if* E *can be covered by a denumerable sequence of closed sets* $E_1, E_2, \ldots$ *such that* $F(x)$ *is* $\underline{AC}$ *on each set* E_1.

If "$> -\epsilon$" is replaced by "$< \epsilon$" in the above definitions, the corresponding definitions of upper and generalized upper semi-absolutely continuous $(\overline{AC})$ and $(\overline{ACG})$ functions are obtained. If $F(x)$ is ACG on (a, b) it is both $\underline{ACG}$ and $\overline{ACG}$ on (a, b).

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Lemma 2. *If $F(x)$ is $\underline{ACG}$ on (a, b) and if $ADF(x) \geqq 0$ almost everywhere on (a, b), then there is an interval (l, m) on (a, b) over which $F(x)$ is non-decreasing.*

Let $E_1, E_2, \ldots$ be the sets over which $F(x)$ is $\underline{ACG}$. Since these sets are closed there exists, by Baire's theorem [5, p. 54], an interval (l, m) on (a, b) such that $E_n(l, m) = (l, m)$ for some n.

Let G be the set of points of (l, m) at which $ADF \geqq 0$. For x a point of G we then have

$$\frac{F(x + h_i) - F(x)}{h_i} > -\epsilon,$$

with $x + h_i$ on a set of density unity at x, $h_i \to 0$. We can then use the Vitali covering theorem [5, p. 109] to get a finite non-overlapping set of intervals (x_k, x'_k) satisfying this relation and such that $\Sigma_k(x'_k - x_k) > mG - \delta = m - l - \delta$, where δ is sufficiently small to ensure that for (a_j, a'_j) a set of non-overlapping intervals with $\Sigma_j(a'_j - a_j) < \delta$ then

$$\Sigma_j\{F(a'_j) - F(a_j)\} > -\epsilon.$$

Let (a_j, a'_j) be the intervals complementary to the intervals (x_k, x'_k). Then

$$\begin{aligned} F(m) - F(l) &= \Sigma_k \{F(x'_k) - F(x_k)\} + \Sigma_j \{F(a'_j) - F(a_j)\} \\ &> -\epsilon(m - l) - \epsilon. \end{aligned}$$

Since ϵ is arbitrary, $F(m) - F(l) \geqq 0$.

In a similar manner it may be shown that $F(m') - F(l') \geqq 0$ for (l', m') any interval on (l, m). Hence $F(x)$ is non-decreasing on (l, m).

Lemma 3_r. *Let $F(x)$ be M_r-continuous and $\underline{ACG}$ on (a, b) and let $ADF(x) \geqq 0$ almost everywhere on (a, b). If P is a perfect set on (a, b) with $F(x)$ non-decreasing on the intervals complementary to P, then there is an interval (l, m) containing points of P with $F(x)$ non-decreasing on (l, m).*

Since P is perfect, by Baire's theorem there exists an interval (l, m) containing points of P and such that $P(l, m)$ is identical with $E_k(l, m)$ for some k, where E_k is one of the sets over which $F(x)$ is $\underline{AC}$.

Let $F_1(x) = F(x)$ on $P(l, m)$ and let $F_1(x)$ be linear in the intervals complementary to $P(l, m)$ in such a way that $F_1(x)$ is continuous on (l, m). Then $F_1(x)$ is $\underline{AC}$ on (l, m). Since $F(x)$ is M_r-continuous and non-decreasing on an interval $a_j < x < b_j$ complementary to P it follows from Theorem I_r that $F(x)$ is continuous and non-decreasing on $a_j \leqq x \leqq b_j$. Hence $ADF_1(x) \geqq 0$ almost everywhere on (l, m). We can therefore use the argument of the preceding lemma to show that $F_1(x)$ is non-decreasing on (l, m) and $F(x)$ therefore non-decreasing on $P(l, m)$. We conclude that $F(x)$ is non-decreasing on (l, m).

Theorem II'_r. *If $F(x)$ is M_r-continuous and $\underline{ACG}$ on (a, b) and if $ADF(x) \geqq 0$ almost everywhere on (a, b) then $F(x)$ is non-decreasing on (a, b).*

If we assume that Theorem II'_r is false we can use Lemmas 2 and 3_r as in the proof of Theorem I, [4, p. 133], to obtain a contradiction. Theorem II_r then follows as a corollary.

THEOREM III_r. *If $f_1(x)$ and $f_2(x)$ are GM_r-integrable on (a, b) and $f_1(x) \geqq f_2(x)$ almost everywhere on (a, b) then $GM_r(f_1, a, b) \geqq GM_r(f_2, a, b)$.*

If we set $F_1(x) = GM_r(f_1, a, x)$ and $F_2(x) = GM_r(f_2, a, x)$, then $F_1(x) - F_2(x)$ is M_r-continuous and ACG on (a, b) and $AD[F_1(x) - F_2(x)] = ADF_1 - ADF_2(x) = f_1(x) - f_2(x) \geqq 0$ almost everywhere on (a, b). Hence, by Theorem II_r, $F_1(x) - F_2(x)$ is non-decreasing and, since $F_1(a) = F_2(a) = 0$, $F_1(b) \geqq F_2(b)$. We have therefore established Property I_r.

THEOREM IV_r. *If $f(x)$ is GM_{r-1}-integrable on (a, b) it is necessarily GM_r-integrable.*

We need only show that $F(x) = GM_{r-1}(f, a, x)$ is M_r-continuous. By hypothesis

$$(GM_{r-2}) \int_x^{x+h} (x + h - t)^{r-2} \{F(t) - F(x)\} \, dt = o(h^{r-1})$$

as $h \to 0$. The equality

$$\frac{r}{h^r} (GM_{r-2}) \int_x^{x+h} (x + h - t)^{r-1} \{F(t) - F(x)\} \, dt$$
$$= \frac{r(r-1)}{h^r} (GM_{r-2}) \int_x^{x+h} dt (GM_{r-3}) \int_x^t (t - \eta)^{r-2} \{F(\eta) - F(x)\} \, d\eta$$

may be established by integrating the integral on the left side by parts $r - 1$ times and the inner integral on the right $r - 2$ times [2, p. 543], operations which are justified by Property III_{r-1}. By Property II_{r-1} the integral on the left side exists in the GM_{r-1}-sense and has the same value. The right hand side is equal to

$$r(r-1)h^{-r} \int_x^{x+h} o\,[(t - x)^{r-1}] \, dt = o(1)$$

as $h \to 0$. We have therefore established Property II_r.

THEOREM V_r. *Let $F(x)$ be M_r-continuous at x and let*

$$g_r(t) = \int_a^t dt_1 \int_a^{t_1} \ldots dt_{r-1} \int_a^{t_{r-1}} g(t) dt,$$

where $g(t)$ is of bounded variation on (a, x). Then $F(x)g_r(x)$ is M_r-continuous at x.

Using the Cesàro-Perron analogue of Property III_{r-1} Burkill [2, p. 549] establishes Theorem V_r for the C_rP-integral. The method of proof applies as well to the GM_r-integral.

THEOREM VI_r (Integration by parts). *Let $f(x)$ be GM_r-integrable on (a, b) and let $g_r(x)$ be defined as in Theorem V_r. Then $f(x)g_r(x)$ is GM_r-integrable on (a, b) and*

$$(GM_r) \int_a^b f(x) g_r(x) = F(b) g_r(b) - (GM_{r-1}) \int_a^b F(x) g_{r-1}(x) dx.$$

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By Property III_{r-1} $F(x)g_{r-1}(x)$ is GM_{r-1}-integrable. We set

$$H(x) = F(x)g_r(x) - (GM_{r-1}) \int_a^x F(t)g_{r-1}(t)dt.$$

By Theorem V_r $F(x)g_r(x)$ is M_r-continuous on (a, b). By Theorem IV_r $F(t)g_{r-1}(t)$ is also GM_r-integrable and to $GM_{r-1}(Fg_{r-1}, a, x)$. Hence $GM_{r-1}(Fg_{r-1}, a, x)$ is M_r-continuous on (a, b). Since the sum of two M_r-continuous functions is M_r-continuous, $H(x)$ is M_r-continuous.

Now $F(x)$ is ACG and $g_r(x)$ is AC on (a, b). Since the product of two AC functions is AC, $F(x)g_r(x)$ is AC over each of the closed sets over which $F(x)$ is AC and therefore $F(x)g_r(x)$ is ACG on (a, b). Since $GM_{r-1}(Fg_{r-1}, a, x)$ is also ACG on (a, b) it follows that $H(x)$ is ACG on (a, b).

As GM-integrals the functions $F(x)$ and $GM_{r-1}(Fg_{r-1}, a, x)$ are approximately derivable almost everywhere on (a, b). Since $ADF(x) = f(x)$ almost everywhere on (a, b), and $AD\,[GM_{r-1}(Fg_{r-1}, a, x)]$ is equal to $F(x)g_{r-1}(x)$ almost everywhere on (a, b), $ADH(x) = ADF(x)g_r(x) = f(x)g_r(x)$ almost everywhere on (a, b).

It follows that $H(x)$ is an indefinite GM_r-integral of $f(x)g_r(x)$ and

$$(GM_r) \int_a^b f(x)g_r(x)dx = H(b) = F(b)g_r(b) - (GM_{r-1})\int_a^b F(x)g_{r-1}(x)dx.$$

We have therefore proved Theorem VI_r and established Properties I-III by induction. Property II shows that the GM-scale of integrals is consistent, i.e. that each integral contains those with lower subscripts. Each integral of the scale is more general than the preceding integral. This is shown by the example $f(x) = 0$, $x = 0$, $f(x) = (d/dx)^p x^q \sin(1/x)$, p, q integers $x \neq 0$ for, given r, $f(x)$ will be a GM_r-integrable but not GM_{r-1}-integrable function if p and q are properly chosen.

A further property of the GM_r-integral that follows easily from Theorem II_r is the following.

The definite GM_r-integral is determined uniquely, and the indefinite GM_r-integral is determined uniquely apart from an additive constant.

We list other properties for which the proofs are essentially the same for higher orders as for the GM_0- or general Denjoy integral.

A function which is GM_r-integrable is necessarily measurable and almost everywhere finite [5, p. 243].

The function $F(x) = GM_r(f, a, x)$, $a \leqq x \leqq b$, satisfies Lusin's condition (N) [5, p. 225].

A function which is GM_r-integrable and almost everywhere non-negative on an interval (a, b) is necessarily Lebesgue integrable on (a, b) [5, p. 242].

Given a non-decreasing sequence of functions $f_n(x)$ which are GM_r-integrable on an interval (a, b) and whose GM_r-integrals over (a, b) constitute a sequence bounded above, then the function $f(x) = \lim f_n(x)$ is itself necessarily GM_r-integrable on (a, b) and

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$$(GM_r)\int_a^b f(x)dx = \lim_{n\to\infty} (GM_r)\int_a^b f_n(x)dx,$$

[5, p. 243].

2. The relation between the GM_r- and the C_rP-, C_rD-integrals. The C_0P-integral is the Perron integral and the C_rP-integral is defined in an analogous manner [1, 2] using major and minor functions but with ordinary continuity and derivatives replaced by mean (Cesàro) continuity and derivatives.

DEFINITION (AC^*_r) [6, p. 221]. *The function $F(x)$ is said to be AC^* (C_r-sense) over a set E if it is $C_{r-1}D$-integrable in an interval which contains E and if to each positive number ϵ corresponds a number δ such that*[2]

$$\sum_{i=1}^{n} \overline{\text{bound}}_{a_i<x<b_i} \left| C_r(F, a_i, x) - F(a_i)\right| < \epsilon,$$

$$\sum_{i=1}^{n} \overline{\text{bound}}_{a_i<x<b_i} \left| C_r(F, b_i, x) - F(b_i)\right| < \epsilon,$$

for all finite sets of non-overlapping intervals (a_i, b_i) with end points on E and such that $\sum_{i=1}^{n} (b_i - a_i) < \delta$.

DEFINITION (ACG^*_r) [6, p. 221]. *The function $F(x)$ will be said to be ACG^* (C_r-sense) over a set E if $F(x)$ is C_r-continuous at points of E and if E is the sum of a denumerable number of sets over each of which $F(x)$ is AC^*(C_r-sense).*

DEFINITION (DI_r) [6, p. 232]. *The function $f(x)$ is said to be C_rD-integrable in (a, b) if there is a function $F(x)$ that is ACG^*(C_r-sense) over the closed interval (a, b) and such that $C_rDF(x) = f(x)$ almost everywhere in (a, b). Then $F(x)$ is an indefinite C_rD-integral of $f(x)$, $F(b) - F(a)$ the definite C_rD-integral in (a, b).*

By basing our generalizations on the general Denjoy integral we were able to obtain a considerably simpler descriptive definition of an r^{th} order integral. The condition that $F(x)$ be C_r-continuous could be separated from Definition (ACG^*_r) and included separately in Definition (DI_r). The concept of (ACG) is then seen to be simpler than the modified Definition (ACG^*_r). Furthermore, in Definition (DI_r) the concepts of both ACG^*(C_r-sense) and C_r-derivatives must be modified for each r. We prove that the simpler GM_r-integral is actually more general than the C_rD- and equivalent C_rP-integrals.

THEOREM VII$_r$. *The GM_r-integral contains the C_rD- and C_rP-integrals.*

Since Miss Sargent [6] proved the equivalence of the C_rD- and C_rP-integrals we need only show that the GM_r-integral contains the C_rD-integral. We proceed by induction. Since it is well known that the GM_0-integral contains the C_0D- or special Denjoy integral we may suppose the theorem true for orders less than r and prove that it is then true for order r.

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[2]The r^{th} Cesàro mean of F on (a, b) is denoted by $C_r(F, a, b)$ and differs from the M_r-mean only in that the integral is required to exist in the C_rD- sense rather than the GM_r-sense.

We suppose that f is C_rD-integrable and set $F(x) = C_rD(f, a, x)$. Then, since $F(x)$ is C_r-continuous,

$$\frac{r}{h^r}(C_{r-1}D)\int_a^{x+h}(x+h-t)^{r-1}F(t)dt \rightarrow F(x)$$

as $h \rightarrow 0$. By hypothesis the integral exists in the GM_{r-1}-sense and has the same value. We conclude that $F(x)$ is M_r-continuous on (a, b).

By the descriptive definition of the C_rD-integral, (a, b) can be covered by a sequence of closed sets (E_n) over each of which $F(x)$ is AC^*(C_r-sense). By Theorem II [6, p. 227] a necessary condition for $F(x)$ to be AC^*(C_r-sense) over a set E_n is that $F(x)$ be AC on E_n. It follows that $F(x)$ is ACG on (a, b).

In [6, p. 228] it is shown that if $F(x)$ is AC^*(C_r-sense) on a set E_n, then the C_r-derivative [2, p. 542] $C_rDF(x)$ exists, is finite and equal to $ADF(x)$ at almost all points x of E_n. Since (a, b) is covered by at most a denumerable sequence of such sets it follows that $C_rDF(x) = ADF(x)$ almost everywhere on (a, b). Since, by the definition of a C_rD-integral, $C_rDF(x) = f(x)$ almost everywhere it follows that $ADF(x) = f(x)$ almost everywhere on (a, b). It then follows that $f(x)$ is GM_r-integrable to $F(x)$.

On the other hand the C_rP-, C_rD-integrals do not contain the GM_r-integral. This is well known for $r = 0$ since the special Denjoy integral is contained in but not equivalent to the general Denjoy integral. A similar relation holds for other values of r. We therefore have two distinct scales: (1) the CD-, CP-scale of integrals similar to and generalizing the Denjoy-Perron integral; and (2) the GM-scale of integrals similar to and generalizing the general Denjoy integral and such that the GM_r-integral contains the C_rD-, C_rP-integrals.

3. The constructive GM_r-integral.

To obtain a constructive definition of the GM_r-integral we modify the definitions and conditions for integrability in the general Denjoy sense by using limits involving M_r-means.

DEFINITION (a). *If the function $f(x)$ is summable over a measurable set E then $GM_r(f, E)$ is $L(f, E)$ the Lebesgue integral of $f(x)$ over E.*

DEFINITION (b). *Let (α_i, β_i) be any interval and suppose that $GM_r(f, \alpha, \beta)$ has been determined for every interval (α, β) interior to (α_i, β_i). Let ξ be a point with $\alpha_i < \xi < \beta_i$ and let $F(t) = GM_r(f, t, \xi)$. Let $K_r(\alpha_i, \xi)$ and $K_r(\xi, \beta_i)$ be the respective limits as $h \rightarrow 0^+$ of*

$$rh^{-r}(GM_{r-1})\int_{\alpha_i}^{\alpha_i+h}(\alpha_i + h - t)^{r-1}F(t)dt,$$

$$rh^{-r}(GM_{r-1})\int_{\beta_i-h}^{\beta_i}(t - \beta_i + h)^{r-1}F(t)dt,$$

where the integrals are supposed to tend to limits which are finite. If $f(x)$ is such that $K_r(\alpha_i, \xi) + K_r(\xi, \beta_i)$ is independent of ξ then

$$GM_r(f, \alpha_i, \beta_i) = K_r(\alpha_i, \xi) + K_r(\xi, \beta_i).$$

DEFINITION (c). *Let E be a closed set over which $f(x)$ is summable, (α_i, β_i) the intervals complementary to E on (a, b); suppose that $GM_r(f, \alpha_i, \beta_i)$ has been determined for all the intervals (α_i, β_i), and that (l, m) is an interval for which $\sum_{(l,m)} |GM_r(f, \alpha_i, \beta_i)|$ converges. Then*

$$GM_r(f, l, m) = \int_{E(l,m)} f(x)dx + \sum_{(l,m)} GM_r(f, \alpha_i, \beta_i),$$

where it is understood that if l is interior to an interval (α_k, β_k) then the term in the sum arising from this interval is $GM_r(f, l, \beta_k)$. A similar understanding holds if m is not a point of E.

The function $f(x)$ is then GM_r-integrable on (a, b) if it satisfies the following conditions:

(1) If E is any closed set on (a, b) there exists an interval (l, m) containing points of E and such that $f(x)$ is summable over $E(l, m)$.

(2) The function $f(x)$ is such that the limits in (b) exist.

(3) If $GM_r(f, l, x)$ exists for all x in an interval (l, m) then it is M_r-continuous as a function of x in (l, m).

(4) The function $f(x)$ is such that if E is any closed set for which $GM_r(f, \alpha_i, \beta_i)$ has been determined for all intervals (α_i, β_i) complementary to E, there exists an interval (l, m) containing points of E and such that $\sum_{(l,m)} |GM_r(f, \alpha_i, \beta_i)|$ converges.

Definitions (a), (b) and (c) together with conditions (1), (2), and (4) permit the determination of $GM_r(f, a, b)$ in a finite or denumerable number of steps as in [3, p. 20 ff.]. Further conditions are needed to ensure that $F(x) = GM_r(f, a, x)$ is M_r-continuous. These conditions are discussed for an integral equivalent to the GM_1-integral in [3]. We have postulated mean continuity by adding condition (3).

If we set $F(a) = 0$, $F(x) = GM_r(f, a, x)$ for $a < x \leqq b$, we can prove Lemmas 4_r and 5_r as in [3, Theorems I and II].

LEMMA 4_r. *The function $F(x) = GM_r(f, a, x)$ is ACG on (a, b).*

LEMMA 5_r. *At almost all points of (a, b) $ADF(x)$ exists and is equal to $f(x)$.*

THEOREM VIII_r. *The constructive and descriptive definitions of the GM_r-integral are equivalent.*

Condition (3) and Lemmas 4_r and 5_r show that the descriptive integral contains the constructive integral. We must therefore show that if the M_r-continuous function $F(x)$ is ACG on (a, b) and such that $ADF(x)$ is finite almost everywhere and equal to $f(x)$, then the constructive definition gives $GM_r(f, a, x) = F(x) - F(a)$. We first prove a lemma.

LEMMA 6_r. *Let $F(x)$ be ACG on (a, b) and let $ADF(x)$ be finite and equal to $f(x)$ almost everywhere on (a, b). If E is any closed set on (a, b) there then exists an interval (l, m) such that $f(x)$ is summable over $E(l, m)$, $\sum_{(l,m)} |F(\beta_i) - F(\alpha_i)|$*

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converges, where (α_i, β_i) *are the intervals complementary to* $E(l, m)$, *and for any such interval*

$$F(m) - F(l) = \int_{E(l,m)} f(x)dx = \sum_{(l,m)} \{F(\beta_i) - F(\alpha_i)\}.$$

Let $A_1, A_2, \ldots$ be the closed sets over which $F(x)$ is AC. There then exists, by Baire's Theorem, an interval (l, m) and an integer k such that A_k and E are identical on (l, m). Since F is AC on A_k, $D_{A_k}F(x)$ exists for almost all points of A_k and is summable over A_k. Then, since for almost all points of A_k we have $D_{A_k}F = ADF = f$, it follows that f is summable over $E(l, m)$. The convergence of $\sum_{(l,m)} |F(\beta_i) - F(\alpha_i)|$ follows from the absolute continuity of F on $E(l, m)$.

Let $G(x) = F(x)$ on $E(l, m)$ and be linear in the intervals (α_i, β_i) in such a way as to be continuous on (l, m). Then $G(x)$ is AC on (l, m) and, at almost all points of $E(l, m)$, $G' = D_{E(l,m)} F = ADF = f$. Hence

$$\begin{aligned} F(m) - F(l) &= (L)\int_l^m G'(x)dx \\ &= \int_{E(l,m)} f(x)dx + \sum_{(l,m)} \int_{\alpha_i}^{\beta_i} G'(x)dx \\ &= \int_{E(l,m)} f(x)dx + \sum_{(l,m)} \{F(\beta_i) - F(\alpha_i)\}. \end{aligned}$$

We return to the proof of the theorem and, as in the existence proof for the constructive GM_r-integral [3, pp. 21-23], we let E_1 be the points of non-summability of F on (a, b). Then E_1 is closed. If we denote by (α_i^1, β_i^1) the intervals complementary to E_1, by (α, β) an interval with $\alpha_i^1 < \alpha < \beta < \beta_i^1$, then

$$F(\beta) - F(\alpha) = L(f, \alpha, \beta) = GM_r(f, \alpha, \beta).$$

Since $F(x)$ is M_r-continuous, $F(\beta) - F(\alpha)$ tends in the M_r-sense to the limit $F(\beta_i^1) - F(\alpha_i^1)$ which is finite and independent of any ξ, $\alpha_i^1 < \xi < \beta_i^1$. It follows that Definition (b) applies to $GM_r(f, \alpha, \beta)$ and

$$F(\beta_i^1) - F(\alpha_i^1) = \lim_{\substack{\alpha \to \alpha_i^1 \\ \beta \to \beta_i^1}} GM_r(f, \alpha, \beta) = GM_r(f, \alpha_i^1, \beta_i^1).$$

By Lemma 6_r there exists at least one interval (l, m) containing points of E_1 and such that f is summable on $E_1(l, m)$ and $\sum_{(l,m)} |F(\beta_i^1) - F(\alpha_i^1)|$ converges. Let E_2 be the points of E_1 that are points of non-summability of f over E_1 and/or points x such that $\sum_{(l,m)} |F(\beta_i^1) - F(\alpha_i^1)|$ diverges for every interval (l, m) containing x. If (α_i^2, β_i^2) are the intervals complementary to E_2 and (α, β) is an interval with $\alpha_i^2 < \alpha < \beta < \beta_i^2$ then, by Lemma 6_r,

$$F(\beta) - F(\alpha) = \int_{E_1(\alpha,\beta)} f(x)dx + \sum_{(\alpha,\beta)} \{F(\beta_i^1) - F(\alpha_i^1)\},$$

and the right side is now equal to $GM_r(f, \alpha, \beta)$ by Definition (c). As before we can use the M_r-continuity of F to determine $GM_r(f, \alpha_i^2, \beta_i^2) = F(\beta_i^2) - F(\beta_i^2)$. Continuing this process we can determine $GM_r(f, l, m) = F(m) - F(l)$ in a finite or denumerable number of steps.

4. A continuity property of GM_r-integrals. We conclude with a proof that an important property of continuous functions extends to mean continuous indefinite integrals. This result has been stated for the C_1P-integral [7, p. 238].

THEOREM IX$_r$. *If $F(x)$ is an indefinite GM_r-integral of $f(x)$ defined on (a, b) and if (l, m) is any closed interval on (a, b), then $F(x)$ takes all values between its upper and lower bounds on (l, m) for $l \leqq x \leqq m$.*

Let E_1 be the points of non-summability of f on (l, m). If (α_i^1, β_i^1) are the intervals complementary to E_1 on (l, m), α_i^1, β_i^1 points of E_1 and (α, β) is an interval with $\alpha_i^1 < \alpha < \beta < \beta_i^1$, then f is Lebesgue integrable on (α, β). It follows that $F(x)$ takes all values between its upper and lower bounds on (α, β) for $\alpha \leqq x \leqq \beta$.

Let β be fixed, let α tend to α_i^1 and consider the intervals (α_i^1, α). There are three possibilities: (i) Every interval (α_i^1, α) contains points x with $F(x) > F(\alpha_i^1)$ and points x' with $F(x') < F(\alpha_i^1)$; (ii) There exists an interval (α_i^1, α) with no point x such that $F(x) > F(\alpha_i^1)$; or (iii) There exists an interval (α_i^1, α) containing no point x such that $F(x) < F(\alpha_i^1)$.

In the first case it is clear that $F(x)$ takes all values between its upper and lower bounds on (α_i^1, β) for $\alpha_i^1 \leqq x \leqq \beta$. In the second case, given an arbitrary $\epsilon > 0$ there exists δ such that $F(\alpha_i^1) - F(x_1) < \epsilon$ for some x_1, $\alpha_i^1 < x_1 < \alpha_i^1 + \delta$. If not, Lemma 1_r would contradict the M_r-continuity of $F(x)$ at α_i^1. Since $F(x)$ takes all values between its upper and lower bounds on (x_1, β) and ϵ is arbitrary, it follows that it takes all values between its upper and lower bounds on (α_i^1, β). Since a similar argument holds for β tending to β_i^1, $F(x)$ takes all values between its upper and lower bounds on (α_i^1, β_i^1) for $\alpha_i^1 \leqq x \leqq \beta_i^1$. A similar argument holds if (iii) applies.

As in the transfinite process by which the GM_r-integral was built up from the constructive definition [3, p. 20], let E_2 be the points x of E_1 such that one or both of the following conditions hold: (i) For every interval (c, d) containing x the function f is not summable over $E_1(c, d)$; (ii) The sum $\sum_{(c,d)} |GM_r(f, \alpha_i^1, \beta_i^1)|$ diverges for every interval (c, d) containing x. Let (α_i^2, β_i^2) be an interval of the set complementary to E_2 with α_i^2, β_i^2 points of E_2; (α, β) an interval with $\alpha_i^2 < \alpha < \beta < \beta_i^2$.

Let $G(x) = F(x)$ for $x = \alpha, \beta$ and at points of $E_1(\alpha, \beta)$ and let $G(x)$ be linear in the intervals (α_i^1, β_i^1) on (α, β) and in the intervals (α, α_k^1), $(\beta_{k'}^1, \beta)$ where $\alpha_k^1, \beta_{k'}^1$, are the upper and lower bounds of points of E_1 on (α, β).

Since α, β are arbitrary it is sufficient to prove that $F(x)$ takes all values between $F(\alpha)$ and $F(\beta)$ on (α, β). Let c be any value between $F(\alpha)$, $F(\beta)$. Then $G(x') = c$ for some x', $\alpha < x' < \beta$. If $x' \epsilon E_1$, then $F(x') = c$. If $G(x) \neq c$ in E_1 there exists some interval (α_j^1, β_j^1) with $\alpha_j^1 < x' < \beta_j^1$ and $F(\alpha_j^1) = G(\alpha_j^1) <$

$c < G(\beta_j^1) = F(\beta_j^1)$ or the reverse inequality. By the first part of the theorem $F(x)$ takes all values between $F(\alpha_j^1)$ and $F(\beta_j^1)$ for $\alpha_j^1 < x < \beta_j^1$ and therefore takes the value c.

As before we can pass from the intervals (α, β) to the intervals (α_i^2, β_i^2) complementary to E_2. Continuing this process we can establish the theorem in a finite or denumerable number of steps.

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