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Modular compactifications of the space of pointed elliptic curves I

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ABSTRACT

We introduce a sequence of isolated curve singularities, the elliptic m -fold points, and an associated sequence of stability conditions, generalizing the usual definition of Deligne–Mumford stability. For every pair of integers $1 \leq m < n$, we prove that the moduli problem of n -pointed m -stable curves of arithmetic genus one is representable by a proper irreducible Deligne–Mumford stack $\overline{\mathcal{M}}_{1,n}(m)$. We also consider weighted variants of these stability conditions, and construct the corresponding moduli stacks $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$. In forthcoming work, we will prove that these stacks have projective coarse moduli and use the resulting spaces to give a complete description of the log minimal model program for $\overline{\mathcal{M}}_{1,n}$.

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1. Introduction

1.1 Why genus one?

One of the most beautiful and influential theorems of modern algebraic geometry is the following theorem of Deligne and Mumford [DM69].

THEOREM. *The moduli stack of stable curves of arithmetic genus $g \geq 2$ is a smooth, proper, Deligne–Mumford stack over $\text{Spec}(\mathbb{Z})$.*

The essential geometric content of the theorem is the identification of a suitable class of singular curves, namely *Deligne–Mumford stable curves*, with the property that every incomplete one-parameter family of smooth curves has a unique ‘limit’ contained in this class. The definition of a Deligne–Mumford stable curve comprises one local condition and one global condition.

DEFINITION (Stable curve). A connected, reduced, complete curve C is *stable* if the following hold.

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- (1) The curve C has only nodes as singularities (local condition).
- (2) The curve C satisfies the following two equivalent conditions (global condition).
 - (a) $H^0(C, \Omega_C^\vee) = 0$.
 - (b) ω_C is ample.

While the class of stable curves gives a natural modular compactification of the space of smooth curves, it is not unique in this respect. Using geometric invariant theory, Schubert constructed a proper moduli space for *pseudostable curves* [Sch91].

DEFINITION (Pseudostable curve). A connected, reduced, complete curve C is *pseudostable* if the following hold.

- (1) The curve C has only nodes and cusps as singularities (local condition).
- (2) If $E \subset C$ is any connected subcurve of arithmetic genus one, then $|E \cap \overline{C \setminus E}| \geq 2$ (global condition).
- (3) The curve C satisfies the following two equivalent conditions (global condition).
 - (a) $H^0(C, \Omega_C^\vee) = 0$.
 - (b) ω_C is ample.

Notice that the definition of pseudostability involves a trade-off: the local condition has been weakened to allow cusps, while the global condition has been strengthened to disallow elliptic tails. It is easy to see how this trade-off comes about: as one ranges over all one-parameter smoothings of a cuspidal curve C , the associated limits in \overline{M}_g are precisely curves of the form $\tilde{C} \cup E$, where \tilde{C} is the normalization of C and E is an elliptic curve (of arbitrary j -invariant) attached to \tilde{C} at the point lying above the cusp. Thus, any separated moduli problem must exclude either cusps or elliptic tails. In light of Schubert’s construction, it is natural to ask the following problem.

Problem. Given a reasonable local condition, e.g. a deformation-open collection of isolated curve singularities, is there a corresponding global condition which yields a proper moduli space?

Any investigation of the above problem should begin by asking: which are the simplest isolated curve singularities? Let C be a reduced curve over an algebraically closed field k , $p \in C$ a singular point, and $\pi : \tilde{C} \rightarrow C$ the normalization of C at p . We have two basic numerical invariants of the singularity:

$$\begin{aligned} \delta &= \dim_k(\pi_* \mathcal{O}_{\tilde{C}} / \mathcal{O}_C), \\ m &= |\pi^{-1}(p)|. \end{aligned}$$

The invariant δ may be interpreted as the number of conditions for a function to descend from \tilde{C} to C , while m is the number of branches. Of course, if a singularity has m branches, there are $m - 1$ obviously necessary conditions for a function $f \in \mathcal{O}_{\tilde{C}}$ to descend: f must have the same value at each point in $\pi^{-1}(p)$. Thus, $\delta - m + 1$ is the number of conditions for a function to descend *beyond the obvious ones*, and we take this as the most basic numerical invariant of a singularity.

DEFINITION. The *genus* of an isolated singularity is $g := \delta - m + 1$.

We use the name ‘genus’ for the following reason: if $\mathcal{C} \rightarrow \Delta$ is a one-parameter smoothing of an isolated curve singularity $p \in C$, then (after a finite base change) one may apply stable reduction around p to obtain a proper birational morphism $\phi : \mathcal{C}^s \rightarrow \mathcal{C}$, where $\mathcal{C}^s \rightarrow \Delta$ is a flat family of curves with nodal special fiber, and $\phi(\text{Exc}(\phi)) = p$. Then it is easy to see that the genus

of the isolated curve singularity $p \in C$ is precisely the arithmetic genus of the curve $\phi^{-1}(p)$. Thus, just as elliptic tails are replaced by cusps in Schubert’s moduli space of pseudostable curves, any separated moduli problem allowing singularities of genus g must disallow certain subcurves of genus g .

The simplest isolated curve singularities are those of genus zero. For each integer $m \geq 2$, there is a unique singularity with m branches and genus zero, namely the union of the m coordinate axes in \mathbb{A}^m . For our purposes, however, these singularities have one very unappealing feature: for $m \geq 3$, they are not *Gorenstein*. This means that the dualizing sheaf of a curve containing such singularities is not invertible. For this reason, we choose to focus upon the next simplest singularities, namely those of genus one. It turns out that, for each integer $m \geq 1$, there is a unique Gorenstein curve singularity with m branches and genus one (Proposition A.3). These are defined below.

DEFINITION (The elliptic m -fold point). We say that p is an *elliptic m -fold point* of C if

$$\hat{O}_{C,p} \simeq \begin{cases} k[[x, y]]/(y^2 - x^3) & m = 1 \text{ (ordinary cusp),} \\ k[[x, y]]/(y^2 - yx^2) & m = 2 \text{ (ordinary tacnode),} \\ k[[x, y]]/(x^2y - xy^2) & m = 3 \text{ (planar triple-point),} \\ k[[x_1, \dots, x_{m-1}]]/I_m & m \geq 4 \text{ (} m \text{ general lines through the origin in } \mathbb{A}^{m-1}\text{),} \end{cases}$$

$$I_m := (x_h x_i - x_h x_j : i, j, h \in \{1, \dots, m - 1\} \text{ distinct}).$$

We will show that if C is a curve with a single elliptic m -fold point p , then, as one ranges over all one-parameter smoothings of C , the associated limits in \overline{M}_g are curves of the form $\tilde{C} \cup E$, where \tilde{C} is the normalization of C and E is any stable curve of arithmetic genus one attached to \tilde{C} at the points lying above p . Following Schubert, one is now tempted to define a sequence of moduli problems in which certain arithmetic genus one subcurves are replaced by elliptic m -fold points.

The idea seems plausible until one considers a one-parameter family of smooth genus g curves specializing to a ring of $g - 1$ elliptic bridges. How can one modify the special fiber to obtain a ‘tacnodal limit’ for this family? Assuming the total space of the family is smooth, one can contract any non-intersecting subset of the elliptic bridges to obtain a tacnodal special fiber, but clearly one cannot obtain in this way a *canonical* tacnodal limit. This example suggests that a systematic handling of mildly non-separated moduli functors, either via the formalism of geometric invariant theory or Artin stacks, will be necessary in order to proceed at this level of generality. (See [HH08] for a geometric invariant theory construction involving tacnodal curves.)

Happily, there is one non-trivial case in which this difficulty of multiple interacting elliptic components does not appear, namely the case of n -pointed stable curves of arithmetic genus one. This leads us to make the definition.

DEFINITION (m -stability). Fix positive integers $m < n$. Let C be a connected, reduced, complete curve of arithmetic genus one, let $p_1, \dots, p_n \in C$ be n distinct smooth points, and let $\Sigma \subset C$ be the divisor $\sum_i p_i$. We say that $(C, \{p_i\}_{i=1}^n)$ is *m -stable* if the following hold.

- (1) The curve C has only nodes and elliptic l -fold points, $l \leq m$, as singularities.
- (2) If $E \subset C$ is any connected subcurve of arithmetic genus one, then

$$|E \cap \overline{C \setminus E}| + |E \cap \Sigma| > m;$$

- (3) The space of global sections $H^0(C, \Omega_C^\vee(-\Sigma)) = 0$.

Remark. Condition (3) says that $(C, \{p_i\}_{i=1}^n)$ has no infinitesimal automorphisms as a marked curve. This condition is *not* simply that every smooth rational component has three distinguished points. Furthermore, while

$$H^0(C, \Omega_C^\vee(-\Sigma)) = 0 \implies \omega_C(\Sigma) \text{ is ample,}$$

these conditions are not equivalent.

The definition of m -stability is compatible with the definition of \mathcal{A} -stability introduced by Hassett [Has03], in which sections of low weight are allowed to collide. More precisely, we have the following definition.

DEFINITION ((m, \mathcal{A}) -stability). Fix positive integers $m < n$, and an n -tuple of rational weights $\mathcal{A} = (a_1, \dots, a_n) \in (0, 1]^n$. Let C be a connected, reduced, complete curve of arithmetic genus one, let $p_1, \dots, p_n \in C$ be smooth (not necessarily distinct) points of C , and let Σ denote the support of the \mathbb{Q} -divisor $\sum_i a_i p_i$. We say that $(C, \{p_i\}_{i=1}^n)$ is (m, \mathcal{A}) -stable if the following hold.

- (1) The curve C has only nodes and elliptic l -fold points, $l \leq m$, as singularities.
- (2) If $E \subset C$ is any connected subcurve of arithmetic genus one, then

$$|E \cap \overline{C \setminus E}| + |E \cap \Sigma| > m.$$

- (3) The space of global sections $H^0(C, \Omega_C^\vee(-\Sigma)) = 0$.
- (4) If $p_{i_1} = \dots = p_{i_k} \in C$ coincide, then $\sum_{j=1}^k a_{i_j} \leq 1$.
- (5) The \mathbb{Q} -divisor $\omega_C(\sum_i a_i p_i)$ is ample.

The definition of an (m, \mathcal{A}) -stable curve extends to a moduli functor in the usual way, and we obtain (Theorem 3.8).

Main result. $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$, the moduli stack of (m, \mathcal{A}) -stable curves, is a proper irreducible Deligne–Mumford stack over $\text{Spec } \mathbb{Z}[1/6]$.

Remark. The restriction to $\text{Spec } \mathbb{Z}[1/6]$ is due to the existence of ‘extra’ infinitesimal automorphisms of cuspidal curves in characteristics two and three, a phenomenon which is addressed in §2.1.

A major impetus for studying alternate compactifications of moduli spaces of curves comes from the program introduced by Hassett [Has05], where one seeks modular descriptions for certain log canonical models of $\overline{M}_{g,n}$. While special cases of this program have been worked out using geometric invariant theory [HH08, HH09, HL07], our construction gives the first example of an infinite sequence of singularities giving rise to alternate stability conditions. Our methods are also quite different from [HH08, HH09, HL07] in that, rather than rely on geometric invariant theory to dictate our choice of moduli problem, we undertake a sufficiently in-depth investigation of the elliptic m -fold point to make moduli problems involving these singularities accessible via standard stack-theoretic techniques.

In forthcoming work, we will study $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$ in the framework of birational geometry. In particular, we will develop techniques for doing intersection theory on $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$ and construct explicit ample divisors on the associated coarse moduli spaces $\overline{M}_{1,\mathcal{A}}(m)$. This will enable us to give a complete description of the log minimal model program for $\overline{M}_{1,n}$, i.e. for all $\alpha \in \mathbb{Q} \cap [0, 1]$

such that $K_{\overline{\mathcal{M}}_{1,n}} + \alpha\Delta$ is big, we will show that

$$\overline{\mathcal{M}}_{1,n}(\alpha) := \text{Proj} \bigoplus_{m \gg 0} H^0(\overline{\mathcal{M}}_{1,n}, m(K_{\overline{\mathcal{M}}_{1,n}} + \alpha\Delta))$$

is the normalization of the coarse moduli space of one of the moduli problems introduced in this paper.

1.2 Outline of results

In § 2, we investigate local properties of the elliptic m -fold point which are necessary for the construction of moduli. In § 2.1, we show that sections of Ω_C^\vee around an elliptic m -fold point $p \in C$ are given by regular vector fields on the normalization which vanish *and* have identical first derivatives at the points lying above p . This will allow us to translate the condition $H^0(C, \Omega_C^\vee(-\Sigma)) = 0$ into a concrete statement involving the number of distinguished points on each irreducible component of C . In § 2.2, we show that ω_C is invertible around an elliptic m -fold point p , and is generated by a rational differential on \tilde{C} with double poles along the points lying above p . This implies that ω_C (twisted by marked points) is ample on any m -stable curve so that our moduli problem is canonically polarized. In § 2.3, we classify the collection of all ‘semistable tails’ obtained by performing semistable reduction on the elliptic m -fold point. This classification will be the key ingredient in verifying that the moduli space of m -stable curves is separated.

In § 3, we construct $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$, the moduli space of (m, \mathcal{A}) -stable curves as a Deligne–Mumford stack over $\text{Spec } \mathbb{Z}[1/6]$. In § 3.1, we prove some elementary topological facts about a reduced, connected, Gorenstein curve C of arithmetic genus one. In § 3.2, we define the moduli problem of (m, \mathcal{A}) -stable curves and prove that it is bounded and deformation-open. Following standard arguments, we obtain a moduli stack $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$. In §§ 3.3 and 3.4, we verify the valuative criterion for $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$.

In Appendix A, we prove that the only isolated Gorenstein singularities that can occur on a reduced curve of arithmetic genus one are nodes and elliptic l -fold points. The proof is pure commutative algebra: we simply classify all one-dimensional complete local rings with the appropriate numerical invariants. The result plays a crucial simplifying role throughout the paper. Using this classification, for example, one does not need any ‘serious’ deformation theory to see that only nodes and elliptic l -fold points, $l \leq m$, can occur as deformations of the elliptic m -fold point.

1.3 Notation

A *curve* is a reduced, connected, one-dimensional scheme of finite type over an algebraically closed field. Starting in § 3, all curves will be assumed complete (this assumption is irrelevant in § 2, which is essentially a local study). An n -pointed curve is a curve C , together with n smooth marked points $p_1, \dots, p_n \in C$ (not necessarily distinct). If $(C, \{p_i\}_{i=1}^n)$ is an n -pointed curve, we say that a point of C is *distinguished* if it is a marked point or a singular point. In addition, if \tilde{C} is the normalization of C , we say that a point of \tilde{C} is distinguished if it lies above a marked point or a singular point of C . An n -pointed curve is *nodal* if it has only nodes as singularities and the marked points are distinct. An n -pointed curve is *semistable* (respectively *stable*) if it is nodal and every rational component of \tilde{C} has at least two (respectively three) distinguished points.

The symbol Δ will always denote the spectrum of a discrete valuation ring R with algebraically closed residue field k and field of fractions K . When we speak of a finite base

change $\Delta' \rightarrow \Delta$, we mean that Δ' is the spectrum of a discrete valuation ring $R' \supset R$ with field of fractions K' , where $K' \supset K$ is a finite separable extension. We use the notation

$$\begin{aligned} 0 &:= \text{Spec } k \rightarrow \Delta, \\ \eta &:= \text{Spec } K \rightarrow \Delta, \\ \bar{\eta} &:= \text{Spec } \bar{K} \rightarrow \Delta, \end{aligned}$$

for the closed point, generic point, and geometric generic point respectively. Families over Δ will be denoted in script, while geometric fibers are denoted in regular font. For example, $C_0, C_\eta, C_{\bar{\eta}}$ and $C'_0, C'_\eta, C'_{\bar{\eta}}$ denote the special fiber, generic fiber, and geometric generic fibers of $C \rightarrow \Delta$ and $C' \rightarrow \Delta$ respectively. We will often omit the subscript '0' for the special fiber, and simply write C, C' .

2. Geometry of the elliptic m -fold point

In this section, we work over a fixed algebraically closed field k . We consider a curve C with a singular point $p \in C$, and let $\pi : \tilde{C} \rightarrow C$ denote the normalization of C at p . The notation $\hat{\mathcal{O}}_{C,p}$ will denote the completion of the local ring of C at p , and $m_p \subset \hat{\mathcal{O}}_{C,p}$ the maximal ideal. In addition, we let $\pi^{-1}(p) = \{p_1, \dots, p_m\}$ and set

$$\hat{\mathcal{O}}_{\tilde{C},\pi^{-1}(p)} := \bigoplus_{i=1}^m \hat{\mathcal{O}}_{\tilde{C},p_i}.$$

Note that a choice of uniformizers $t_i \in m_{p_i}$ induces an identification

$$\hat{\mathcal{O}}_{\tilde{C},\pi^{-1}(p)} \simeq k[[t_1]] \oplus \dots \oplus k[[t_m]].$$

We will be concerned with the following sequence of singularities.

DEFINITION 2.1 (The elliptic m -fold point). We say that p is an *elliptic m -fold point* of C if

$$\hat{\mathcal{O}}_{C,p} \simeq \begin{cases} k[[x, y]]/(y^2 - x^3) & m = 1 \quad (\text{ordinary cusp}), \\ k[[x, y]]/(y^2 - xy^2) & m = 2 \quad (\text{ordinary tacnode}), \\ k[[x, y]]/(x^2y - xy^2) & m = 3 \quad (\text{planar triple-point}), \\ k[[x_1, \dots, x_{m-1}]]/I_m & m \geq 4 \quad (\text{cone over } m \text{ general points in } \mathbb{A}^{m-2}), \end{cases}$$

$$I_m := (x_h x_i - x_h x_j : i, j, h \in \{1, \dots, m-1\} \text{ distinct}).$$

One checks immediately that, for an appropriate choice of uniformizers, the map $\pi^* : \hat{\mathcal{O}}_{C,p} \hookrightarrow \hat{\mathcal{O}}_{\tilde{C},\pi^{-1}(p)}$ is given by

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &\rightarrow \begin{pmatrix} t_1^2 \\ t_1^3 \end{pmatrix} & m = 1, \\ \begin{pmatrix} x \\ y \end{pmatrix} &\rightarrow \begin{pmatrix} t_1 & t_2 \\ t_1^2 & 0 \end{pmatrix} & m = 2, \\ \begin{pmatrix} x_1 \\ \vdots \\ x_{m-1} \end{pmatrix} &\rightarrow \begin{pmatrix} t_1 & 0 & \dots & 0 & t_m \\ 0 & t_2 & \ddots & \vdots & t_m \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & t_{m-1} & t_m \end{pmatrix} & m \geq 3. \end{aligned} \tag{†}$$

It will also be useful to have the following coordinate-free characterization of the elliptic m -fold point.

LEMMA 2.2. *The singularity $p \in C$ is an elliptic m -fold point $\iff \pi^* : \hat{\mathcal{O}}_{C,p} \hookrightarrow \hat{\mathcal{O}}_{\tilde{C},\pi^{-1}(p)}$ satisfies the following.*

- (1) *The subspace $\pi^*(m_p/m_p^2) \subset \bigoplus_{i=1}^m m_{p_i}/m_{p_i}^2$ is a codimension-one subspace.*
- (2) *The subspace $\pi^*(m_p/m_p^2) \not\supseteq m_{p_i}/m_{p_i}^2$ for any $i = 1, \dots, m$.*
- (3) *The subspace $\pi^*(m_p) \supseteq \bigoplus_{i=1}^m m_{p_i}^2$.*

Furthermore, if $m \geq 3$, then parts (1) and (2) automatically imply part (3).

It is useful to think of the lemma as describing when a function f on \tilde{C} descends to C . Part (3) says that if f vanishes to order at least two along p_1, \dots, p_m , then f descends to C . Part (1) says that if f vanishes at p_1, \dots, p_m , then the derivatives of f at p_1, \dots, p_m must satisfy one additional condition in order for f to descend. We leave the proof of Lemma 2.2 as an exercise to the reader.

2.1 The tangent sheaf Ω_C^\vee

The tangent sheaves of C and \tilde{C} are defined as

$$\begin{aligned} \Omega_C^\vee &:= \mathcal{H}om_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C), \\ \Omega_{\tilde{C}}^\vee &:= \mathcal{H}om_{\mathcal{O}_{\tilde{C}}}(\Omega_{\tilde{C}}, \mathcal{O}_{\tilde{C}}), \end{aligned}$$

respectively. Let $K(\tilde{C})$ denote the constant sheaf of rational functions on \tilde{C} . Then we have a natural inclusion

$$\Omega_C^\vee \hookrightarrow \pi_* \Omega_{\tilde{C}}^\vee \otimes K(\tilde{C}),$$

given by restricting a regular vector field on C to $C \setminus \{p\} \simeq \tilde{C} \setminus \{p_1, \dots, p_m\}$, and then extending to a rational section of $\pi_* \Omega_{\tilde{C}}^\vee$. If p is an ordinary node then this inclusion induces an isomorphism

$$\Omega_C^\vee \simeq \pi_* \Omega_{\tilde{C}}^\vee(-p_1 - p_2).$$

In other words, a regular vector field on \tilde{C} descends to C if and only if it vanishes at the points lying above the node.

In Proposition 2.3, we give a similar description of Ω_C^\vee when $p \in C$ is an elliptic m -fold point. In this case, $\Omega_C^\vee \subset \pi_* \Omega_{\tilde{C}}^\vee$ is precisely the sheaf of regular vector fields on \tilde{C} which vanish at p_1, \dots, p_m , and have the same first derivative at p_1, \dots, p_m . This allows us to say when a curve with an elliptic m -fold point has infinitesimal automorphisms, and in particular to conclude that m -stable curves have none.

PROPOSITION 2.3 (Tangent sheaf of the elliptic m -fold point). *Suppose that one of the following three conditions holds.*

- (1) *The singularity p is a cusp and characteristic $k \neq 2, 3$.*
- (2) *The singularity p is a tacnode and characteristic $k \neq 2$.*
- (3) *The singularity p is an elliptic m -fold point and $m \geq 3$.*

Consider the exact sequence

$$0 \rightarrow \pi_* \Omega_{\tilde{C}}^\vee(-\sum_i 2p_i) \rightarrow \pi_* \Omega_{\tilde{C}}^\vee(-\sum_i p_i) \rightarrow \bigoplus_{i=1}^m \Omega_{\tilde{C}}^\vee(-p_i)|_{p_i} \rightarrow 0.$$

Since we have a canonical isomorphism

$$\bigoplus_{i=1}^m \Omega_{\tilde{C}}^{\vee}(-p_i)|_{p_i} \simeq \bigoplus_{i=1}^m k,$$

there is a well-defined diagonal map

$$\Delta : k \hookrightarrow \bigoplus_{i=1}^m \Omega_{\tilde{C}}^{\vee}(-p_i)|_{p_i},$$

and $\Omega_{\tilde{C}}^{\vee} \subset \pi_*\Omega_{\tilde{C}}^{\vee}$ is the inverse image of $\Delta \subset \bigoplus_{i=1}^m \Omega_{\tilde{C}}^{\vee}(-p_i)|_{p_i}$. Equivalently, if we let

$$\bigoplus_{i=1}^m f_i(t_i) \frac{d}{dt_i} \quad \text{with } f_i = a_{i0} + a_{i1}t_i + g_i, g_i \in (t_i)^2$$

be the local expansion of a section of $\Omega_{\tilde{C}}^{\vee}$ around p_1, \dots, p_m , then $\Omega_{\tilde{C}}^{\vee} \subset \pi_*\Omega_{\tilde{C}}^{\vee}$ is the subsheaf generated by those sections which satisfy

$$\begin{aligned} a_{10} &= \dots = a_{m0} = 0, \\ a_{11} &= \dots = a_{m1}. \end{aligned}$$

Proof. A section of $\Omega_{\tilde{C}}^{\vee} \otimes K(\tilde{C})$ is contained in $\Omega_{\tilde{C}}^{\vee}$ if and only if its image under the push forward map

$$\pi_* : \pi_*\mathcal{H}om(\Omega_{\tilde{C}}, K(\tilde{C})) \rightarrow \mathcal{H}om(\Omega_C, \pi_*K(\tilde{C})),$$

lies in the subspace

$$\mathcal{H}om(\Omega_C, \mathcal{O}_C) \subset \mathcal{H}om(\Omega_C, \pi_*K(\tilde{C})).$$

Thus, we must write out the push forward map in local coordinates. We may work formally around p and use the coordinates introduced in (†).

- (1) (The cusp) The section $f(t_1)(d/dt_1) \in \pi_*\Omega_{\tilde{C}}^{\vee} \otimes K(\tilde{C})$ pushes forward to

$$\pi_* \left(f(t_1) \frac{d}{dt_1} \right) = 2t_1 f(t_1) \frac{d}{dx} + 3t_1^2 f(t_1) \frac{d}{dy}.$$

Since $\hat{\mathcal{O}}_{C,p} = k[[t_1^2, t_1^3]] \subset k[[t_1]]$, we see that if characteristic $k \neq 2, 3$, then

$$2t_1 f(t_1), 3t_1^2 f(t_1) \in \hat{\mathcal{O}}_{C,p} \iff f(t_1) \in (t_1).$$

Thus,

$$\Omega_{\tilde{C}}^{\vee} = \pi_*\Omega_{\tilde{C}}^{\vee}(-p_1).$$

- (2) (The tacnode) The section $f_1(t_1)(d/dt_1) \oplus f_2(t_2)(d/dt_2) \in \pi_*\Omega_{\tilde{C}}^{\vee} \otimes K(\tilde{C})$ pushes forward to

$$\pi_* \left(f_1(t_1) \frac{d}{dt_1} \oplus f_2(t_2) \frac{d}{dt_2} \right) = (f_1(t_1) \oplus f_2(t_2)) \frac{d}{dx} + (2t_1 f_1(t_1) \oplus 0) \frac{d}{dy}.$$

If characteristic $k \neq 2$, then

$$(2t_1 f_1(t_1) \oplus 0) \in \hat{\mathcal{O}}_{C,p} \iff f_1(t_1) \in (t_1).$$

Furthermore, once we know $f_1(t_1) \in (t_1)$, then

$$(f_1(t_1) \oplus f_2(t_2)) \in \hat{\mathcal{O}}_{C,p} \iff f_1(t_1) \oplus f_2(t_2) = a(t_1 \oplus t_2) + (g_1 \oplus g_2)$$

for some $a \in k$ and $(g_1 \oplus g_2) \in (t_1^2) \oplus (t_2^2)$, which is precisely the conclusion of the proposition.

(3) ($m \geq 3$) The section $\bigoplus_{i=1}^m f_i(t_i)(d/dt_i) \in \pi_*\Omega_{\tilde{C}}^\vee \otimes K(\tilde{C})$ pushes forward to

$$\pi_*\left(\bigoplus_{i=1}^m f_i(t_i) \frac{d}{dt_i}\right) = \sum_{i=1}^{m-1} (f_i(t_i) \oplus f_m(t_m)) \frac{d}{dx_i}.$$

Note that the function $(f_i(t_i) \oplus f_m(t_m))$ vanishes identically on all branches except the i th and m th. It follows that, for each $i = 1, \dots, m - 1$,

$$(f_i(t_i) \oplus f_m(t_m)) \in \hat{\mathcal{O}}_{C,p} \iff (f_i(t_i) \oplus f_m(t_m)) = a(t_i \oplus t_m) + (g_1 \oplus g_2),$$

for some $a \in k$ and $(g_i \oplus g_m) \in (t_i^2) \oplus (t_m^2)$. Thus,

$$\bigoplus_{i=1}^m f_i(t_i) = a(t_1 \oplus \dots \oplus t_m) + (g_1(t_1) \oplus \dots \oplus g_m(t_m)),$$

for some $a \in k$ and $g_i \in (t_i^2)$. This completes the proof of the proposition. □

Our only use for Proposition 2.3 is the following corollary, which translates the condition of having no infinitesimal automorphisms into a condition on distinguished points.

COROLLARY 2.4. *Suppose characteristic $k \neq 2, 3$. Let C be a complete n -pointed curve (C, q_1, \dots, q_n) , and let Σ denote the support of the divisor $\sum_i q_i$. Suppose C has an elliptic m -fold point $p \in C$, and that the normalization of C at p consists of m distinct connected components:*

$$\tilde{C} = \tilde{C}_1 \cup \dots \cup \tilde{C}_m,$$

where each \tilde{C}_i is a nodal curve of arithmetic genus zero. Then we have

$$H^0(C, \Omega_C^\vee(-\Sigma)) = 0 \iff \text{conditions (1)–(3) hold.}$$

- (1) The components $\tilde{B}_1, \dots, \tilde{B}_m$ each have at least two distinguished points, where $\tilde{B}_i \subset \tilde{C}_i$ is the unique irreducible component of \tilde{C}_i lying above p .
- (2) At least one of $\tilde{B}_1, \dots, \tilde{B}_m$ has at least three distinguished points.
- (3) Every other component of \tilde{C} has at least three distinguished points.

Proof. First, let us check that these conditions are necessary. For (1), suppose that \tilde{B}_i has only one distinguished point. Then this distinguished point is necessarily p_i , the point lying above p , so $\tilde{B}_i = \tilde{C}_i$, and \tilde{C}_i has a non-zero vector field which vanishes to order two at p_i . One may extend this section (by zero) to a section of $\Omega_{\tilde{C}}^\vee(-\Sigma)$, and Proposition 2.3 implies that it descends to give a non-zero section of $\Omega_C^\vee(-\Sigma)$.

For (2), suppose that each \tilde{B}_i has exactly two distinguished points, say p_i and r_i . Then the restriction map

$$\bigoplus_{i=1}^m \Omega_{\tilde{B}_i}^\vee(-p_i - r_i) \rightarrow \bigoplus_{i=1}^m \Omega_{\tilde{B}_i}^\vee(-p_i - r_i)|_{p_i} \rightarrow 0$$

is surjective on global sections. Thus we can find sections of $\Omega_{\tilde{B}_i}^\vee$ which vanish at p_i and r_i , and whose first derivatives at p_1, \dots, p_m agree. We can extend these (by zero) to a section of $\Omega_{\tilde{C}}^\vee(-\Sigma)$, and Proposition 2.3 implies that this descends to give a non-vanishing section of $\Omega_C^\vee(-\Sigma)$.

Finally, if any other component of \tilde{C} has fewer than three distinguished points, then there exists a vector field on that component which vanishes at all distinguished points. Since this component necessarily meets the rest of \tilde{C} nodally, such a section can be extended (by zero) to a section of $\Omega_{\tilde{C}}^\vee(-\Sigma)$ which descends to $\Omega_C^\vee(-\Sigma)$.

Now let us check that conditions (1)–(3) are sufficient. One easily checks that conditions (1) and (3) imply

$$H^0(\tilde{C}_1, \Omega_{\tilde{C}_1}^\vee(-2p_1 - \Sigma|_{\tilde{C}_1})) = \cdots = H^0(\tilde{C}_m, \Omega_{\tilde{C}_m}^\vee(-2p_m - \Sigma|_{\tilde{C}_m})) = 0,$$

while conditions (2) and (3) imply that, for some i , we have

$$H^0(\tilde{C}_i, \Omega_{\tilde{C}_i}^\vee(-p_i - \Sigma|_{\tilde{C}_i})) = 0.$$

This latter condition says that any section of $\Omega_{\tilde{C}_i}^\vee(-\Sigma|_{\tilde{C}_i})$ which vanishes at p_i must vanish identically. It follows, by Proposition 2.3, that any section of $\Omega_{\tilde{C}}^\vee(-\Sigma)$ which descends to a section of $\Omega_C^\vee(-\Sigma)$ must vanish at p_1, \dots, p_m and have vanishing first derivative at p_1, \dots, p_m . However, since

$$H^0(\tilde{C}_1, \Omega_{\tilde{C}_1}^\vee(-2p_2 - \Sigma|_{\tilde{C}_1})) = \cdots = H^0(\tilde{C}_m, \Omega_{\tilde{C}_m}^\vee(-2p_m - \Sigma|_{\tilde{C}_m})) = 0,$$

any section of $\Omega_{\tilde{C}}^\vee$ satisfying these conditions is identically zero. □

2.2 The dualizing sheaf ω_C

In the following proposition, we describe the dualizing sheaf ω_C locally around an elliptic m -fold point. If $p \in C$ is a singular point on a reduced curve, then, locally around p , ω_C admits the following explicit description. Let $\pi : \tilde{C} \rightarrow C$ be the normalization of C at p and consider the sheaf $\Omega_{\tilde{C}} \otimes K(\tilde{C})$ of rational differentials on \tilde{C} . Let $K_{\tilde{C}}(\Delta) \subset \Omega_{\tilde{C}} \otimes K(\tilde{C})$ be the subsheaf of rational differentials ω satisfying the following condition: for every function $f \in \mathcal{O}_{C,p}$

$$\sum_{p_i \in \pi^{-1}(p)} \text{Res}_{p_i}((\pi^* f) \omega) = 0.$$

Then, locally around p , we have $\omega_C = \pi_* K_{\tilde{C}}(\Delta)$ [AK70, ch. 8, Proposition 1.16(ii)]. Using this description, we can show the following proposition.

PROPOSITION 2.5. *If $p \in C$ is an elliptic m -fold point, then the following hold.*

- (1) *The sheaf ω_C is invertible near p , i.e. the elliptic m -fold point is Gorenstein.*
- (2) *There exists an isomorphism $\pi^* \omega_C \simeq \omega_{\tilde{C}}(2p_1 + \cdots + 2p_m)$.*

Proof. We will prove the proposition when $m \geq 3$ and leave the details of $m = 1, 2$ to the reader. By the previous discussion, sections of ω_C near p are given by rational sections $\omega \in \omega_{\tilde{C}} \otimes K(\tilde{C})$ satisfying

$$\sum_{i=1}^m \text{Res}_{p_i}((\pi^* f) \omega) = 0 \quad \text{for all } f \in \mathcal{O}_{C,p}.$$

By Lemma 2.2(3), every function vanishing to order at least two along p_1, \dots, p_m descends to C , so any differential ω which satisfies this condition can have at most double poles along p_1, \dots, p_m . Now consider the polar part of ω around p_1, \dots, p_m , i.e. write

$$\omega - \omega' = \left(a_1 \frac{dt_1}{t_1^2} + b_1 \frac{dt_1}{t_1} \right) \oplus \cdots \oplus \left(a_m \frac{dt_m}{t_m^2} + b_m \frac{dt_m}{t_m} \right)$$

with $a_i, b_i \in k$ and $\omega' \in \omega_{\tilde{C}}$. It suffices to check the residue condition for scalars and a basis of m_p/m_p^2 . Taking $f \in \mathcal{O}_C$ to be non-zero scalar, the residue condition gives

$$b_1 + \cdots + b_m = 0.$$

Working in the coordinates (†) introduced at the beginning of the section, we see that

$$\{(t_1, 0, \dots, 0, t_m), (0, t_2, 0, \dots, 0, t_m), \dots, (0, \dots, 0, t_{m-1}, t_m)\}$$

gives a basis for m_p/m_p^2 , so the residue condition forces

$$a_i - a_m = 0 \quad \text{for } i = 1, \dots, m - 1.$$

From this, one checks immediately that

$$\left\{ \left(\frac{dt_1}{t_1^2} + \dots + \frac{dt_{m-1}}{t_{m-1}^2} - \frac{dt_m}{t_m^2} \right), \left(\frac{dt_1}{t_1} - \frac{dt_m}{t_m} \right), \left(\frac{dt_2}{t_2} - \frac{dt_m}{t_m} \right), \dots, \left(\frac{dt_{m-1}}{t_{m-1}} - \frac{dt_m}{t_m} \right) \right\}$$

gives a basis of sections for $\omega_C/\pi_*\omega_{\tilde{C}}$. It follows that multiplication by

$$\frac{dt_1}{t_1^2} + \dots + \frac{dt_{m-1}}{t_{m-1}^2} - \frac{dt_m}{t_m^2}$$

gives an isomorphism

$$\mathcal{O}_C \simeq \omega_C,$$

so ω_C is invertible. Since a local generator for ω_C pulls back to a differential with a double pole along each of p_1, \dots, p_m , we also have

$$\pi^*\omega_C = \omega_{\tilde{C}}(2p_1 + \dots + 2p_m). \quad \square$$

2.3 Semistable limits

Our aim in this section is to classify those ‘tails’ that occur when performing semistable reduction to a one-parameter smoothing of the elliptic m -fold point. This will be the key ingredient in the verification of the valuative criterion for $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$. Throughout the section, C denotes a connected curve (not necessarily complete), and for simplicity we will assume that C has a unique singular point p .

DEFINITION 2.6. A *smoothing* of (C, p) consists of a morphism $\pi : C \rightarrow \Delta$, where Δ is the spectrum of a discrete valuation ring with residue field k , and a distinguished closed point $p \in C$ satisfying the following.

- (1) The morphism π is quasiprojective and flat of relative dimension one.
- (2) The morphism π is smooth on $U := C \setminus p$.
- (3) The special fiber of π is isomorphic to (C, p) .

DEFINITION 2.7. If C/Δ is a smoothing of (C, p) , a *semistable limit* of C/Δ consists of a finite base change $\Delta' \rightarrow \Delta$, and a diagram

$$\begin{array}{ccc} \mathcal{C}^s & \xrightarrow{\phi} & C \times_{\Delta} \Delta' \\ & \searrow \pi^s & \swarrow \\ & \Delta' & \end{array}$$

satisfying the following.

- (1) The morphism π^s is quasiprojective and flat of relative dimension one.
- (2) The morphism ϕ is proper, birational, and $\phi(\text{Exc}(\phi)) = p$.
- (3) The total space \mathcal{C}^s is regular, and the special fiber C^s is nodal.
- (4) The exceptional locus $\text{Exc}(\phi)$ contains no (-1) -curves.

The *exceptional curve* of the semistable limit is the pair (E, Σ) where

$$E := \phi^{-1}(p),$$

$$\Sigma := \{E \cap \overline{C^s \setminus E}\}.$$

We think of Σ as a reduced effective Weil divisor on E . Note that (E, Σ) is necessarily semistable, i.e. nodal and each rational component of \tilde{E} has two distinguished points.

DEFINITION 2.8. We say that a pointed curve $(E, \{p_i\}_{i=1}^m)$ is a *semistable tail* of (C, p) if it arises as the exceptional curve of a semistable limit of a smoothing of (C, p) .

In Proposition 2.12, we classify the semistable tails of the elliptic m -fold point. In order to state the result, we need a few easy facts about the dual graph of a nodal curve of arithmetic genus one. (These remarks will be generalized to arbitrary Gorenstein curves of arithmetic genus one in §3.2.) First, observe that if E is any complete, connected, nodal curve of arithmetic genus one, E contains a connected, arithmetic genus one subcurve $Z \subset E$ with no disconnecting nodes. (If E has a disconnecting node q , then the normalization of E at q will comprise two connected components, one of which has arithmetic genus one. Proceed by induction on the number of disconnecting nodes.) There are two possibilities for Z : either it is irreducible or a ring of smooth genus zero curves. By genus considerations, the connected components of $\overline{E \setminus Z}$ will each have arithmetic genus zero and will meet Z in a unique point. We record these observations in the following definition.

DEFINITION 2.9. If E is a connected, complete, nodal curve of arithmetic genus one, there exists a decomposition

$$E := Z \cup R_1 \cup \dots \cup R_m,$$

where Z is either irreducible or a ring of \mathbb{P}^1 entities, and each R_i has arithmetic genus zero and meets Z in exactly one point. We call Z the *minimal elliptic subcurve* of E .

Next, we must introduce notation to talk about the distance between various irreducible components of E .

DEFINITION 2.10. If $F_1, F_2 \subset E$ are subcurves of E , we define $l(F_1, F_2)$ to be the minimum length of any path in the dual graph of E that connects an irreducible component of F_1 to an irreducible component of F_2 . If $p \in E$ is any smooth point, then there is a unique irreducible component $F_p \subset E$ containing p , and we abuse notation by writing $l(p, -)$ instead of $l(F_p, -)$.

DEFINITION 2.11. If $(E, \{p_i\}_{i=1}^m)$ is a semistable curve of arithmetic genus one, we say that $(E, \{p_i\}_{i=1}^m)$ is *balanced* if

$$l(Z, p_1) = l(Z, p_2) = \dots = l(Z, p_m),$$

where $Z \subset E$ is the minimal elliptic subcurve of E .

Now we can state the main result of this section.

PROPOSITION 2.12 (Semistable tails of the elliptic m -fold point). *Suppose $p \in C$ is an elliptic m -fold point, and $(E, \{p_i\}_{i=1}^m)$ is a semistable curve of arithmetic genus one. Then $(E, \{p_i\}_{i=1}^m)$ is a semistable tail of (C, p) if and only if $(E, \{p_i\}_{i=1}^m)$ is balanced.*

The intuition behind Proposition 2.12 is that $(E, \{p_i\}_{i=1}^m)$ is a semistable tail for an elliptic m -fold point if and only if there exists a line bundle $\mathcal{L} := \omega_{C^{ss}/\Delta}(D)$, with $\text{Supp } D \subset E$

and $\omega_{\mathcal{C}/\Delta}(D)|_E \simeq \mathcal{O}_E$ on some semistable curve \mathcal{C}^{ss}/Δ containing E in the special fiber. Clearly, if we are given a semistable limit $\phi : \mathcal{C}^{ss} \rightarrow \mathcal{C}$, where \mathcal{C} is a smoothing of an elliptic m -fold point, then $\phi^*\omega_{\mathcal{C}/\Delta}$ must be a line bundle with the stated properties. On the other hand, if there exists a line bundle with these properties on a semistable curve $\mathcal{C}^{ss} \rightarrow \Delta$, then the birational map $\mathcal{C}^{ss} \rightarrow \mathcal{C}$ induced by a high multiple of \mathcal{L} exhibits E as a semistable tail of the elliptic m -fold point $\phi(E) \in \mathcal{C}$.

To prove Proposition 2.12, we need the following lemma. In conjunction with the classification of singularities in Appendix A, it tells us that whenever we contract an elliptic curve E in the special fiber of a one-parameter family using a line bundle of the form $\omega_{\mathcal{C}/\Delta}(D)$ with $\text{Supp } D \subset E$, then the resulting special fiber has an elliptic m -fold point.

LEMMA 2.13 (Contraction lemma). *Let $\pi : \mathcal{C} \rightarrow \Delta$ be projective and flat of relative dimension one, with connected, reduced geometric fibers. Let \mathcal{L} be a line bundle on \mathcal{C} with positive degree on each irreducible component of the generic fiber and non-negative degree on each irreducible component of the special fiber C . Set*

$$E = \{\text{Irreducible components } F \subset C \mid \text{deg } \mathcal{L}|_F = 0\},$$

and assume that the following hold.

- (1) The subcurve E is connected of arithmetic genus one.
- (2) There is an isomorphism $\mathcal{L}|_E \simeq \mathcal{O}_E$.
- (3) Each point $p \in \overline{C \setminus E} \cap E$ is a node of C .
- (4) Each point $p \in \overline{C \setminus E} \cap E$ is a regular point of C .

Then \mathcal{L} is π -semiample and there exists a diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\phi} & \mathcal{C}' := \text{Proj} \left(\bigoplus_{m \geq 0} \pi_* \mathcal{L}^m \right) \\ & \searrow \pi & \swarrow \pi' \\ & \Delta & \end{array}$$

where ϕ is proper, birational, and $\text{Exc}(\phi) = E$. Furthermore, we have the following.

- (1) The morphism π' is flat and projective, with connected reduced special fiber.
- (2) The restriction $\phi|_{\overline{C \setminus E}} : \overline{C \setminus E} \rightarrow \mathcal{C}'$ is the normalization of \mathcal{C}' at $p := \phi(E)$.
- (3) The number of branches and the δ -invariant of the isolated curve singularity $p \in \mathcal{C}'$ are given by

$$\begin{aligned} m &= |\overline{C \setminus E} \cap E|, \\ \delta &= p_a(E) + m - 1. \end{aligned}$$

If, in addition, we assume that $\omega_{\mathcal{C}/\Delta}$ is invertible and that

$$\mathcal{L} \simeq \omega_{\mathcal{C}/\Delta}(D + \Sigma),$$

where D is a Cartier divisor supported on E , and Σ is a Cartier divisor disjoint from E , then the following holds.

- (4) The sheaf $\omega_{\mathcal{C}'/\Delta}$ is invertible. Equivalently, $p \in \mathcal{C}'$ is a Gorenstein curve singularity.

Proof. To prove that \mathcal{L} is π -semiample, we must show that the natural map

$$\pi^* \pi_* \mathcal{L}^m \rightarrow \mathcal{L}^m$$

is surjective for $m \gg 0$. Since \mathcal{L} is ample on the general fiber of π , it suffices to prove that for each point $x \in C$ there exists a section

$$s_x \in \pi_* \mathcal{L}^m|_0 \subset H^0(C, L^m)$$

which is non-zero at x . Our assumptions imply that E is a Cartier divisor on \mathcal{C} , so we have an exact sequence

$$0 \rightarrow \mathcal{L}^m(-E) \rightarrow \mathcal{L}^m \rightarrow \mathcal{O}_E \rightarrow 0.$$

Pushing forward, we obtain

$$0 \rightarrow \pi_* \mathcal{L}^m(-E) \rightarrow \pi_* \mathcal{L}^m \rightarrow \pi_* \mathcal{O}_E \rightarrow R^1 \pi_* \mathcal{L}^m(-E),$$

and we claim that $R^1 \pi_* \mathcal{L}^m(-E) = 0$ for $m \gg 0$. Since $\mathcal{L}^m(-E)$ is flat over Δ , it is enough to prove that this line bundle has vanishing H^1 on fibers for $m \gg 0$. Since \mathcal{L} is ample on the generic fiber, we only need to consider the special fiber, where we have an exact sequence

$$0 \rightarrow \mathcal{L}^m(-E)|_C \otimes I_{E/C} \rightarrow \mathcal{L}^m(-E)|_C \rightarrow \mathcal{O}_E(-E) \rightarrow 0.$$

We have $H^1(E, \mathcal{O}_E(-E)) = 0$, since $E^2 < 0$ and E has arithmetic genus one. On the other hand, since $\mathcal{L}^m(-E)|_C \otimes I_{E/C}$ is supported on $\overline{C \setminus E}$, we have

$$H^1(C, \mathcal{L}^m(-E)|_C \otimes I_{E/C}) = H^1(\overline{C \setminus E}, (\mathcal{L}^m(-E)|_C \otimes I_{E/C})|_{\overline{C \setminus E}}) = 0$$

for $m \gg 0$, since $\mathcal{L}|_{\overline{C \setminus E}}$ is ample. Thus, $H^1(C, \mathcal{L}^m(-E)|_C) = 0$. This vanishing has two consequences: first, we have a surjection

$$\pi_* \mathcal{L}^m|_0 \rightarrow \pi_* \mathcal{O}_E|_0 \simeq k,$$

so there exists a section $s \in \pi_* \mathcal{L}^m|_0$ which is constant and non-zero along E . Second, we have

$$\pi_* \mathcal{L}^m(-E)|_0 = H^0(C, \mathcal{L}^m(-E)|_C) \subset \pi_* \mathcal{L}^m|_0,$$

which implies the existence of non-vanishing sections at any point of $\overline{C \setminus E}$.

Since \mathcal{L} is π -semiample, we obtain a proper, birational contraction $\phi : \mathcal{C} \rightarrow \mathcal{C}'$ with $\text{Exc}(\phi) = E$ and $\phi_* \mathcal{O}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}'}$. Since \mathcal{C} satisfies Serre's condition S_2 , \mathcal{C}' does as well. The special fiber \mathcal{C}' is a Cartier divisor in \mathcal{C}' , and is therefore S_1 . Equivalently, \mathcal{C}' has no embedded points. No component of \mathcal{C}' can be generically non-reduced because it is the birational image of some component of $\overline{C \setminus E}$. Thus, \mathcal{C}' is reduced. The curve \mathcal{C}' is connected because it is the continuous image of C , which is connected. Finally, Δ is a discrete valuation ring and each component of \mathcal{C}' dominates Δ , so the flatness of π' is automatic. This proves conclusion (1).

Conclusion (2) is immediate from the observation that $\overline{C \setminus E}$ is smooth along the points $E \cap \overline{C \setminus E}$ and maps isomorphically to \mathcal{C}' elsewhere. Since the number of branches of the singular point $p \in \mathcal{C}'$ is, by definition, the number of points lying above p in the normalization, we have

$$m = |\overline{C \setminus E} \cap E|.$$

To obtain $\delta = p_a(E) + m - 1$, note that

$$\begin{aligned} \delta &= \chi(C, \mathcal{O}_{\overline{C \setminus E}}) - \chi(\mathcal{C}', \mathcal{O}_{\mathcal{C}'}) \\ &= \chi(C, \mathcal{O}_{\overline{C \setminus E}}) - \chi(C, \mathcal{O}_C) \\ &= -\chi(C, I_{\overline{C \setminus E}}). \end{aligned}$$

The first equality is just the definition of δ since $\overline{C \setminus E}$ is the normalization of C' at p . The second equality follows from the fact that C and C' occur in flat families with the same generic fiber, and the third equality is just the additivity of Euler characteristic on exact sequences. Since $I_{\overline{C \setminus E}}$ is supported on E , we have

$$\chi(C, I_{\overline{C \setminus E}}) = \chi(E, I_{\overline{C \setminus E}}|_E) = \chi(E, \mathcal{O}_E(-E \cap \overline{C \setminus E})) = 1 - m - p_a(E).$$

This completes the proof of conclusion (3).

Finally, to prove conclusion (4), note that we have a line bundle $\mathcal{O}_{C'}(1)$ such that

$$\phi^* \mathcal{O}_{C'}(1) \simeq \omega_{C/\Delta}(D + \Sigma).$$

Since Σ is a Cartier divisor on \mathcal{C} disjoint from $\text{Exc}(\phi)$, its image is a Cartier divisor on C' , and we have

$$\phi^*(\mathcal{O}_{C'}(1)(-\Sigma)) \simeq \omega_{C/\Delta}(D).$$

Since D is supported on $\text{Exc}(\phi)$, we have

$$\mathcal{O}_{C'}(1)(-\Sigma)|_{C' \setminus p} \simeq \omega_{C'/\Delta}|_{C' \setminus p}.$$

Since $\omega_{C'/\Delta}$ and $\mathcal{O}_{C'}(1)(-\Sigma)$ are both S_2 -sheaves on a S_2 -surface and they are isomorphic in codimension one, we conclude

$$\mathcal{O}_{C'}(1)(-\Sigma) \simeq \omega_{C'/\Delta},$$

i.e. the dualizing sheaf $\omega_{C'/\Delta}$ is actually a line bundle. Since the formation of the dualizing sheaf commutes with base change, $\omega_C = \omega_{C'/\Delta}|_C$ is invertible. Equivalently, $p \in C'$ is a Gorenstein singularity. □

Now we are ready to prove Proposition 2.12.

Proof. (E, p_1, \dots, p_m) balanced implies (E, p_1, \dots, p_m) is a semistable tail. We must show that if $(E, \{p_i\}_{i=1}^m)$ is a balanced semistable curve, then it arises as a semistable tail for some smoothing of the elliptic m -fold point. To construct this smoothing, start by taking $(\tilde{C}, \{p_i\}_{i=1}^m)$ to be any complete smooth m -pointed curve of genus at least two, and attach \tilde{C} and E along $\{p_1, \dots, p_m\}$ to form a nodal curve

$$C^s = \tilde{C} \cup E.$$

Now let C^s/Δ be any smoothing of C^s with smooth total space. We will exhibit a birational morphism $C^s \rightarrow \mathcal{C}$ collapsing E to an elliptic m -fold point $p \in C$. To do this, we must build a line bundle on C^s which is trivial on E , but has positive degree on \tilde{C} . We define

$$\mathcal{L} := \omega_{C^s/\Delta}(D),$$

where

$$D = \sum_{F \subset E} (l + 1 - l(F, Z))F,$$

with $l := l(p_1, Z) = \dots = l(p_m, Z)$. The multiplicities of D are illustrated in Figure 1. Suppose we can show the following.

- (A) The line bundle \mathcal{L} has positive degree on the general fiber of π .
- (B) The restriction $\mathcal{L}|_{\tilde{C}}$ has positive degree on \tilde{C} .
- (C) The restriction $\mathcal{L}|_F \simeq \mathcal{O}_F$ for all irreducible components $F \subset E$.

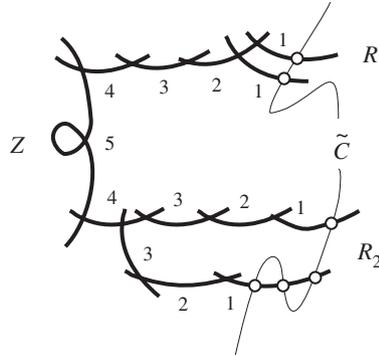


FIGURE 1. A balanced curve E , with minimal elliptic subcurve Z , appearing in the special fiber of a semistable family. We have labeled the multiplicities of a Cartier divisor D such that $\omega_{\mathcal{C}/\Delta}(D)$ is trivial on every component of E .

Then \mathcal{L} satisfies the hypotheses of Lemma 2.13, so a suitably high multiple of \mathcal{L} defines a morphism $\phi : \mathcal{C}^s \rightarrow \mathcal{C}$ contracting E to a single point p . Furthermore, the lemma implies that $p \in \mathcal{C}$ is a Gorenstein singularity with m branches and δ -invariant m . By Proposition A.3, there is a unique such singularity: p must be an elliptic m -fold point. It follows that \mathcal{C}/Δ is a smoothing of the elliptic m -fold point, and hence that $(E, \{p_i\}_{i=1}^m)$ is a semistable tail as desired.

Since the genus of \tilde{C} is at least two, conditions (A) and (B) are automatic. For condition (C), we write

$$E = Z \cup R_1 \cup \dots \cup R_m$$

as in Definition 2.9, and consider the cases $F \subset R_i$ and $F \subset Z$ separately. Suppose first that $F \subset R_i$ for some i , and let G_1, \dots, G_k be the irreducible components of E adjacent to F . Since the dual graph of R_i is a tree meeting Z in a single point, we may order the $\{G_i\}$ so that

$$\begin{aligned} l(G_1, Z) &= l(F, Z) - 1, \\ l(G_i, Z) &= l(F, Z) + 1, \quad 2 \leq i \leq k. \end{aligned}$$

Since F is rational and the total space \mathcal{C}^s is regular, we have

$$\begin{aligned} \deg \omega_{\mathcal{C}^s/\Delta}|_F &= -k - 2, \\ F \cdot F &= -k, \\ G_i \cdot F &= 1, \quad 1 \leq i \leq k. \end{aligned}$$

Now, since F, G_1, \dots, G_k are the only components of D meeting F , we have

$$\begin{aligned} \deg \omega_{\mathcal{C}^s/\Delta}(D)|_F &= \deg \omega_{\mathcal{C}^s/\Delta}|_F + (l + 1 - l(F, Z))(F \cdot F) \\ &\quad + (l + 2 - l(F, Z))(G_1 \cdot F) + (l - l(F, Z))((G_2 + \dots + G_k) \cdot F) \\ &= (-k - 2) + (l + 1 - l(F, Z))(-k) + (l + 2 - l(F, Z)) + (l - l(F, Z))(k - 1) \\ &= 0. \end{aligned}$$

Since F is rational, this implies $\omega_{\mathcal{C}^s/\Delta}(D)|_F \simeq \mathcal{O}_F$.

It remains to show that $\omega_{\mathcal{C}^s/\Delta}(D)|_Z \simeq \mathcal{O}_Z$. First, note that $\omega_Z = \mathcal{O}_Z$ (since Z is irreducible of arithmetic genus one, or a ring of smooth genus zero curves). If G_1, \dots, G_k are the components

of E adjacent to Z , then $l(G_i, Z) = 1$, so we have

$$\begin{aligned} \omega_{\mathcal{C}^s/\Delta}(D)|_Z &\simeq \omega_{\mathcal{C}^s/\Delta}((l+1)Z + lG_1 + \cdots + lG_k)|_Z \\ &\simeq \omega_Z \otimes \mathcal{O}_{\mathcal{C}^s}(lZ + lG_1 + \cdots + lG_k)|_Z \\ &\simeq \mathcal{O}_Z. \end{aligned} \quad \square$$

Proof. $(E, \{p_i\}_{i=1}^m)$ a semistable tail implies $(E, \{p_i\}_{i=1}^m)$ balanced. Suppose $(E, \{p_i\}_{i=1}^m)$ is a semistable tail of the elliptic m -fold point. Then we have a smoothing \mathcal{C}/Δ , a semistable limit \mathcal{C}^s/Δ , and a birational morphism $\phi: \mathcal{C}^s \rightarrow \mathcal{C}$ with exceptional curve E . (Replacing \mathcal{C}/Δ by $\mathcal{C} \times_{\Delta} \Delta'/\Delta'$, we may assume that the semistable limit is defined over the same base as the smoothing.) Set

$$\tilde{C} := \overline{\mathcal{C}^s \setminus E},$$

and note that the restriction of ϕ to \tilde{C} is precisely the normalization of C .

Since $\mathcal{C} \rightarrow \Delta$ and $\mathcal{C}^s \rightarrow \Delta$ are Gorenstein morphisms, they are equipped with relative dualizing sheaves and we may consider the discrepancy of ϕ , i.e. we have

$$\phi^* \omega_{\mathcal{C}/\Delta} = \omega_{\mathcal{C}^s/\Delta}(D),$$

where D is a Cartier divisor supported on E . We may write

$$D = \sum_{F \subset E} d(F)F,$$

and we claim that the coefficients $d(F)$ must satisfy the following conditions.

- (A) If F meets \tilde{C} , then $d(F) = 1$.
- (B) If F, G are adjacent and $l(F, Z) = l(G, Z) - 1$, then $d(F) = d(G) + 1$.

Condition (A) is easy to see: we have

$$\omega_{\mathcal{C}^s/\Delta}(D)|_{\tilde{C}} \simeq (\phi^* \omega_{\mathcal{C}/\Delta})|_{\tilde{C}} \simeq \phi|_{\tilde{C}}^* (\omega_{\mathcal{C}/\Delta}|_C) \simeq \phi|_{\tilde{C}}^* \omega_C.$$

Furthermore, since $\phi|_{\tilde{C}}$ is just the normalization of C , Proposition 2.5 implies that

$$\phi|_{\tilde{C}}^* \omega_C \simeq \omega_{\tilde{C}}(2p_1 + \cdots + 2p_m).$$

Putting these two equations together, we get $\omega_{\mathcal{C}^s/\Delta}(D)|_{\tilde{C}} \simeq \omega_{\tilde{C}}(2p_1 + \cdots + 2p_m)$. Since $\omega_{\mathcal{C}^s/\Delta}|_{\tilde{C}} \simeq \omega_{\tilde{C}}(p_1 + \cdots + p_m)$, D must contain each component that meets \tilde{C} with multiplicity one. This proves (A).

Condition (B) comes from the observation that

$$\omega_{\mathcal{C}^s/\Delta}(D)|_G \simeq \mathcal{O}_G$$

for each irreducible component $G \subset E$, since E is contracted by ϕ . Indeed, suppose condition (B) fails for a pair of adjacent components F, G with $l(F, Z) = l(G, Z) - 1$. Let H_1, \dots, H_k be the remaining components of E adjacent to G and note that

$$l(H_i, Z) = l(G, Z) + 1, \quad 1 \leq i \leq k.$$

By choosing the pair F, G with $l(F, Z)$ maximal, we may assume (B) holds for each of the pairs G, H_i . Thus,

$$d(H_i, Z) = d(G, Z) - 1, \quad 1 \leq i \leq k.$$

Since $\omega_{\mathcal{C}^s/\Delta}(D)|_G \simeq \mathcal{O}_G$, we obtain

$$\begin{aligned} 0 &= \deg \omega_{\mathcal{C}^s/\Delta}(D)|_G \\ &= \deg \omega_{\mathcal{C}^s/\Delta}|_G + d(F)(F \cdot G) + d(G)(G \cdot G) + (d(G) - 1)(H_1 \cdot F + \dots + H_k \cdot F) \\ &= -2 + (k + 1) + d(F) + d(G)(-k - 1) + (d(G) - 1)k, \\ &= d(F) - d(G) - 1, \end{aligned}$$

which gives $d(F) = d(G) + 1$ as desired.

Now we will show that conditions (A) and (B) imply that (E, p_1, \dots, p_m) is balanced. Suppose first that Z is irreducible. Pick a point $p_i \in E \cap \tilde{C}$, and consider a minimum-length path from the irreducible component containing p_i to Z . Then $l(-, Z)$ decreases by one as we move along each consecutive component, so conditions (A) and (B) imply that

$$d(Z) = l(p_i, Z).$$

Since this holds for each point $p_i \in E \cap \overline{C^s \setminus E}$, we have

$$d(Z) = l(p_1, Z) = \dots = l(p_m, Z),$$

so $(E, \{p_i\}_{i=1}^m)$ is balanced.

If Z is a ring of smooth genus zero curves, and $Z_i \subset Z$ is any irreducible component, then the same argument shows that

$$d(Z_i) = l(p_j, Z),$$

for any point $p_j \in E \cap \tilde{C}$ which lies on a connected component of $\overline{C \setminus Z}$ meeting Z_i . Since every connected component of $\overline{C \setminus Z}$ meets some irreducible component of Z , (E, p_1, \dots, p_k) will be balanced if we can show that $d(Z_i) = d(Z_j)$ for all irreducible components $Z_i, Z_j \subset Z$. Since Z is a ring, it suffices to show that for each triple of consecutive components Z_1, Z_2, Z_3 , we have

$$2d(Z_2) = d(Z_1) + d(Z_3).$$

To see this, let H_1, \dots, H_k be the components of R adjacent to Z_2 . By condition (B) we have $d(H_i) = d(Z_2) - 1$ for each H_i . Using $\omega_{\mathcal{C}/\Delta}(D)|_{Z_2} \simeq \mathcal{O}_{Z_2}$, we obtain

$$\begin{aligned} 0 &= \deg \omega_{\mathcal{C}/\Delta}(D)|_{Z_2} \\ &= \deg \omega_{\mathcal{C}/\Delta}|_{Z_2} + d(Z_1)(Z_1 \cdot Z_2) + d(Z_3)(Z_3 \cdot Z_2) + d(Z_2)(Z_2 \cdot Z_2) \\ &\quad + (d(Z_2) - 1)(H_1 \cdot Z_2 + \dots + H_k \cdot Z_2) \\ &= k + d(Z_1) + d(Z_3) + d(Z_2)(-k - 2) + (d(Z_2) - 1)k, \\ &= d(Z_1) + d(Z_3) - 2d(Z_2) \end{aligned}$$

which gives $2d(Z_2) = d(Z_1) + d(Z_3)$ as desired. □

3. Construction of $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$

In this section, we turn from local results concerning the elliptic m -fold point to global considerations of moduli.

3.1 Fundamental decomposition of a genus one curve

The reason that we can formulate a separated moduli problem for pointed curves of genus one, but not for higher genus, boils down to the following observation.

LEMMA 3.1 (Fundamental decomposition). *Let C be a Gorenstein curve of arithmetic genus one. Then C contains a unique subcurve $Z \subset C$ satisfying the following.*

- (1) *The subcurve Z is connected;*
- (2) *The subcurve Z has arithmetic genus one;*
- (3) *The subcurve Z has no disconnecting nodes.*

We call Z the minimal elliptic subcurve of C . We write

$$C = Z \cup R_1 \cup \dots \cup R_k,$$

where R_1, \dots, R_k are the connected components of $\overline{C \setminus Z}$, and call this the fundamental decomposition of C . Each R_i is a nodal curve of arithmetic genus zero meeting Z in a single point, and $Z \cap R_i$ is a node of C .

Proof. First, we show that $Z \subset C$ exists. If C itself has no disconnecting nodes, take $Z = C$. If C has a disconnecting node p , then the normalization of Z at p will comprise two connected components, one of which has arithmetic genus one. Proceed by induction on the number of disconnecting nodes.

Now let R_1, \dots, R_k be the connected components of $\overline{C \setminus Z}$. By genus considerations, each R_i meets Z in a single point $p_i := R_i \cap Z$. We claim that p_1, \dots, p_k are distinct nodes of Z . By genus considerations, each $p_i \in C$ must be a genus zero singularity, so Corollary A.4 implies p_i is a node. Since a node only has two branches, R_i, R_j , and Z cannot meet at a single node, so the points p_1, \dots, p_k are distinct. Finally, each R_i must be Gorenstein of arithmetic genus zero so Proposition A.2 implies that R_i is nodal.

It remains to show that Z is unique. By symmetry, it is enough to show that if Z' satisfies conditions (1)–(3) then $Z' \subset Z$. If this fails then $Z' \cap R_i \neq \emptyset$ for some i . Since $p_a(Z') = 1$, Z' cannot be contained in R_i , so Z' meets Z . Then, however, since Z' is connected, Z' contains the disconnecting node $R_i \cap Z$, a contradiction. □

COROLLARY 3.2. *Let C be a Gorenstein curve of arithmetic genus one with minimal elliptic subcurve Z . If $E \subset C$ is any connected arithmetic genus one subcurve of C , then $Z \subset E$.*

Proof. The minimal elliptic subcurve of E is necessarily the minimal elliptic subcurve of C , namely Z . Thus, $Z \subset E$. □

The following lemma gives an exact characterization of the ‘minimal elliptic subcurves’ appearing in Lemma 3.1.

LEMMA 3.3. *Suppose Z is a Gorenstein curve of arithmetic genus one and has no disconnecting nodes. Then either Z is one of the following:*

- (1) *a smooth elliptic curve;*
- (2) *an irreducible rational nodal curve;*
- (3) *a ring of smooth genus zero curves; or it satisfies the following.*
- (4) *The curve Z has an elliptic m -fold point p and the normalization of Z at p consists of m distinct smooth rational curves.*

Furthermore, in all four cases, $\omega_Z \simeq \mathcal{O}_Z$.

Proof. If Z is nodal, it is clear that possibilities (1)–(3) are the only possibilities, and that for these, $\omega_Z \simeq \mathcal{O}_Z$.

Now suppose Z has a non-nodal singular point p . Then, by Corollary A.4, p is an elliptic m -fold point for some integer m and the normalization of Z at p consists of m distinct connected nodal curves of arithmetic genus zero. But a nodal curve of arithmetic genus zero with no disconnecting nodes must be smooth, so possibility (4) holds.

To prove $\omega_Z \simeq \mathcal{O}_Z$, we will write down a nowhere vanishing global section of ω_Z . Let $\tilde{Z}_1, \dots, \tilde{Z}_m$ be the connected components of \tilde{Z} and $p_i \in \tilde{Z}_i$ the point lying over p . We may choose local coordinates t_i at p_i so that the map $\tilde{Z} \rightarrow Z$ is given by the expression (†) in Definition 2.1. Since each $\tilde{Z}_i \simeq \mathbb{P}^1$, the rational differential

$$\frac{dt_1}{t_1^2} \in H^0(\tilde{Z}_i, \omega_{\tilde{Z}_i}(2p_i))$$

gives a global section of $\omega_{\tilde{Z}_i}(2p_i)$, regular and non-vanishing away from p_i . The proof of Proposition 2.5 shows that

$$\frac{dt_1}{t_1^2} + \dots + \frac{dt_{m-1}}{t_{m-1}^2} - \frac{dt_m}{t_m^2} \in H^0(\omega_{\tilde{Z}}(2p_1 + \dots + 2p_m))$$

descends to a section of ω_Z which generates ω_Z locally around p . Thus, it generates ω_Z globally. □

In order to define and work with the moduli problem of m -stable curves, it is useful to have the following terminology.

DEFINITION 3.4 (Level). Let $(C, \{p_i\}_{i=1}^n)$ be an n -pointed curve of arithmetic genus one, let $Z \subset C$ be the minimal elliptic subcurve of C , and let $\Sigma \subset C$ denote the support of the divisor $\sum_i p_i$. The *level* of $(C, \{p_i\}_{i=1}^n)$ is defined to be the integer

$$|Z \cap \overline{C \setminus Z}| + |Z \cap \Sigma|.$$

LEMMA 3.5. Suppose $(C, \{p_i\}_{i=1}^n)$ is an n -pointed curve of arithmetic genus one and suppose every smooth rational component of C has at least two distinguished points. Let $Z \subset C$ be the minimal elliptic subcurve, and $\Sigma \subset C$ the support of the divisor $\sum_i p_i$. If E is any connected subcurve of arithmetic genus one, then

$$|E \cap \overline{C \setminus E}| + |E \cap \Sigma| \geq |Z \cap \overline{C \setminus Z}| + |Z \cap \Sigma|.$$

Proof. Let $C = Z \cup R_1 \cup \dots \cup R_k$ be the fundamental decomposition of C , and order the R_i so that E contains R_1, \dots, R_j , but does not contain R_{j+1}, \dots, R_k . The assumption that each smooth rational component has at least two distinguished points implies that each of R_1, \dots, R_j contains at least one marked point so

$$|E \cap \Sigma| \geq |Z \cap \Sigma| + j.$$

On the other hand, since E does not contain R_{j+1}, \dots, R_k , we must have

$$|E \cap \overline{C \setminus E}| \geq |Z \cap \overline{C \setminus Z}| - j.$$

Thus,

$$|E \cap \overline{C \setminus E}| + |E \cap \Sigma| \geq |Z \cap \overline{C \setminus Z}| + |Z \cap \Sigma|. \quad \square$$

COROLLARY 3.6. Let $(C, \{p_i\}_{i=1}^n)$ be an n -pointed curve of arithmetic genus one, and suppose that every smooth rational component has at least two distinguished points. Then $(C, \{p_i\}_{i=1}^n)$ has level greater than m if and only if

$$|E \cap \overline{C \setminus E}| + |E \cap \Sigma| > m$$

for every connected arithmetic genus one subcurve $E \subset C$.

3.2 Definition of the moduli problem

We are ready to define the moduli problem of (m, \mathcal{A}) -stable curves.

DEFINITION 3.7 ((m, \mathcal{A}) -stability). Fix positive integers $m < n$, and a vector of rational weights $\mathcal{A} = (a_1, \dots, a_n) \in (0, 1]^n$. Let $(C, \{p_i\}_{i=1}^n)$ be an n -pointed curve of arithmetic genus one, and let $\Sigma \subset C$ denote the support of the divisor $\sum_i p_i$. We say that C is (m, \mathcal{A}) -stable if the following hold.

- (1) The singularities of C are nodes or elliptic l -fold points, $l \leq m$.
- (2) The level of $(C, \{p_i\}_{i=1}^n)$ is $> m$. Equivalently, by Corollary 3.6,

$$|E \cap \overline{C \setminus E}| + |E \cap \Sigma| > m$$

for every connected arithmetic genus one subcurve $E \subset C$.

- (3) The space $H^0(C, \Omega_C^\vee(-\Sigma)) = 0$. Equivalently, by Corollary 2.4, we have the following.
 - (a) If C is nodal, then every rational component of \tilde{C} has at least three distinguished points.
 - (b) If C has a (unique) elliptic m -fold point p , and $\tilde{B}_1, \dots, \tilde{B}_m$ denote the components of the normalization whose images contain p , then the following hold.
 - (b1) $\tilde{B}_1, \dots, \tilde{B}_m$ each have at least two distinguished points.
 - (b2) At least one of $\tilde{B}_1, \dots, \tilde{B}_m$ has each have at least three distinguished points.
 - (b3) Every other component of \tilde{C} has at least three distinguished points.
- (4) If $p_{i_1} = \dots = p_{i_k} \in C$ coincide, then $\sum_{j=1}^k a_{i_j} \leq 1$.
- (5) The \mathbb{Q} -divisor $\omega_C(\sum_i a_i p_i)$ is ample.

Remark. When $\mathcal{A} = (1, \dots, 1)$, then we say simply that $(C, \{p_i\}_{i=1}^n)$ is m -stable. In this case, condition (4) merely asserts that the marked points are distinct, and condition (5) follows automatically from condition (3). Indeed, conditions (b1) and (b3) above, combined with Proposition 2.5, imply that $\omega_C(\sum_i p_i)$ has positive degree on every component of C .

The definition of an (m, \mathcal{A}) -stable curve extends to a moduli functor in the usual way, and the main theorem of this paper is the following.

THEOREM 3.8. $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$ is a proper irreducible Deligne–Mumford stack over $\text{Spec } \mathbb{Z}[1/6]$.

We will prove that the moduli problem of (m, \mathcal{A}) -stable curves is bounded and deformation-open in Lemmas 3.9 and 3.10, and we verify the valuative criterion in §§ 3.3 and 3.4. Everything else follows by standard arguments which we outline below.

Proof. To say that $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$ is an algebraic stack of finite type over $\text{Spec } \mathbb{Z}[1/6]$ means the following [LM00, Definition 4.1].

- (1) The diagonal $\Delta : \overline{\mathcal{M}}_{1,\mathcal{A}}(m) \rightarrow \overline{\mathcal{M}}_{1,\mathcal{A}}(m) \times \overline{\mathcal{M}}_{1,\mathcal{A}}(m)$ is representable, quasicompact, and of finite type.
- (2) There exists an irreducible scheme U , of finite type over $\text{Spec } \mathbb{Z}[1/6]$, with a smooth, surjective morphism $U \rightarrow \overline{\mathcal{M}}_{1,\mathcal{A}}(m)$.

Since m -stable curves are canonically polarized, the Isom-functor for any pair of m -stable curves over S is representable by a quasiprojective scheme over S , which gives result (1).

For result (2), fix an integer $N > n + \max\{2m, 4\}$ as in the boundedness statement of Lemma 3.9, and assume that N is sufficiently divisible so that each $Na_i \in \mathbb{Z}$. Set

$$\begin{aligned} d &= N(\sum_i a_i), \\ r &= N(\sum_i a_i) - 1. \end{aligned}$$

If $(C, \{p_i\}_{i=1}^n)$ is any m -stable curve, Riemann–Roch implies

$$\begin{aligned} d &= \deg \omega_C(\sum_i p_i)^{\otimes N}, \\ r &= \dim H^0(C, \omega_C(\sum_i p_i)^{\otimes N}) - 1, \end{aligned}$$

so Lemma 3.9 implies that every N -canonically polarized m -stable curve appears in the Hilbert scheme of curves of degree d and arithmetic genus one in $\mathbb{P}^r := \mathbb{P}_{\mathbb{Z}[1/6]}^r$. Let \mathcal{H} denote this Hilbert scheme and consider the locally closed subscheme

$$Z = \{(C, \{p_i\}_{i=1}^n) \in \mathcal{H} \times (\mathbb{P}^r)^n \mid p_1, \dots, p_n \text{ are smooth points of } C\}.$$

By Lemma 3.10, there exists an open subscheme of Z defined by

$$V = \{(C, \{p_i\}_{i=1}^n) \in \mathcal{H} \times (\mathbb{P}^r)^n \mid (C, \{p_i\}_{i=1}^n) \text{ is } m\text{-stable}\}.$$

Using the representability of the Picard scheme as in [MFK94, Proposition 5.1], there exists a locally closed subscheme $U \subset V$, such that

$$U = \{(C, \{p_i\}_{i=1}^n) \in V \mid \omega_C(\sum_i a_i p_i)^{\otimes N} \simeq \mathcal{O}_C(1)\}.$$

Now the classifying map $U \rightarrow \overline{\mathcal{M}}_{1,n}(m)$ is smooth and surjective.

To show that $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$ is Deligne–Mumford over $\text{Spec } \mathbb{Z}[1/6]$, it suffices to show that if k is an algebraically closed field and characteristic $k \neq 2, 3$, then the group scheme $\text{Aut}_k(C, \{p_i\}_{i=1}^n)$ is unramified over k . There is a natural identification of $k[\epsilon]/(\epsilon^2)$ -points of $\text{Aut}_k(C, \{p_i\}_{i=1}^n)$ with global sections of $\Omega_C^\vee(-\Sigma)$ [Has03, § 3.3], so this follows from condition (3) in the definition of an (m, \mathcal{A}) -stable curve.

Finally, to show that $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$ is irreducible, it is sufficient to show that $\mathcal{M}_{1,n} \subset \overline{\mathcal{M}}_{1,\mathcal{A}}(m)$ is dense, i.e. that every m -stable curve is smoothable. Since a curve is smoothable if and only if each of its singularities is smoothable [Kol96, II.6.3], and the only singularities on an m -stable curve are elliptic l -fold points and nodes, it suffices to see that the elliptic l -fold point is smoothable. This is an old result going back to Pinkham [Pin74, Example 11.13], but we may also note that we have constructed explicit smoothings of the elliptic l -fold point in our proof of Proposition 2.12. □

LEMMA 3.9 (Boundedness). *If $(C, \{p_i\}_{i=1}^n)$ is any (m, \mathcal{A}) -stable curve, then the line bundle*

$$L^N := \omega_C(\sum_i a_i p_i)^{\otimes N}$$

is very ample on C for any $N > n + \max\{2m, 4\}$ and sufficiently divisible.

Proof. Throughout this argument, we will assume that N is chosen sufficiently divisible so that $\omega_C(\sum_i a_i p_i)^{\otimes N}$ is integral. With this caveat, it is enough to show that $N > n + \max\{2m, 4\}$ implies the following.

- (1) The space $H^1(C, L^N \otimes I_p) = 0$ for any point $p \in C$.
- (2) The space $H^1(C, L^N \otimes I_p I_q) = 0$ for any pair of points $p, q \in C$.

Condition (1) says that the complete linear series $H^0(C, L^N)$ is basepoint-free, while condition (2) says that it separates points ($p \neq q$) and tangent vectors ($p = q$). Clearly condition (2) implies condition (1). Using Serre duality, it is enough to show that

$$H^0(C, \omega_C \otimes L^{-N} \otimes (I_p I_q)^\vee) = 0.$$

Let $\pi : \tilde{C} \rightarrow C$ be the normalization of C at p and q , with p_1, \dots, p_k the points of \tilde{C} lying above p , and q_1, \dots, q_l the points lying above q . Define

$$D := \sum_{i=1}^m 2p_i + \sum_{j=1}^l 2q_j$$

as a Cartier divisor on \tilde{C} , and note that $\deg D \leq \max\{4, 2m\}$ (since any singular point of C has at most $\max\{2, m\}$ branches). By Lemma 2.2,

$$\pi_* \mathcal{O}_{\tilde{C}}(-D) \subset I_p I_q,$$

and the quotient is torsion, supported at $\{p\} \cup \{q\}$. Thus, we obtain injections

$$\mathcal{H}om(I_p I_q, \mathcal{O}_C) \hookrightarrow \mathcal{H}om(\pi_* \mathcal{O}_{\tilde{C}}(-D), \mathcal{O}_C) \hookrightarrow \pi_* \mathcal{H}om(\mathcal{O}_{\tilde{C}}(-D), \mathcal{O}_{\tilde{C}}).$$

Tensoring by $\omega_C \otimes L^{-N}$, we obtain

$$(I_p I_q)^\vee \otimes (\omega_C \otimes L^{-N}) \hookrightarrow \pi_* \mathcal{O}_{\tilde{C}}(D) \otimes (\omega_C \otimes L^{-N}),$$

so that

$$H^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D) \otimes \pi^*(\omega_C \otimes L^{-N})) = 0 \implies H^0(C, \omega_C \otimes L^{-N} \otimes (I_p I_q)^\vee) = 0.$$

We claim that $N > n + \max\{4, 2m\}$ forces the line bundle $\mathcal{O}_{\tilde{C}}(D) \otimes \pi^*(\omega_C \otimes L^{-N})$ to have negative degree on each component (and hence no sections). Since π^*L has degree at least one on every component of \tilde{C} , and $\deg D \leq \max\{4, 2m\}$, it is enough to show that $\pi^*\omega_C$ has degree at most n on any irreducible component $F \subset \tilde{C}$. To see this, simply observe

$$\deg_F \pi^* \omega_C \leq \deg_F \pi^* \omega_C(\sum_i p_i) \leq n,$$

where the last inequality follows from the fact that $\pi^*\omega_C(\sum_i p_i)$ has total degree n and non-negative degree on each component. □

LEMMA 3.10 (Deformation-openness). *Let S be a Noetherian scheme and let $(\phi : \mathcal{C} \rightarrow S, \sigma_1, \dots, \sigma_n)$ be a flat, projective morphism of relative dimension one with n sections $\sigma_1, \dots, \sigma_n$. The set*

$$T = \{s \in S \mid (C_{\bar{s}}, \sigma_1(\bar{s}), \dots, \sigma_n(\bar{s})) \text{ is } m\text{-stable}\}$$

is Zariski-open in S .

Proof. We may assume that the fibers of ϕ are reduced, connected, and of arithmetic genus one, since these are all open conditions. We may also assume that the geometric fibers are Gorenstein (the locus in S over which the geometric fibers are Gorenstein is the same as the locus over which the relative dualizing sheaf $\omega_{\mathcal{C}/S}$ is invertible, hence open). Finally, the conditions that the sections lie in the smooth locus of ϕ , that $\sigma_{i_1}, \dots, \sigma_{i_k}$ collide only if $\sum_{j=1}^k a_{i_j} \leq 1$, and that $\omega_{\mathcal{C}/S}(\sum_i a_i)$ is relatively ample are obviously open. It only remains to check conditions (1)–(3) of Definition 3.7.

For condition (1), suppose that $s \in S$ is a geometric point and that the fiber C_s has an elliptic m -fold point p . We must show there exists an open neighborhood of s over which the fibers of \mathcal{C}

have only elliptic l -fold points, $l \leq m$, and nodes. Suppose first that $m \geq 3$. Since the dimension of the Zariski tangent space of the elliptic m -fold point is m when $m \geq 3$, we have

$$\dim_{k(x)} m_x/m_x^2 \leq m \quad \text{for every } x \in C_s,$$

where m_x refers to the maximal ideal of x in the local ring of the fiber. Thus, there is an open neighborhood of the fiber $C \subset V \subset X$ such that

$$\dim_{k(x)} m_x/m_x^2 \leq m \quad \text{for every } x \in V.$$

Since π is proper, we may take V to be of the form $\pi^{-1}(U)$ for some open set $U \subset S$. Now, for any $s \in U$, the fiber is a Gorenstein curve of arithmetic genus one whose Zariski tangent space dimension is everywhere no greater than m . By Proposition A.3, the only singularities appearing on fibers over U are elliptic l -fold points, $l \leq m$, and nodes.

It remains to consider the case $m = 2$ or $m = 3$, i.e. $p \in C_s$ is cusp or tacnode. For this, we need a bit of deformation theory [Ser06, §3.1]. Recall that the cusp and tacnode, being local complete intersections, admit versal deformations given by:

$$\begin{aligned} \text{Spec } A[x, y, a, b]/(y^2 = x^3 + ax + b) &\rightarrow \text{Spec } A[a, b], \\ \text{Spec } A[x, y, a, b, c]/(y^2 = x^4 + ax^2 + bx + c) &\rightarrow \text{Spec } A[a, b, c], \end{aligned}$$

where $A = k(s)$ if characteristic $k(s) = 0$ or the unique complete local ring with residue field $k(s)$ and maximal ideal pA if characteristic $k(s) = p$. If $p \in C$ is a cusp (respectively tacnode), there is an étale neighborhood $(U, 0) \rightarrow (S, s)$, and a map

$$U \rightarrow \text{Spec } A[a, b] \ (\text{Spec } A[a, b, c]),$$

such that, étale-locally around $p \in C$, $\mathcal{C} \times_S U$ is pulled back from the versal family. Since the only singularities appearing in fibers of the versal deformation of the cusp (respectively tacnode) are nodes (respectively nodes and cusps), we are done.

For condition (2), we must show that the locus in S over which the fibers have level greater than m is open in S . Since S is Noetherian, it suffices to show that this locus is constructible and stable under generalization. It is clearly constructible, since we may stratify T into locally closed subsets corresponding to the topological type of the fiber and observe that the level of a fiber depends only on the topological type. To see that it is stable under generalization, we may assume that S is the spectrum of a discrete valuation ring with closed point $0 \in S$ and generic point $\eta \in S$. We must show that if $(C_0, \sigma_1(0), \dots, \sigma_n(0))$ has level greater than m , then so does $(C_{\bar{\eta}}, \sigma_1(\bar{\eta}), \dots, \sigma_n(\bar{\eta}))$.

Let $E_{\bar{\eta}}$ be a connected arithmetic genus one subcurve of the geometric generic fiber $C_{\bar{\eta}}$. The limit of $E_{\bar{\eta}}$ in the special fiber is a connected arithmetic genus one subcurve $E_0 \subset C_0$ satisfying

$$\begin{aligned} |E_{\bar{\eta}} \cap \overline{C_{\bar{\eta}} \setminus E_{\bar{\eta}}}| &= |E_0 \cap \overline{C_0 \setminus E_0}|, \\ |E_{\bar{\eta}} \cap \Sigma_{\bar{\eta}}| &= |E_0 \cap \Sigma_0|. \end{aligned}$$

Since $(C_0, \sigma_1(0), \dots, \sigma_n(0))$ has level greater than m , we have

$$|E_{\bar{\eta}} \cap \overline{C_{\bar{\eta}} \setminus E_{\bar{\eta}}}| + |E_{\bar{\eta}} \cap \Sigma| = |E \cap \overline{C \setminus E}| + |E \cap \Sigma| > m.$$

Thus, $(C_{\bar{\eta}}, \sigma_1(\bar{\eta}), \dots, \sigma_n(\bar{\eta}))$ has level greater than m , as desired.

For condition (3), using the natural identification between $k[\epsilon]/\epsilon^2$ -points of $\text{Aut}_k(C, \{p_i\}_{i=1}^n)$ and global sections of $\Omega_C^\vee(-\Sigma)$ [Has03, §3.3], it suffices to show that the locus $U \subset S$ over which the group scheme $\text{Aut}_S(\mathcal{C}, \sigma_1, \dots, \sigma_n) \rightarrow S$ is unramified is open in S . However, this is

a general fact about group schemes: Suppose that $\pi : G \rightarrow S$ is any finite type group scheme over a Noetherian base with identity section $e : S \rightarrow G$, and suppose that π is unramified over a point $s \in S$. Since the condition of being unramified is open on the domain, there is an open neighborhood $e(s) \in W \subset G$ such that $\pi|_W$ is unramified. Setting $U := e^{-1}(W) \subset S$, we may use translations to cover $\pi^{-1}(U)$ by open sets over which π is unramified. \square

3.3 Valuative criterion for $\overline{\mathcal{M}}_{1,n}(m)$

To show that $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$ is proper, it suffices to verify the valuative criterion for discrete valuation rings with algebraically closed residue fields, whose generic point maps into the open dense substack $\mathcal{M}_{1,n}$ [LM00, Remark 7.12.4]. Thus, the required statement is the following.

THEOREM 3.11 (Valuative criterion for properness of $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$). *Let Δ be the spectrum of a discrete valuation ring with an algebraically closed residue field, and let $\eta \in \Delta$ be the generic point.*

- (1) *(Existence of (m, \mathcal{A}) -stable limits). If $(\mathcal{C}, \sigma_1, \dots, \sigma_n)|_\eta$ is a smooth n -pointed curve of arithmetic genus one over η , there exists a finite base change $\Delta' \rightarrow \Delta$, and an (m, \mathcal{A}) -stable curve $(\mathcal{C}' \rightarrow \Delta', \sigma'_1, \dots, \sigma'_n)$, such that*

$$(\mathcal{C}', \sigma'_1, \dots, \sigma'_n)|_{\eta'} \simeq (\mathcal{C}, \sigma_1, \dots, \sigma_n)|_\eta \times_{\eta} \eta'.$$

- (2) *(Uniqueness of (m, \mathcal{A}) -stable limits). Suppose that $(\mathcal{C} \rightarrow \Delta, \sigma_1, \dots, \sigma_n)$ and $(\mathcal{C}' \rightarrow \Delta, \sigma'_1, \dots, \sigma'_n)$ are (m, \mathcal{A}) -stable curves with smooth generic fiber. Then any isomorphism over the generic fiber*

$$(\mathcal{C}, \sigma_1, \dots, \sigma_n)|_\eta \simeq (\mathcal{C}', \sigma'_1, \dots, \sigma'_n)|_\eta$$

extends to an isomorphism over Δ :

$$(\mathcal{C}, \sigma_1, \dots, \sigma_n) \simeq (\mathcal{C}', \sigma'_1, \dots, \sigma'_n).$$

In this section, we will prove existence and uniqueness of m -stable limits, i.e. we will restrict ourselves to the special case $\mathcal{A} = (1, \dots, 1)$. This will allow us to exhibit the main ideas of the proof with a minimum of notational obfuscation. In § 3.4, we will show that the existence and uniqueness of (m, \mathcal{A}) -stable limits can be deduced from the corresponding statement for m -stable limits, in the same way that the existence and uniqueness of \mathcal{A} -stable limits are deduced from the corresponding statement for Deligne–Mumford stable limits.

3.3.1 Existence of m -stable limits. Given a one-parameter family of smooth curves over η , we construct the m -stable limit in three steps: first, we extend this family to a semistable curve with smooth total space. Second, we blow up marked points on the minimal elliptic subcurve of the special fiber, and then contract the strict transform of the minimal elliptic subcurve using Lemma 2.13. Repeating this process, one eventually reaches a stage where the minimal elliptic subcurve Z satisfies

$$|Z \cap \overline{C \setminus Z}| + |\{p_i \mid p_i \in Z\}| > m.$$

At this point, we ‘stabilize’, i.e. blow-down all smooth \mathbb{P}^1 which meet the rest of the fiber in two nodes and have no marked points, or meet the rest of the fiber in a single node and have one marked point. The entire process is pictured in Figure 2.

Step 1. Pass to a semistable limit with smooth total space.

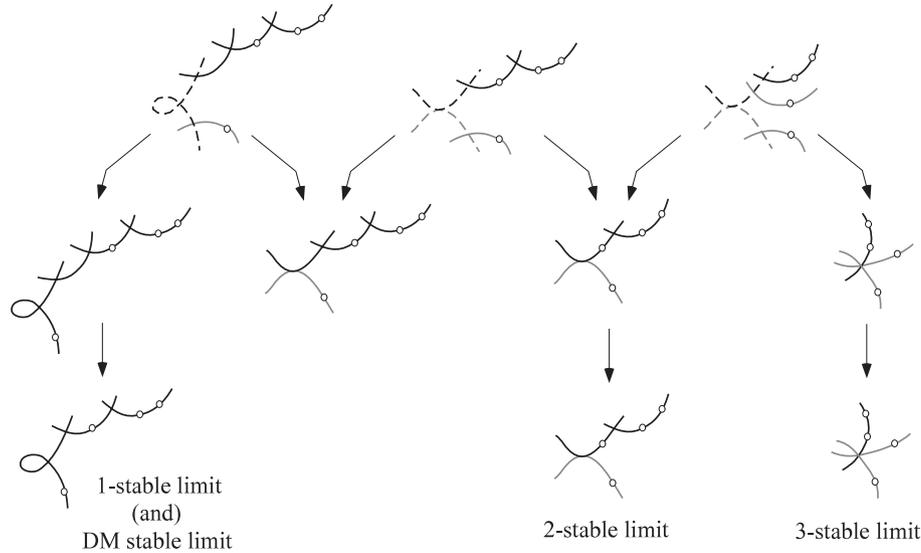


FIGURE 2. The process of blow-up/contraction/stabilization in order to extract the m -stable limit for each $m = 1, 2, 3$. Every irreducible component pictured above is rational. The left-diagonal maps are simple blow-ups along the marked points of the minimal elliptic subcurve, and exceptional divisors of these blow-ups are colored grey. The right-diagonal maps contract the minimal elliptic subcurve of the special fiber, and exceptional components of these contractions are dotted. The vertical maps are stabilization morphisms, blowing down all semistable components of the special fiber.

By the semistable reduction theorem [DM69, Corollary 2.7], there exists a finite base change $\Delta' \rightarrow \Delta$, and a semistable curve $(\mathcal{C}^{ss} \rightarrow \Delta', \sigma'_1, \dots, \sigma'_n)|_{\eta}$ such that

$$(\mathcal{C}^{ss}, \sigma'_1, \dots, \sigma'_n)|_{\eta'} \simeq (C, \sigma_1, \dots, \sigma_n) \times_{\eta} \eta'.$$

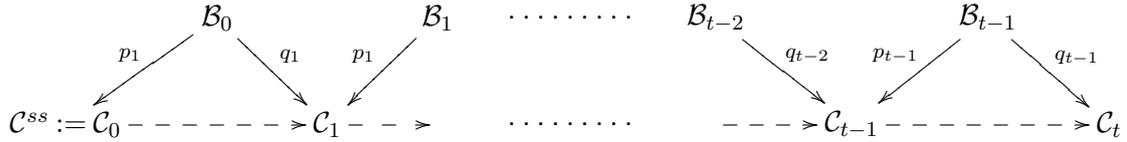
After taking a minimal resolution of singularities, we may assume that the total space of \mathcal{C}^{ss} is regular. For notational simplicity, we will continue to denote our base by Δ , and the given sections by $\sigma_1, \dots, \sigma_n$.

Step 2. Alternate between blowing up marked points contained on the minimal elliptic subcurve and contracting the minimal elliptic subcurve.

Starting from $\mathcal{C}_0 := \mathcal{C}^{ss}$, we construct a sequence $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_t$ of flat proper families over Δ satisfying the following.

- (i) The special fiber $C_i \subset \mathcal{C}_i$ is a Gorenstein curve of arithmetic genus one.
- (ii) The total space \mathcal{C}_i is regular at every node of C_i .
- (iii) The strict transforms of $\sigma_1, \dots, \sigma_n$ on \mathcal{C}_i are contained in the smooth locus of π_i , so we may consider the special fiber as an n -pointed curve (C_i, p_1, \dots, p_n) .
- (iv) Every component of C_i has at least two distinguished points.
- (v) For $i \geq 1$, C_i has an elliptic l_{i-1} -fold point p , where l_i denotes the level of the special fiber C_i (Definition 3.4).
- (vi) For $i \geq 1$, we have $l_i \geq l_{i-1}$. Furthermore, $l_i = l_{i-1}$ if and only if each irreducible component of Z_i has exactly two distinguished points, where Z_i is the minimal elliptic subcurve of C_i .
- (vii) The curve C_t has no disconnecting nodes.

These families fit into the following diagram of birational morphisms over Δ .



Indeed, given \mathcal{C}_i satisfying conditions (i)–(vi), we construct \mathcal{C}_{i+1} as follows. The special fiber $C_i \subset \mathcal{C}_i$ is Gorenstein by condition (i), so it possesses a minimal elliptic subcurve $Z_i \subset C_i$, and we define $p_i : \mathcal{B}_i \rightarrow \mathcal{C}_i$ to be the simple blow-up of \mathcal{C}_i at the finite set of smooth points $\{p_j \mid p_j \in Z_i\}$. We define $q_i : \mathcal{B}_i \rightarrow \mathcal{C}_{i+1}$ to be the contraction of \tilde{Z}_i , the strict transform of Z_i in \mathcal{B}_i (q_i is uniquely characterized by the properties that $\text{Exc}(q_i) = \tilde{Z}_i$ and $q_{i*} \mathcal{O}_{\mathcal{B}_i} = \mathcal{O}_{\mathcal{C}_{i+1}}$).

To prove that q_i exists, consider the line bundle

$$\mathcal{L} := \omega_{\mathcal{B}_i/\Delta}(\tilde{Z}_i + \sigma_1 + \cdots + \sigma_n).$$

Note that $Z_i \subset C_i$ is Cartier by condition (ii), so $\tilde{Z}_i \subset \mathcal{B}_i$ is Cartier. Furthermore, $\sigma_1, \dots, \sigma_n$ are Cartier divisors on \mathcal{B}_i by condition (iii). Adjunction and Lemma 3.3 give

$$\mathcal{L}|_{\tilde{Z}_i} \simeq \omega_{\tilde{Z}_i} \simeq \mathcal{O}_{\tilde{Z}_i}.$$

Let $B_i \subset \mathcal{B}_i$ be the special fiber. By condition (iv), \mathcal{L} has non-negative degree on every irreducible component of B_i not contained in \tilde{Z}_i , and the subcurve $E \subset C_i$ on which \mathcal{L} has degree zero is precisely

$$E = \tilde{Z}_i \cup F,$$

where F is the union of irreducible components of B_i which are disjoint from \tilde{Z}_i and have exactly two distinguished points. Now Lemma 2.13 applies to the line bundle \mathcal{L} , so $\tilde{Z}_i \cup F$ is a contractible subcurve of the special fiber. Since \tilde{Z}_i is disjoint from F , we may certainly contract \tilde{Z}_i on its own; this shows that $q_i : \mathcal{B}_i \rightarrow \mathcal{C}_{i+1}$ exists.

Now we must show that \mathcal{C}_{i+1} satisfies conditions (i)–(vi), and that after finitely many steps we achieve condition (vii).

- (i) Locally around $q(\tilde{Z}_i)$, \mathcal{C}_{i+1} is isomorphic to the contraction given by a high power of \mathcal{L} , so Lemma 2.13 implies that \mathcal{C}_{i+1} is Gorenstein.
- (ii) Since \mathcal{C}_i is regular around every node of the special fiber, so is \mathcal{B}_i . Since $q(\tilde{Z}_i) \in \mathcal{C}_{i+1}$ is not a node, the same is true for \mathcal{C}_{i+1} .
- (iii) Condition (iii) follows immediately from the fact that none of the sections $\sigma_1, \dots, \sigma_n$ of \mathcal{B}_i pass through \tilde{Z}_i .
- (iv) Since every component of C_i has at least two distinguished points, and every exceptional divisor of p_i has two distinguished points, every component of B_i has at least two distinguished points. Since q_i maps distinguished points to distinguished points, every component of \mathcal{C}_{i+1} has at least two distinguished points.
- (v) Write out the fundamental decomposition of C_i :

$$C_i = Z_i \cup R_1 \cup \cdots \cup R_k.$$

Then we can decompose the special fiber B_i as

$$B_i = \tilde{Z}_i \cup \tilde{R}_1 \cup \cdots \cup \tilde{R}_k \cup F_1 \cup \cdots \cup F_j,$$

where \tilde{Z}_i, \tilde{R}_i are the strict transforms of the corresponding subcurves in C_i , and F_1, \dots, F_j are the exceptional curves of the blow-up. Note that $l_i = j + k$. Lemma 2.13 implies

that $q(\tilde{Z}_i) \in C_{i+1}$ is a Gorenstein singularity with l_i branches and $\delta = l_i$. By Proposition A.3, there is a unique such singularity: the elliptic l_i -fold point.

- (vi) With notation as above, let $G_i \subset \tilde{R}_i$ be the unique irreducible component meeting \tilde{Z}_i for each $i = 1, \dots, k$. When \tilde{Z}_i is contracted, the minimal elliptic subcurve of C_{i+1} consists of the smooth rational components

$$q(G_1) \cup \dots \cup q(G_k) \cup q(F_1) \cup \dots \cup q(F_j),$$

meeting along an elliptic l_i -fold point, i.e. one component of each subcurve \tilde{R}_i is absorbed into the minimal elliptic subcurve. It is easy to see that the level l_{i+1} is just the number of distinguished points of $q(G_1), \dots, q(G_k), q(F_1), \dots, q(F_j)$ minus $j + k$. Indeed, each component $q(G_1), \dots, q(F_j)$ has a distinguished point where it meets the elliptic l_i -fold point and these do not contribute to l_{i+1} , while the remaining distinguished points are either disconnecting nodes or marked points and these each contribute one to l_{i+1} . Since q maps distinguished points of $G_1, \dots, G_k, F_1, \dots, F_j$ bijectively to distinguished points of $q(G_1), \dots, q(G_k), q(F_1), \dots, q(F_j)$, and since each $G_1, \dots, G_k, F_1, \dots, F_j$ has at least two distinguished points, we have $l_{i+1} \geq l_i$. Furthermore, equality holds if and only if each $G_1, \dots, G_k, F_1, \dots, F_j$ has exactly two distinguished points.

- (vii) In the previous paragraph, we saw that if

$$C_i = Z_i \cup R_1 \cup \dots \cup R_k,$$

then one irreducible component from each subcurve R_i is absorbed into the minimal elliptic subcurve $E_{i+1} \subset C_{i+1}$. It follows that the number of irreducible components of $C_{i+1} \setminus E_{i+1}$ is less than the number of irreducible components of $C_i \setminus E_i$. Thus, after finitely many steps, we have $C_t = E_t$, i.e. C_t has no disconnecting nodes.

Step 3. Stabilize to obtain an m -stable limit.

By condition (vii), C_t has no disconnecting nodes so $l_t = n$. Since $m < n$, we may set

$$e := \min\{j \mid l_j > m\}.$$

Let $\phi: C_e \rightarrow C$ be the ‘stabilization’ contraction uniquely determined by the properties that $\phi_* \mathcal{O}_{C_e} = \mathcal{O}_C$, and

$$\text{Exc}(\phi) = \left\{ \bigcup_{F \subset C_e} F \mid F \not\subseteq Z_e \text{ and } F \text{ has exactly two distinguished points} \right\}.$$

Since each component $F \subset C_e$ satisfying the above condition is a smooth rational curve meeting the rest of the special fiber in one or two nodes, and the total space C_e is regular around F , the existence of ϕ_i follows by standard results on the contractibility of rational cycles [Lip69, Theorem 27.1]. Furthermore, the images of the sections $\sigma_1, \dots, \sigma_n$ on C_e lie in the smooth locus of C , so we may consider the special fiber $(C, \{p_i\}_{i=1}^n)$ as an n -pointed curve. To show that $(C, \{p_i\}_{i=1}^n)$ is m -stable, we must verify conditions (1)–(3) of Definition 3.7.

- (1) C has only nodes and elliptic- l fold points, $l \leq m$, as singularities. By conditions (i) and (v) above, C_e has only nodes and an elliptic l_{e-1} -fold point as singularities, where $l_{e-1} < m$ by our choice of e . The same is true of C , since the only singularities produced by contracting semistable chains of rational curves are nodes.

- (2) C has level greater than m . The level of C_e is greater than m by our choice of e , so it suffices to see that the level of C is the same as the level of C_e . Let

$$C_e = Z_e \cup R_1 \cup \dots \cup R_k,$$

be the fundamental decomposition of C_e . Order the R_i so that R_1, \dots, R_j consist entirely of components with two distinguished points, while R_{j+1}, \dots, R_k each contain a component with at least three distinguished points. Then ϕ contracts each of R_1, \dots, R_j to a point, so that the fundamental decomposition of C is

$$C = \phi(Z_e) \cup \phi(R_{j+1}) \cup \dots \cup \phi(R_k).$$

Thus,

$$|\overline{C \setminus \phi(Z_e)}| = |\overline{C_e \setminus Z_e}| - j.$$

On the other hand, since each R_1, \dots, R_j must be a chain of \mathbb{P}^1 whose final component carries a marked point, $\phi(R_1), \dots, \phi(R_j)$ will be marked points on the minimal elliptic subcurve $\phi(Z_e)$, i.e. we have

$$|\{p_i \mid p_i \in \phi(Z_e)\}| = |\{p_i \mid p_i \in Z_e\}| + j.$$

Thus, $|\overline{C \setminus \phi(Z_e)}| + |\{p_i \mid p_i \in \phi(Z_e)\}| = |\overline{C_e \setminus Z_e}| + |\{p_i \mid p_i \in Z_e\}|$ as desired.

- (3) $(C, \{p_i\}_{i=1}^n)$ satisfies the stability condition. Since ϕ contracts every component of $R_1 \cup \dots \cup R_k$ with two distinguished points, every component of $\phi(R_1) \cup \dots \cup \phi(R_k)$ has at least three distinguished points. It remains to check the stability condition for irreducible components of $\phi(Z_e)$.

We may assume that $e \geq 1$, so Z_e consists of l_{e-1} smooth rational branches meeting in an elliptic l_{e-1} -fold point. Since no component of Z_e is contained in $\text{Exc}(\phi)$, Z_e maps isomorphically onto $\phi(Z_e)$ and condition (iv) implies that every component of $\phi(Z_e)$ has at least two distinguished points. Finally, if every component of $\phi(Z_e)$ had exactly two distinguished points, the same would be true of Z_e and condition (vi) would imply that $l_i = l_{i-1}$. This contradicts our choice of e ; we conclude that some component of $\phi(Z_e)$ has at least three distinguished points.

3.3.2 Uniqueness of m -stable limits. In order to prove that an isomorphism

$$(C, \sigma_1, \dots, \sigma_n)|_\eta \simeq (C', \sigma'_1, \dots, \sigma'_n)|_\eta$$

extends to an isomorphism over Δ , it suffices to check that the rational map $C \dashrightarrow C'$ extends to an isomorphism after a finite base change. Thus, we may assume that there exists a flat proper nodal curve $(C^{ss} \rightarrow \Delta, \tau_1, \dots, \tau_n)$ with regular total space and a diagram

$$\begin{array}{ccc} & (C^{ss}, \tau_1, \dots, \tau_n) & \\ \phi \swarrow & & \searrow \phi' \\ (C, \sigma_1, \dots, \sigma_n) & & (C', \sigma'_1, \dots, \sigma'_n) \end{array}$$

where ϕ and ϕ' are proper birational morphisms over Δ . In fact, we may further assume that $(C^{ss} \rightarrow \Delta, \tau_1, \dots, \tau_n)$ is Deligne–Mumford semistable. Indeed, any unmarked (-1) -curve in the special fiber C^{ss} must be contracted by both ϕ and ϕ' since neither C nor C' contain unmarked smooth rational components meeting the rest of the curve in a single point. Thus, ϕ and ϕ' both factor through the minimal model of C^{ss} , obtained by successively blowing down unmarked (-1) -curves.

The strategy of the proof is to show that $\text{Exc}(\phi) = \text{Exc}(\phi')$. Since \mathcal{C} and \mathcal{C}' are normal, this immediately implies $\mathcal{C} \simeq \mathcal{C}'$.

REDUCTION 1. *We may assume that C and C' each contain a non-nodal singular point.*

Proof. If C and C' are both nodal, then they are Deligne–Mumford stable, so $\mathcal{C} \simeq \mathcal{C}'$ by the usual stable reduction theorem. Next, suppose that C' is nodal, but that C contains an elliptic l -fold point p for some $l \leq m$. Set

$$E := \phi^{-1}(p) \subset C^{ss},$$

and note that $p_a(E) = 1$ and $|E \cap \overline{C^{ss} \setminus E}| = l \leq m$. It follows that $\phi'(E) \subset C'$ is an unmarked connected arithmetic genus one subcurve meeting $\overline{C' \setminus \phi'(E')}$ in no more than m points, which contradicts the m -stability of C' . Thus, we may assume that C and C' each have a non-nodal singular point. \square

Now we may assume that C has an elliptic j -fold point p , and C' has an elliptic k -fold p' , where $j, k \leq m$. We set

$$\begin{aligned} E &:= \phi^{-1}(p) \subset C^{ss}, \\ E' &:= \phi^{-1}(p') \subset C'^{ss}, \end{aligned}$$

and we claim that $E = E'$. By Proposition 2.12, $(E, \{q_i\}_{i=1}^j)$ and $(E', \{q'_i\}_{i=1}^k)$ are *balanced*, where

$$\begin{aligned} \{q_1, \dots, q_j\} &:= \{E \cap \overline{C^{ss} \setminus E}\}, \\ \{q'_1, \dots, q'_k\} &:= \{E' \cap \overline{C'^{ss} \setminus E'}\}. \end{aligned}$$

Let $Z \subset C^{ss}$ be the minimal elliptic subcurve of C^{ss} . By Corollary 3.2, we have $Z \subset E$ and $Z \subset E'$. Proposition 2.12 implies there exist integers l and l' such that

$$\begin{aligned} l &:= l(Z, q_1) = \dots = l(Z, q_j), \\ l' &:= l(Z, q'_1) = \dots = l(Z, q'_k). \end{aligned}$$

Put differently, this says that E comprises all components in C^{ss} whose length from Z is less than l , while E' comprises all irreducible components in C'^{ss} whose length from Z is less than l' . If $l = l'$, then we have $E = E'$ and we are done. Otherwise, we may assume that $l < l'$, and we have a strict containment $E \subset E'$. Then, however, since E' meets $\overline{C'^{ss} \setminus E'}$ in no more than m points, $\phi(E') \subset C$ is a connected arithmetic genus one subcurve meeting $\overline{C \setminus \phi(E')}$ in no more than m points. This contradicts the m -stability of C . Thus, we have $E = E'$.

Finally, we claim that E and E' determine $\text{Exc}(\phi)$ and $\text{Exc}(\phi')$ in the following sense:

$$\begin{aligned} \text{Exc}(\phi) &:= E \cup \{F \mid F \cap E = \emptyset \text{ and } F \text{ has two distinguished points}\}, \\ \text{Exc}(\phi') &:= E' \cup \{F \mid F \cap E' = \emptyset \text{ and } F \text{ has two distinguished points}\}. \end{aligned}$$

This will imply $\text{Exc}(\phi) = \text{Exc}(\phi')$ and hence $\mathcal{C} \simeq \mathcal{C}'$ as desired. Let us argue the first equality (the argument for the second is identical).

It is clear that no irreducible component of C^{ss} which meets E can be contracted by ϕ . Such a component would be contracted to the point p and hence contained in $E := \phi^{-1}(p)$. It remains to see that an irreducible component $F \subset C^{ss}$ with $F \cap E = \emptyset$ is contracted if and only if F has exactly two distinguished points. If F has at least three distinguished points, then it cannot be contracted without introducing a singular point with more than three branches, a section passing through a node, or two sections colliding, any one of which contradicts the m -stability of C .

On the other hand, if F has two distinguished points, then F must be contracted or else $\phi(F) \subset C$ is an irreducible component lying outside the minimal elliptic subcurve and containing only two distinguished points. This completes the proof of the uniqueness of m -stable limits.

3.4 Valuative criterion for $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$

In this section, we complete the proof of Theorem 3.11 by handling the case when $\mathcal{A} \neq (1, \dots, 1)$. The key idea, following Hassett [Has03, Proposition 3.7], is that we can construct the (m, \mathcal{A}) -stable limit from the m -stable limit by running a relative minimal model program with respect to $\omega_{\mathcal{C}/\Delta}(\sum_i a_i \sigma_i)$.

3.4.1 Existence of limits. Given a family of smooth n -pointed curves over the generic point of the spectrum of a discrete valuation ring Δ , we may (after a finite base change) complete this family to an m -stable curve $(\pi : \mathcal{C} \rightarrow \Delta, \sigma_1, \dots, \sigma_n)$. To obtain the (m, \mathcal{A}) -stable limit, we construct a sequence of birational contractions

$$\mathcal{C} := \mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots \rightarrow \mathcal{C}_N,$$

where each special fiber C_i satisfies conditions (1)–(4) of Definition 3.7, and such that $\omega_{\mathcal{C}_N/\Delta}(\sum_i a_i \sigma_i)$ is relatively ample. Thus, $\mathcal{C}_N \rightarrow \Delta$ is the desired (m, \mathcal{A}) -stable limit.

To construct this sequence, we proceed by induction on i . If $\omega_{\mathcal{C}_i/\Delta}(\sum_i a_i \sigma_i)$ is ample, we are done. If not, then $\omega_{\mathcal{C}_i/\Delta}(\sum_i a_i \sigma_i)$ has non-positive degree on some component of the special fiber, and we claim that this component must be a smooth rational curve meeting the rest of the fiber in a single node. To see this, note that condition (3) of Definition 3.7 implies that every component $F \subset C_i$ satisfies one of the following.

- (I) The component F has arithmetic genus one and at least one distinguished point.
- (II) The component F is a smooth rational component meeting an elliptic l -fold point and has at least one additional distinguished point.
- (III) The component F is a smooth rational component meeting the rest of the fiber in at least two nodes and has at least one additional distinguished point.
- (IV) The component F is a smooth rational component meeting the rest of the fiber in one node.

On components of types (I)–(III), the restriction of the dualizing sheaf $\omega_{\mathcal{C}/\Delta}|_F$ has non-negative degree. Since the weights $\{a_i\}$ are positive, each distinguished point contributes a positive amount to the degree, and we conclude that $\omega_{\mathcal{C}_i/\Delta}(\sum_i a_i \sigma_i)$ has positive degree on all such components. Thus, if $\omega_{\mathcal{C}_i/\Delta}(\sum_i a_i \sigma_i)$ fails to be ample, it has non-positive degree on a component of type (IV). If $F \subset C_i$ is such a component, standard results on the contractibility of rational cycles imply the existence of a projective birational contraction $\phi : \mathcal{C}_i \rightarrow \mathcal{C}_{i+1}$ contracting F to a smooth point [Lip69, Theorem 27.1]. We leave it to the reader to check that \mathcal{C}_{i+1} still satisfies conditions (1)–(4) of Definition 3.7.

Since there are only finitely many components in the special fiber of \mathcal{C} , and the total degree of $\omega_{\mathcal{C}/\Delta}(\sum_i a_i \sigma_i)$ is positive, we must achieve ampleness of $\omega_{\mathcal{C}_i/\Delta}(\sum_i a_i \sigma_i)$ after finitely many repetitions of this procedure.

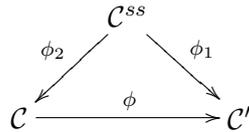
3.4.2 Uniqueness of limits. To prove uniqueness of (m, \mathcal{A}) -stable limits, it suffices (by uniqueness of m -stable limits) to show the following: given an (m, \mathcal{A}) -stable curve $(\pi' : \mathcal{C}' \rightarrow \Delta, \sigma_1, \dots, \sigma_n)$ with a smooth generic fiber, there exists an m -stable curve $(\pi : \mathcal{C} \rightarrow \Delta, \sigma_1, \dots, \sigma_n)$

and a birational morphism $\mathcal{C} \rightarrow \mathcal{C}'$ such that

$$\mathcal{C}' = \text{Proj} \bigoplus_{m \geq 0} \pi_*(\omega_{\mathcal{C}/\Delta}(\sum_i a_i \sigma_i)^m),$$

where the sum is taken over m sufficiently divisible so that $\omega_{\mathcal{C}/\Delta}(\sum_i a_i \sigma_i)^m$ is integral.

To obtain $\mathcal{C} \rightarrow \mathcal{C}'$, simply apply stable reduction locally around the points of \mathcal{C}' where marked points coincide. This gives a diagram of birational morphisms



satisfying the following.

- (1) The morphism ϕ_1 is a composition of blow-ups along smooth points of the special fiber.
- (2) The image $\phi_1(\text{Exc}(\phi_1)) \in \mathcal{C}'$ is the locus where two or more marked points coincide.
- (3) The morphism ϕ_2 is the contraction of all unmarked (-2) -curves in $\text{Exc}(\phi_1)$.
- (4) The strict transforms of $\sigma_1, \dots, \sigma_n$ on \mathcal{C} are disjoint.
- (5) The line bundle $\omega_{\mathcal{C}/\Delta}(\sum_i \sigma_i)$ is ϕ -ample.

We claim that $(\mathcal{C} \rightarrow \Delta, \sigma_1, \dots, \sigma_n)$ is an m -stable curve. By construction, the sections $\sigma_1, \dots, \sigma_n$ are distinct, and $\omega_{\mathcal{C}/\Delta}(\sum_i \sigma_i)$ is relatively ample, so it suffices to check conditions (1)–(3) of Definition 3.7. For condition (1), since \mathcal{C}' has only nodes and elliptic l -fold points, the same is true of \mathcal{C} . For condition (2), we will show that the level of \mathcal{C} is the same as the level of \mathcal{C}' . To see this, let $Z \subset \mathcal{C}$ be the minimal elliptic subcurve of \mathcal{C} , and we write the fundamental decomposition

$$\mathcal{C} = Z \cup R_1 \cup \dots \cup R_k.$$

We may order the R_i so that R_1, \dots, R_j are contracted to a point by ϕ , while R_{j+1}, \dots, R_k are not. Then

$$\mathcal{C}' = \phi(Z) \cup \phi(R_{j+1}) \cup \dots \cup \phi(R_k)$$

is the fundamental decomposition of \mathcal{C}' , so we have

$$|\phi(Z) \cap \overline{\mathcal{C}' \setminus \phi(Z)}| = |Z \cap \overline{\mathcal{C} \setminus Z}| - j.$$

On the other hand, since each rational chain R_i must support at least one marked point, the points $\phi(R_1), \dots, \phi(R_j)$ are now marked distinguished points on $\phi(Z)$. Thus,

$$|\phi(Z) \cap \Sigma'| = |Z \cap \Sigma| + j.$$

In sum, we get,

$$|\phi(Z) \cap \overline{\mathcal{C}' \setminus \phi(Z)}| + |\Sigma' \cap \phi(Z)| = |Z \cap \overline{\mathcal{C} \setminus Z}| + |\Sigma \cap Z|,$$

as desired. Finally, condition (3) is immediate from the fact that each irreducible component of $\text{Exc}(\phi)$ has at least three distinguished points.

To see that $\mathcal{C}' = \text{Proj} \bigoplus_{m \geq 0} \pi_*(\omega_{\mathcal{C}/\Delta}(\sum_i a_i \sigma_i)^m)$, we only need to check that

$$\begin{aligned}
 \omega_{\mathcal{C}^{ss}/\Delta}(\sum_i a_i \sigma_i) - \phi_1^* \omega_{\mathcal{C}'/\Delta}(\sum_i a_i \sigma_i) &\geq 0, \\
 \omega_{\mathcal{C}^{ss}/\Delta}(\sum_i a_i \sigma_i) - \phi_2^* \omega_{\mathcal{C}/\Delta}(\sum_i a_i \sigma_i) &\geq 0.
 \end{aligned}$$

Indeed, this implies that

$$\pi'_*(\omega_{\mathcal{C}'/\Delta}(\sum_i a_i \sigma_i)^m) = \pi_*^{ss}(\omega_{\mathcal{C}^{ss}/\Delta}(\sum_i a_i \sigma_i)^m) = \pi_*(\omega_{\mathcal{C}/\Delta}(\sum_i a_i \sigma_i)^m)$$

for all $m \gg 0$ sufficiently divisible. Since $\omega_{C'}(\sum_i a_i \sigma_i)$ is an ample \mathbb{Q} -divisor, this gives

$$\begin{aligned} C' &= \text{Proj} \bigoplus_{m \geq 0} \pi'_* (\omega_{C'/\Delta}(\sum_i a_i \sigma_i)^m) \\ &= \text{Proj} \bigoplus_{m \geq 0} \pi_*^{ss} (\omega_{C^{ss}/\Delta}(\sum_i a_i \sigma_i)^m) \\ &= \text{Proj} \bigoplus_{m \geq 0} \pi_* (\omega_{C/\Delta}(\sum_i a_i \sigma_i)^m). \end{aligned}$$

Since $p_{i_1}, \dots, p_{i_k} \in C'$ coincide only if $\sum_{j=1}^k a_{i_j} \leq 1$, ϕ_1 is composed of blow-ups at smooth points where the total multiplicity of $\sum_i a_i \sigma_i$ is less than or equal to one, which gives

$$\omega_{C^{ss}/\Delta}(\sum_i a_i \sigma_i) - (\phi_1)^* \omega_{C'/\Delta}(\sum_i a_i \sigma_i) \geq 0.$$

On the other hand, since ϕ_2 is simply a contraction of unmarked (-2) -curves, we have

$$\omega_{C^{ss}/\Delta}(\sum_i a_i \sigma_i) - (\phi_2)^* \omega_{C/\Delta}(\sum_i a_i \sigma_i) = 0.$$

This completes the proof of Theorem 3.8, and hence of the main result, Theorem 3.11.

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Appendix A. Gorenstein curve singularities of genus one

Let C be a curve over an algebraically closed field k , $p \in C$ a singular point, and $\pi : \tilde{C} \rightarrow C$ be the normalization of C at p . We have the following basic numerical invariants.

DEFINITION A.1. Define

$$\begin{aligned} \delta(p) &:= \dim_k \pi_* \mathcal{O}_{\tilde{C},p} / \mathcal{O}_{C,p}, \\ m(p) &:= |\pi^{-1}(p)|, \\ g(p) &:= \delta(p) - m(p) + 1. \end{aligned}$$

We call $g(p)$ the *genus* of the singularity. Note that if C is complete and has arithmetic genus g , then $g(p) \leq g$. The purpose of this appendix is to classify (up to analytic isomorphism) Gorenstein singularities of genus zero and one. The main results are as follows.

PROPOSITION A.2. *If $p \in C$ has m branches and genus zero, then*

$$\hat{\mathcal{O}}_{C,p} \simeq k[[x_1, \dots, x_m]]/I,$$

where

$$I := (x_i x_j : 1 \leq i < j \leq m).$$

Furthermore, p is Gorenstein if and only if $m = 2$ (i.e. when p is an ordinary node).

PROPOSITION A.3. *If $p \in C$ is Gorenstein with m branches and genus one, then p is an elliptic m -fold point, i.e.*

$$\hat{\mathcal{O}}_{C,p} \simeq \begin{cases} k[[x, y]]/(y^2 - x^3) & m = 1, \\ k[[x, y]]/y(y - x^2) & m = 2, \\ k[[x, y]]/xy(y - x) & m = 3, \\ k[[x_1, \dots, x_{m-1}]]/I_m & m \geq 4, \end{cases}$$

where I_m is the ideal generated by all quadrics of the form

$$x_h(x_i - x_j) \quad \text{with } i, j, h \in \{1, \dots, m - 1\} \text{ distinct.}$$

Remark. There are many non-isomorphic non-Gorenstein singularities of genus one with fixed number branches. Furthermore, in higher genera, there are many non-isomorphic Gorenstein singularities with fixed number branches.

Combining these two propositions, we conclude the following corollary.

COROLLARY A.4. *If C is a Gorenstein curve with $p_a(C) = 1$, and $p \in C$ is a singular point, then p is either an ordinary node or an elliptic m -fold point for some integer m .*

In order to prove the propositions, it will be useful to switch to ring-theoretic notation. Set

$$\begin{aligned} R &:= \hat{\mathcal{O}}_{C,p}, \\ \tilde{R} &:= \widetilde{R/P_1} \oplus \dots \oplus \widetilde{R/P_{k(p)}}, \end{aligned}$$

where P_1, \dots, P_m are the minimal primes of R , and $\widetilde{R/P_i}$ denotes the integral closure of R/P_i . Note that

$$\tilde{R} \simeq k[[t_1]] \oplus \dots \oplus k[[t_m]],$$

since each $\widetilde{R/P_i}$ is a complete, regular local ring of dimension one over k . Let m_R be the maximal ideal of R , and let $m_{\tilde{R}}$ be the ideal $(t_1) \oplus \dots \oplus (t_m)$. Since R is reduced, we have an embedding

$$\begin{aligned} R &\hookrightarrow \tilde{R}, \\ m_R &= (m_{\tilde{R}} \cap R). \end{aligned}$$

In these terms, the *conductor ideal* of the singularity is given by

$$I_p := \text{Ann}_R(\tilde{R}/R),$$

and R is *Gorenstein* if and only if [AK70, Proposition 1.16(iv)]

$$\dim_k(R/I_p) = \dim_k(\tilde{R}/R).$$

Note that the R -module \tilde{R}/R has a natural grading given by powers of $m_{\tilde{R}}$; we define

$$(\tilde{R}/R)^i = m_{\tilde{R}}^i / ((m_{\tilde{R}}^i \cap R) + m_{\tilde{R}}^{i+1}).$$

Now we have the following trivial observations:

- (1) $\delta(p) = \sum_{i \geq 0} \dim_k(\tilde{R}/R)^i$;
- (2) $g(p) = \sum_{i \geq 1} \dim_k(\tilde{R}/R)^i$;
- (3) $(\tilde{R}/R)^i = (\tilde{R}/R)^j = 0 \implies (\tilde{R}/R)^{i+j} = 0$ for any $i, j \geq 1$.

Having dispensed with these preliminaries, the proofs of Propositions A.2 and A.3 are straightforward, albeit somewhat tedious. The basic idea is to find a basis for m_R/m_R^2 in terms of the local coordinates t_1, \dots, t_m .

Proof of Proposition A.2. If $g(p) = 0$, then $(\tilde{R}/R)^i = 0$ for all $i > 0$, so $m_R = m_{\tilde{R}}$. Thus, we may define a local homomorphism of complete local rings

$$\begin{aligned} k[[x_1, \dots, x_m]] &\rightarrow R \subset k[[t_1]] \oplus \dots \oplus k[[t_m]] \\ x_i &\rightarrow (0, \dots, 0, t_i, 0, \dots, 0). \end{aligned}$$

This homomorphism is surjective since it is surjective on tangent spaces, and the kernel is precisely the ideal

$$I_m = (x_i x_j, i < j).$$

To see that R is Gorenstein if and only if $m = 2$, note that the conductor ideal is

$$I_p = m_R.$$

Thus, the Gorenstein condition

$$\dim_k(\tilde{R}/R) = \dim_k(R/I_p)$$

is satisfied if and only if $\dim_k(\tilde{R}/R) = 1$, i.e. when $m = 2$. □

Proof of Proposition A.3. Since $g(p) = 1$, observations (2) and (3) imply that

$$\begin{aligned} \dim_k(\tilde{R}/R)^1 &= 1, \\ \dim_k(\tilde{R}/R)^i &= 0 \quad \text{for all } i \geq 1. \end{aligned}$$

Put differently, this says that

$$m_R \supset m_{\tilde{R}}^2,$$

while

$$m_R/m_R^2 \subset m_{\tilde{R}}/m_{\tilde{R}}^2$$

is a codimension-one subspace. By Gaussian elimination, we may choose elements $f_1, \dots, f_{m-1} \in m_R$ such that

$$\begin{pmatrix} f_1 \\ \vdots \\ \vdots \\ f_{m-1} \end{pmatrix} \equiv \begin{pmatrix} t_1 & 0 & \dots & 0 & a_1 t_{m-1} \\ 0 & t_2 & \ddots & \vdots & a_2 t_{m-1} \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & t_{m-2} & a_{m-1} t_{m-1} \end{pmatrix} \pmod{m_{\tilde{R}}^2}$$

for some $a_1, \dots, a_{m-1} \in k$.

CLAIM. *If R is Gorenstein, we may take $a_1, \dots, a_{m-1} = 1$.*

Proof of Claim. First, let us show that R Gorenstein implies $I_p = m_{\tilde{R}}^2$. Since $m_R \supset m_{\tilde{R}}^2$, we certainly have $I_p \supset m_{\tilde{R}}^2$. Thus,

$$\dim(R/I_p) \leq \dim(R/m_{\tilde{R}}^2) = m.$$

On the other hand, we have $\dim(\tilde{R}/R) = m$, so the Gorenstein equality $\dim(R/I_p) = \dim(\tilde{R}/R)$ implies $\dim(R/I_p) = \dim R/m_{\tilde{R}}^2$, i.e. $I_p = m_{\tilde{R}}^2$.

In particular, we have $f_1, \dots, f_{m-1} \notin I_p$. Now if $a_i = 0$ then

$$f_i g \in (f_i) + m_{\tilde{R}}^2 \subset R \quad \text{for all } g \in \tilde{R},$$

i.e. $f \in I_p$. We conclude that $a_i \in k^*$ for each $i = 1, \dots, m$. Making a change of coordinates $t'_i = a_i t_i$, we may assume that each $a_i = 1$. □

At this point, the proof breaks into three cases.

- (I) ($m \geq 3$). We claim that f_1, \dots, f_{m-1} give a basis for m_R/m_R^2 . Clearly, it is enough to show that $m_R^2 = m_{\tilde{R}}^2$. Since $m_R^2 \supset m_{\tilde{R}}^4$, it is enough to show that

$$m_R^2/m_{\tilde{R}}^4 \hookrightarrow m_{\tilde{R}}^2/m_{\tilde{R}}^4$$

is surjective. Using the matrix expressions for the $\{f_i\}$, one easily verifies that $f_1^2, \dots, f_{m-1}^2, f_1 f_2$ map to a basis of $m_{\tilde{R}}^2/m_{\tilde{R}}^3$, and $f_1^3, \dots, f_{m-1}^3, f_1^2 f_2$ map to a basis of $m_{\tilde{R}}^3/m_{\tilde{R}}^4$.

Since f_1, \dots, f_{m-1} give a basis of m_R/m_R^2 , we have a surjective homomorphism

$$\begin{aligned} k[[x_1, \dots, x_{m-1}]] &\rightarrow R \subset k[[t_1]] \oplus \dots \oplus k[[t_m]] \\ x_i &\rightarrow (0, \dots, 0, t_i, 0, \dots, 0, t_{m-1}), \end{aligned}$$

and the kernel is precisely $I = (x_h(x_i - x_j))$ with $i, j, h \in \{1, \dots, m-1\}$ distinct).

- (II) ($m = 2$). By the preceding analysis, there exists $f_1 \in m_R$ such that

$$f_1 \equiv (t_1 \ t_2) \pmod{m_{\tilde{R}}^2}.$$

Since $m_R \supset m_{\tilde{R}}^2$, we may choose $f_2 \in m_R$ such that f_1^2, f_2 map to a basis of $m_{\tilde{R}}^2/m_{\tilde{R}}^3$. After Gaussian elimination, we may assume that

$$\begin{pmatrix} f_1^2 \\ f_2 \end{pmatrix} \equiv \begin{pmatrix} t_1^2 & t_2^2 \\ t_1^2 & 0 \end{pmatrix} \pmod{m_{\tilde{R}}^3}.$$

We claim that f_1 and f_2 form a basis for m_R/m_R^2 . Since f_1, f_2, f_1^2 form a basis for $m_R/m_{\tilde{R}}^3$, it suffices to show that $m_R^2 \cap m_{\tilde{R}}^3 = m_{\tilde{R}}^3$. Since $m_R^2 \supset m_{\tilde{R}}^4$, it is enough to show that

$$(m_R^2 \cap m_{\tilde{R}}^3)/m_{\tilde{R}}^4 \hookrightarrow m_{\tilde{R}}^3/m_{\tilde{R}}^4$$

is surjective. From the matrix expression for the $\{f_i\}$, one easily sees that f_1^3 and $f_1 f_2$ give a basis of $m_{\tilde{R}}^3/m_{\tilde{R}}^4$.

Since f_1, f_2 give a basis of m_R/m_R^2 , we have a surjective homomorphism of complete local rings

$$\begin{aligned} k[[x, y]] &\rightarrow R \subset k[[t_1]] \oplus k[[t_2]] \\ x &\rightarrow (t_1, t_2), \\ y &\rightarrow (t_1^2, 0), \end{aligned}$$

with kernel $y(y - x^2)$.

- (III) ($m = 1$). Since $m_R/m_R^2 \subset m_{\tilde{R}}/m_{\tilde{R}}^2$ is codimension-one, we have $m_R = m_{\tilde{R}}^2$. Thus, we may pick $f_1, f_2 \in m_R$ so that

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \equiv \begin{pmatrix} t_1^2 \\ t_1^3 \end{pmatrix} \pmod{m_{\tilde{R}}^4}.$$

Since $m_R^2 = m_R^4$, f_1 and f_2 give a basis for m_R/m_R^2 . Thus, the homomorphism

$$\begin{aligned} k[[x, y]] &\rightarrow R \subset k[[t_1]], \\ x &\rightarrow (t_1^2), \\ y &\rightarrow (t_1^3), \end{aligned}$$

is surjective, with kernel $y^2 - x^3$. □

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