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MODULAR GROUP ACTING ON REAL QUADRATIC FIELDS

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Coset diagrams for the orbit of the modular group $G = \langle x, y \colon x^2 = y^3 = 1\rangle$ acting on real quadratic fields give some interesting information. By using these coset diagrams, we show that for a fixed value of n, a non-square positive integer, there are only a finite number of real quadratic irrational numbers of the form $\theta = \left(a + \sqrt{n}\right)/c$, where θ and its algebraic conjugate $\left(a - \sqrt{n}\right)/c$ have different signs, and that part of the coset diagram containing such numbers forms a single circuit (closed path) and it is the only circuit in the orbit of θ.

We adopt a standard group theoretical notation as used in [1] and [4]. We shall denote by G the group with presentation $\langle x, y \colon x^2 = y^3 = 1\rangle$. It is well-known that this is isomorphic to the modular group, $PSL(2, Z)$ in which the generators x and y can be taken to be the maps $z \to -1/z$ and $z \to (z-1)/z$. (A proof of this, using coset diagrams, is given in [2].)

The natural action of G on real quadratic fields gives some interesting information. We have used coset diagrams, as defined in [3], to study this action.

A coset diagram depicts a permutation representation of the modular group: the 3-cycles of the transformation y are denoted by three vertices of a triangle permuted anti-clockwise by y and two vertices which are interchanged by x are joined by an edge. Fixed points of x and y are denoted by heavy dots.

For instance, in the case of $PSL(2, 13)$, where we can take x as the transformation $z \to -1/z$ and y as the transformation $z \to (z-1)/z$, the diagram is given by Figure 1 on the next page. We have labelled each vertex to give a fuller illustration.

Let α denote a real quadratic irrational number $(a + \sqrt{n})/c$, where n is a non-square positive integer and a, $\left(a^2 - n\right)/c$ and c are relatively prime integers. We denote the algebraic conjugate $(a - \sqrt{n})/c$ of α by $\bar{\alpha}$.

Consider the coset diagram for the natural action of $PSL(2, Z)$ on any subset of the real projective line. If $k \neq 0$ and ∞ is a vertex of a triangle in the coset diagram,

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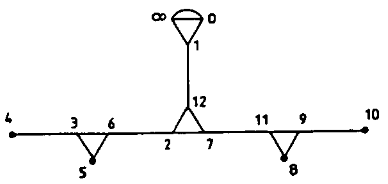

Fig. 1

then k and kx are endpoints of an edge. Since $kx = -1/k$, one of k and kx is positive and one is negative. We will use an arrow-head on an edge to indicate its direction from a negative to a positive vertex. Since $y: z \to (z-1)/z$ if:

(i) $z < 0$ then $zy > 1$

(ii) $0 < z < 1$ then $zy < 0$, and

(iii) $1 < z$ then $0 < zy < 1$.

That is, if vertices k, ky, ky^2 of a triangle are not 1, 0, ∞ then just one lies in each of the intervals: $z < 0$, $0 < z < 1$ and $1 < z$. Thus, in particular, of the vertices k, ky, ky^2, one is negative and two are positive. In the case of the modular group acting on real quadratic irrational number fields, there are two possible interpretations of this. For some fixed non-square positive integer n, an element $\alpha = (a+\sqrt{n})/c$ and its conjugate $\bar{\alpha} = (a-\sqrt{n})/c$ may have different signs. If such is the case then we shall call such an α an *ambiguous number.* If α and $\bar{\alpha}$ are both negative (positive), then we shall call α a *totally negative (positive)* number.

We will show that for a fixed value of n there are only a finite number of real quadratic irrational ambiguous numbers of the form $\alpha = (a+\sqrt{n})/c$ and that part of the coset diagram containing ambiguous numbers forms a single closed path and it is the only closed path in the orbit of α.

We begin with the following well-known result with a modified proof.

LEMMA 1. *Every real quadratic irrational number can be written uniquely as* $(a+\sqrt{n})/c$, *where* n *is a non-square positive integer and* a, $(a^2-n)/c$ *and* c *are relatively prime integers.*

PROOF: It is well-known that any real quadratic irrational number α can be written as $(a+\sqrt{n})/c$, where n is a non-square positive integer and a and c are integers not necessarily positive. If $(a^2-n)/c$ is not an integer then we write $\alpha = \left(ka+\sqrt{kn}\right)/kc = \left(a'+\sqrt{n'}\right)/c'$ for some positive integer k. This means that $\left(a'^2-n'\right)/c' = \left(k^2a^2-k^2n\right)/kc = k\left(a^2-n\right)/c$. Thus replacing a, n and c by a', n' and c' for some suitable k we can suppose that $\left(a^2-n\right)/c = b$ is an integer. If

a, b and c have common factor r, $(r > 1)$, then $n = a^2 - bc$ and r^2 divides n. So $a = a'r$, $c = c'r$, $n = n'r^2$ and $\alpha = \left(a' + \sqrt{n'}\right)/c'$. Here a', $b' = \left(a'^2 - n\right)/c'$ and c' are relatively prime integers, so that we can write α in the desired form. Also, if $\alpha = \left(a'' + \sqrt{n''}\right)/c''$, then $a'' = as$, $c'' = cs$ and $n'' = ns^2$ for some positive rational number s and so $b'' = \left(a''^2 - n''\right)/c'' = bs$.

Since a, b and c have no common factor, a'', b'' and c'' are not all integers unless s is an integer and they have no common factor only if $s = 1$. So every real quadratic irrational number is uniquely expressible as $(a + \sqrt{n})/c$, where n is a non-square positive integer and a, $\left(a^2 - n\right)/c$, c are relatively prime integers.

Henceforth, by α we shall mean a real quadratic irrational number $(a + \sqrt{n})/c$ where n is a non-square positive integer and a, $\left(a^2 - n\right)/c$, c are relatively prime integers. By αG, we shall mean the orbit of α under the modular group G. ∎

We need the following lemma:

LEMMA 2. *For every real quadratic irrational number in αG the non-square positive integer n has the same value.*

PROOF: Let $\alpha = (a + \sqrt{n})/c$, where a, $a^2 - n$ and c are relatively prime and n is a non-square positive integer. Since $\alpha x = -1/\alpha = -c/(a + \sqrt{n}) = -c(a - \sqrt{n})/\left(a^2 - n\right) = (-a + \sqrt{n})/b$, we can replace a, b and c by $-a$, c and b respectively and n remains the same. Similarly, $\alpha y = 1 - c/(a + \sqrt{n}) = (b - a + \sqrt{n})/b$. Also, $(b - a)^2 - n = a^2 + b^2 - 2ab - n = bc + b^2 - 2ab$ implies that the new a, b and c are respectively $b - a$, $b - 2a + c$ and b. As $b - a$, $b - 2a + c$ and b are relatively prime integers, it shows that every element of the orbit αG has the same value of the non-square positive integer n. ∎

THEOREM 3. *For a fixed value of n there is only a finite number of ambiguous numbers and, in particular, in αG there is only a finite number.*

PROOF: Suppose β is an ambiguous number $(a + \sqrt{n})/c$ such that n is a fixed non-square positive integer, and a, $\left(a^2 - n\right)/c$ and c are relatively prime integers. If both a and c are positive then $\beta > 0$ and $\bar{\beta} < 0$ together imply that $a^2 < n$ and so there is only a finite number of possibilities for a. Moreover, if a and n are fixed then $bc = a^2 - n$ implies that there is only a finite number of choices for b and c. The cases where a or c or both are negative follow similarly. Thus, for a fixed non-square positive integer n there is only a finite number of choices for a, b and c. Hence, for this fixed number n, there is a finite number of ambiguous numbers. ∎

If α is an ambiguous number, then, by Lemma 1 and Theorem 3, every element of αG is of the form $(a + \sqrt{n})/c$, where n is a fixed non-square positive integer. So, as a particular case of the preceding argument, αG will contain a finite number of ambiguous numbers.

THEOREM 4. *In a coset diagram for αG the ambiguous numbers form a set of closed paths.*

PROOF: Let p, q and r be the vertices of a triangle in a coset diagram for αG. Let p be an ambiguous number such that p is positive and $\bar{p}$ is negative. Then because of the argument preceding Lemma 1, each of the sets $\{p,q,r\}$ and $\{\bar{p},\bar{q},\bar{r}\}$ contains just one negative number. So the only possibilities for the signs of these numbers are:

p	q	r	$\bar{p}$	$\bar{q}$	$\bar{r}$
+	+	−	−	+	+
+	−	+	−	+	+

Thus in a case like this, the triangle will have one vertex positive and the other two vertices ambiguous.

It is important to note that since p is an ambiguous number, px is also ambiguous and so the adjacent triangle, one of whose vertices is px, will also have one positive vertex and two ambiguous vertices. Since by Theorem 3 there are only a finite number of ambiguous numbers in αG, there will be only a finite number of such triangles in a coset diagram for αG. As the ambiguous numbers are mapped onto ambiguous numbers by the transformation x, the triangles with one positive vertex and two ambiguous vertices will form a closed path because these ambiguous vertices are joined together by an edge corresponding to the transformation x. Thus, the ambiguous numbers form closed paths such as in a coset diagram for αG. In Figure 2 below, a denotes an ambiguous number.

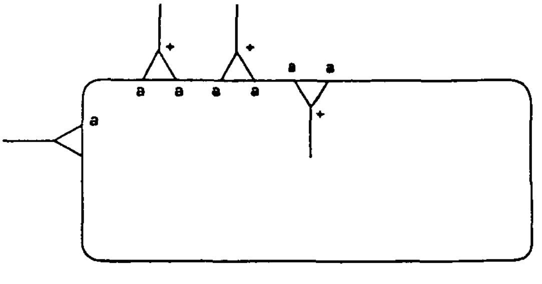

Fig. 2

As an illustration, consider the following closed path in which $n = 1892$. The transformation

$$g = (xy)^2(xy^2)^2(xy)(xy^2)^2(xy)^3(xy^2)^2,$$

corresponding to the closed path given below, fixes k and so gives the quadratic equation $62k^2 + 120k - 64 = 0$. The zeros, $(-30 \pm \sqrt{1892})/31$, of this equation are fixed points of the transformation g.

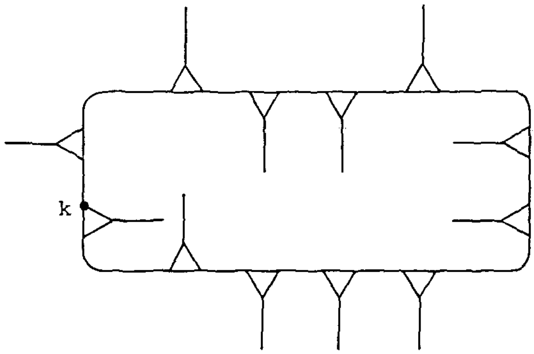

Fig. 3

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THEOREM 5. *If k is a totally negative real quadratic irrational number, then there is a unique sequence $k = k_0, k_1, \ldots, k_m$ such that k_i is totally negative for $i = 0, 1, \ldots, (m-1)$ and k_m is ambiguous and k_i, k_{i+1} are in adjacent triangles in the coset diagram for αG.*

PROOF: Let one of the vertices of a triangle in the coset diagram given below

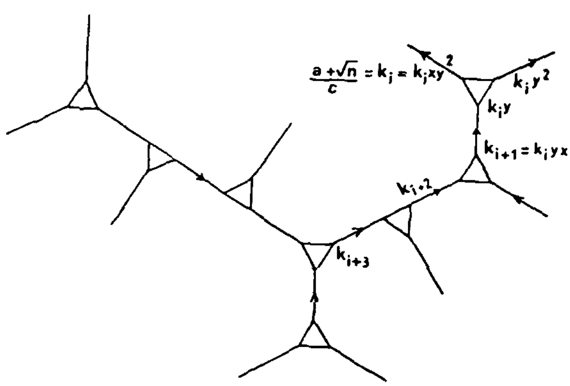

Fig. 4

be a totally negative number $(a + \sqrt{n})/c$. Let $c > 0$ and $a < -\sqrt{n}$. Since $(a + \sqrt{n})/c$ and $(a - \sqrt{n})/c$ are both negative, $c > 0$ and $a < -\sqrt{n}$ together imply that $a < 0$. Moreover, $a^2 - n = bc$, $a^2 > n$ and $c > 0$ implies that $b > 0$.

Consider Figure 4 and let $k_i = (a+\sqrt{n})/c$. Then by the argument preceding Lemma 1, k_iy and k_iy^2 are positive. Let $k_{i+1} = k_iyx$ or k_iy^2x. In this case then, either k_{i+1} is $(a-b+\sqrt{n})/(c-2a+b)$ or $(a-c+\sqrt{n})/c$. Moreover, since k_iy and k_iy^2 are positive, therefore k_{i+1} is negative. That is $(a-b+\sqrt{n})/(c-2a+b) < 0$ or $(a-c+\sqrt{n})/c < 0$ and so because $c > 0$, $a < 0$ and $b > 0$ we have either $c-2a+b > c$ or $|a-c| > |a|$. If we start from the vertex k_{i+1} we get the totally negative vertex k_{i+2}, and so, if we let $k_i = k_0$ and continue in this way, we get a sequence of such steps and consequently a sequence $k_0, k_1, k_2, \ldots$ of totally negative numbers.

The arrangement of arrows depicts that each step is uniquely determined and hence if we start from a totally negative vertex, we will get the unique sequence $k_0, k_1, k_2, \ldots$ of totally negative numbers. Let us now define $\|k\| = \max(|a|, |c|)$. Then $(c-2a+b) > c$ or $|a-c| > |a|$, implies that we cannot continue the sequence $\|k_0\| < \|k_1\| < \|k_2\| < \ldots$ indefinitely but come to an end when we reach a triangle with an ambiguous vertex. Thus, the sequence $k_0, k_1, k_2, \ldots$ of totally negative numbers terminates after a finite number of steps, say $i = m-1$, and the terminal point, k_m, must be an ambiguous number. ∎

THEOREM 6. *The ambiguous numbers in the coset diagram for the orbit αG form a single closed path and it is the only closed path contained in it.*

PROOF: If k is an ambiguous vertex of a triangle in the coset diagram for αG, then kx is ambiguous and so is one but not both of ky and ky^2. That is, each ambiguous vertex is joined by an x-edge or a y-edge to just two other ambiguous vertices. Hence the ambiguous vertices form a path in the diagram; and this path is closed, by Theorem 4. Of course, the edges in this closed path are alternately x-edges and y-edges, so that the closed path contains an even number of vertices.

If we follow the path from a positive vertex, then by Theorem 5, we do not come across any ambiguous number and so, any closed path in the coset diagram for that particular orbit. So the ambiguous numbers in the coset diagram for the orbit αG form a single closed path and it is the only closed path in the diagram. ∎

We conclude with the following observations.

If we are given a real quadratic irrational number $\alpha = (a+\sqrt{n})/c$, we can find the closed path in the orbit αG. If α is totally positive then $\bar{\alpha}$ is totally negative and so we can use Theorem 5 to find an ambiguous number in the same orbit. When we have an ambiguous number, the proof of Theorem 6 shows how to construct the closed path. This means that if α and β are two real quadratic irrational numbers then we can test whether or not they belong to the same orbit. We can find closed paths in the orbits αG and βG and see if they are the same or not.

For some values of n, there is more than one orbit containing numbers of discriminant n. For instance, the closed paths associated with the transformations:

$$(xy)^2(xy^2)^3xyxy^2 \text{ and } (xy)^{18}xy^2$$

contain numbers of discriminant 99.

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