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A NOTE ON ASYMPTOTIC NONBASES

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Abstract

Let A be a subset of $\mathbb{N}$, the set of all nonnegative integers. For an integer $h \geq 2$, let hA be the set of all sums of h elements of A. The set A is called an asymptotic basis of order h if hA contains all sufficiently large integers. Otherwise, A is called an asymptotic nonbasis of order h. An asymptotic nonbasis A of order h is called a maximal asymptotic nonbasis of order h if $A \cup \{a\}$ is an asymptotic basis of order h for every $a \notin A$. In this paper, we construct a sequence of asymptotic nonbases of order h for each $h \geq 2$, each of which is not a subset of a maximal asymptotic nonbasis of order h.

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1. Introduction

Let $\mathbb{N}$ denote the set of all nonnegative integers and let A be a subset of $\mathbb{N}$. For an integer $h \geq 2$, let hA denote the set of all sums of h elements of A. The set A is called an asymptotic basis of order h if hA contains all sufficiently large integers. Otherwise, A is called an asymptotic nonbasis of order h. An asymptotic basis A of order h is *minimal* if $A \setminus \{a\}$ is not an asymptotic basis of order h for every $a \in A$. Dual to the idea of a minimal asymptotic basis is that of a maximal asymptotic nonbasis. A set A is called a *maximal asymptotic nonbasis* of order h if A is an asymptotic nonbasis of order h but $A \cup \{a\}$ is an asymptotic basis of order h for every $a \notin A$.

The question of whether every asymptotic nonbasis of order h is a subset of some maximal asymptotic nonbasis of order h was originally posed by Nathanson [5] and repeated by Erdős and Nathanson [2, 3]. Hennefeld [4] constructed an asymptotic nonbasis, A, of order h for $h \geq 2$ which is not a subset of a maximal asymptotic nonbasis of order h. The example is

$$A = \{1\} \cup \{h\} \cup \{\text{all multiples of } h, \text{ except for } q_1, q_2, \ldots\},$$

where $\{q_i\}$ is an increasing sequence of multiples of h, with $\lim(q_{i+1} - q_i) = \infty$. Recently, Alladi and Krantz [1] remarked that the set of even nonnegative integers

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is a trivial example of a maximal nonbasis of order h for every $h \geq 2$, and one can construct many other examples that are unions of the nonnegative parts of congruence classes. It is difficult to construct nontrivial examples. In this paper, we extend the result of Hennefeld.

THEOREM 1.1. *Suppose $h \geq 2$, $r \in \{0, 1, \ldots, h-1\}$ and $(r, h) = 1$. Let $\{q_i\}_{i=1}^{\infty}$ be a sequence of positive integers with $0 \leq q_1 < q_2 < \ldots$ and $\lim(q_{i+1} - q_i) = \infty$. Then $A = \{r\} \cup \{h\} \cup \{$all multiples of h, except for $q_1 h, q_2 h, \ldots\}$ is an asymptotic nonbasis of order h which is not a subset of a maximal asymptotic nonbasis of order h.*

THEOREM 1.2. *Suppose $h \geq 2$, $r \in \{0, 1, \ldots, h-1\}$ and $(r, h) = 1$. Let $\{q_i\}_{i=1}^{\infty}$ be a sequence of positive integers with $1 < q_1 < q_2 < \ldots$ and $\lim(q_{i+1} - q_i) = \infty$ and let $x_i = q_i h + r, i = 1, 2, \ldots$. Then*

$$A = \{0\} \cup \{\textit{all positive integers congruent to } r \pmod{h}, \textit{ except for } x_1, x_2, \ldots\}$$

is an asymptotic nonbasis of order h which is not a subset of a maximal asymptotic nonbasis of order h.

2. Proof of Theorem 1.1

By Hennefeld's theorem [4], it is sufficient to prove Theorem 1.1 for $h \geq 3$. Clearly, since r is the only element of A not congruent to $0 \pmod{h}$, hA will miss all integers of the form $q_i h + (h-1)r$. Thus A is an asymptotic nonbasis of order h. Suppose $n > hr$ and $n \not\equiv (h-1)r \pmod{h}$. Since $(r, h) = 1$, the set $\{0, r, 2r, \ldots, (h-1)r\}$ is a complete set of residues modulo h and so there exists a positive integer k with $0 \leq k \leq h-2$ such that

$$n \equiv kr \pmod{h},$$

say, $n = qh + kr$. If $q - (h-k-1) \notin \{q_1, q_2, \ldots\}$, then

$$n = qh + kr = [q - (h-k-1)]h + (h-k-1)h + kr.$$

Thus n is the sum of h elements of A. Suppose $q - (h-k-1) \in \{q_1, q_2, \ldots\}$. Since $\lim(q_{i+1} - q_i) = \infty$, there exist $m, M \in \mathbb{N}$ such that $q_m + 1 < q_{m+1}$ and $q_i - q_{i-1} > q_m$ for all $i \geq M$. Thus, when $q - (h-k-1) > q_M$,

$$\begin{aligned} n &= qh + kr = [q - (h-k-1)]h + (h-k-1)h + kr \\ &= [q - (h-k-1) - q_m]h + (q_m + 1)h + (h-k-2)h + kr \end{aligned}$$

and, again, n is the sum of h elements of A. Also, for any positive integer $q \notin \{q_1, q_2, \ldots\}$, $qh + (h-1)r \in hA$. So all but a finite number of positive integers that hA misses are integers of the form $q_i h + (h-1)r$.

We claim that if x is any positive integer greater than zero, which is not congruent to $0 \pmod{h}$, then $A \cup \{x\}$ is an asymptotic basis of order h. Let $x = qh + kr$ with $1 \leq k \leq h-1$. If $k = 1$, then, for any sufficiently large i,

$$q_i h + (h-1)r = (h-1)x + [q_i - (h-1)q]h.$$

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Thus $q_ih + (h-1)r$ is the sum of h elements of $A \cup \{x\}$. If $2 \le k \le h-1$, then

$$q_ih + (h-1)r - x = (q_i - q)h + (h-1-k)r.$$

Note that $0 \le h-1-k \le h-3$. If $q_i - q - (k-1) \notin \{q_1, q_2, \ldots\}$, then

$$\begin{aligned} q_ih + (h-1)r - x &= (q_i - q)h + (h-1-k)r \\ &= [q_i - q - (k-1)]h + (k-1)h + (h-1-k)r. \end{aligned}$$

Thus $q_ih + (h-1)r - x$ is the sum of $h-1$ elements of A. If $q_i - q - (k-1) \in \{q_1, q_2, \ldots\}$, then, since $\lim(q_{i+1} - q_i) = \infty$, there exist $m, M \in \mathbb{N}$ such that $q_m + 1 < q_{m+1}$ and $q_i - q_{i-1} > q_m$ for all $i \ge M$. Thus, when $q_i - q - (k-1) > q_M$,

$$\begin{aligned} q_ih + (h-1)r - x &= (q_i - q)h + (h-1-k)r \\ &= [q_i - q - (k-1)]h + (k-1)h + (h-1-k)r \\ &= [q_i - q - (k-1) - q_m]h + (q_m + 1)h + (k-2)h + (h-1-k)r. \end{aligned}$$

Thus $q_ih + (h-1)r - x$ is the sum of $h-1$ elements of A and $q_ih + (h-1)r = x + [q_ih + (h-1)r - x]$ is the sum of h elements of $A \cup \{x\}$.

Therefore, the only possibility for extending A to a maximal asymptotic nonbasis is by adjoining B, a subset of $\{q_1h, q_2h, \ldots\}$. However, if there are an infinite number of q_ih missing from B, then $A \cup B$ will be a nonbasis which is not maximal. If there are only a finite number of q_ih missing from B, then $A \cup B$ will be a basis. In neither case will $A \cup B$ be a maximal asymptotic nonbasis. This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

Since $(r, h) = 1, kr \not\equiv r \pmod h$ for any k with $2 \le k \le h$. Clearly, since zero is the only element of A not congruent to $r \pmod h$, hA will miss all integers x_i. Let $n > hr$ and $n \not\equiv r \pmod h$. Since $(r, h) = 1$, the set $\{r, 2r, \ldots, hr\}$ is a complete set of residues modulo h and so there exists a positive integer k with $2 \le k \le h$ such that

$$n \equiv kr \pmod h.$$

Let $n = qh + kr$. If $q \notin \{q_1, q_2, \ldots\}$, then

$$n = (k-1)r + (qh + r) \in kA \subseteq hA.$$

If $q \in \{q_1, q_2, \ldots\}$, take $m \in \mathbb{N}$ such that $q_i - q_{i-1} \ge 2$ for all $i \ge m$. Then, except for a finite number of n,

$$n = (k-2)r + (h + r) + [(q-1)h + r] \in kA \subseteq hA.$$

Thus all but a finite number of positive integers that hA misses are integers x_i.

We claim that if x is any positive integer greater than zero which is not congruent to $r \pmod h$, then $A \cup \{x\}$ is an asymptotic basis of order h.

If $x \equiv 0 \pmod{h}$, then there exists a positive integer q such that $x = qh$. Since $\lim(q_{i+1} - q_i) = \infty$, there exists an $m \in \mathbb{N}$ such that $q_i - q_{i-1} > q$ for all $i \geq m$. So, for all $i \geq m$,

$$x_i = x + (x_i - x) = x + [(q_i - q)h + r],$$

that is, $x_i \in 2(A \cup \{x\}) \subseteq h(A \cup \{x\})$.

If $x \not\equiv 0 \pmod{h}$, then $h \geq 3$ and, for all i, $x_i - x \not\equiv 0, r \pmod{h}$. Thus there exists a positive integer $2 \leq l \leq h - 1$ such that $x_i - x \equiv lr \pmod{h}$. Let $x_i - x = th + lr$. If $t \notin \{q_1, q_2, \ldots\}$, then

$$x_i - x = (l - 1)r + (th + r).$$

Thus $x_i - x$ is the sum of l elements of A. If $t \in \{q_1, q_2, \ldots\}$, choose $m \in \mathbb{N}$ such that $q_i - q_{i-1} \geq 2$ for all $i \geq m$. For all large enough i,

$$x_i - x = (l - 2)r + (h + r) + [(t - 1)h + r]$$

and so $x_i - x$ is the sum of l elements of A. Hence

$$x_i = x + (x_i - x) \in (l + 1)(A \cup \{x\}) \subseteq h(A \cup \{x\}).$$

Therefore, the only possibility for extending A to a maximal asymptotic nonbasis is by adjoining B, a subset of $\{x_1, x_2, \ldots\}$. However, if there are an infinite number of x_i missing from B, then $A \cup B$ will be a nonbasis which is not maximal. If there are only a finite number of x_i missing from B, then $A \cup B$ will be a basis. In neither case will $A \cup B$ be a maximal asymptotic nonbasis. This completes the proof of Theorem 1.2.

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