BULL. AUSTRAL. MATH. SOC. 11D61
VOL. 59 (1999) [477–480]

A NOTE ON JEŚMANOWICZ' CONJECTURE CONCERNING PYTHAGOREAN TRIPLES

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Let n be a positive integer, and let (a, b, c) be a primitive Pythagorean triple. In this paper we give certain conditions for the equation $(an)^x + (bn)^y = (cn)^z$ to have positive integer solutions (x, y, z) with $(x, y, z) \neq (2, 2, 2)$. In particular, we show that x, y and z must be distinct.

Let $\mathbb{N}$ be the set of all positive integers. Let n be a positive integer, and let (a, b, c) be a primitive Pythagorean triple such that

$$(1) \qquad a^2 + b^2 = c^2, \quad a, b, c \in \mathbb{N}, \quad gcd(a, b, c) = 1, \quad 2 \mid b.$$

Then we have

$$(2) \qquad a = u^2 - v^2, \quad b = 2uv, \quad c = u^2 + v^2,$$

where u, v are positive integers satisfying $u > v$, $gcd(u, v) = 1$ and $2 \mid uv$. In 1956, Jeśmanowicz [2] conjectured that the equation

$$(3) \qquad (an)^x + (bn)^y = (cn)^z, \quad x, y, \ z \in \mathbb{N}$$

has only the solution $(x, y, z) = (2, 2, 2)$ for any n. This conjecture has been proved to be true in many special cases for $n = 1$.

Recently, Deng and Cohen [1] considered this conjecture for $n > 1$. For any positive integer t with $t > 1$, let $P(t)$ denote the product of distinct prime factors of t. Further let $P(1) = 1$. Deng and Cohen proved that if $n > 1$, $u = v + 1$, a is a prime power and either $P(b) \mid n$ or $P(n) \nmid b$, then (3) has only the solution $(x, y, z) = (2, 2, 2)$. In this paper we prove a general result as follows.

Received 26th November, 1998
Supported by the National Science Foundation of China, the Guangdong Provincial Natural Science Foundation and the Natural Science Foundation of the Higher Education Department of Guangdong Province.

Theorem. *If (x,y,z) is a solution of (3) with $(x,y,z) \neq (2,2,2)$, then one of the following conditions is satisfied:*

(i) $\max(x,y) > \min(x,y) > z$, $P(n) \mid c$ *and* $P(n) < P(c)$,

(ii) $x > z > y$ *and* $P(n) \mid b$.

(iii) $y > z > x$ *and* $P(n) \mid a$.

By the above result, we can obtain the following corollaries immediately.

Corollary 1. *If (x,y,z) is a solution of (3) with $(x,y,z) \neq (2,2,2)$, then x, y and z are distinct.*

Corollary 2. *If $P(n)$ does not divide any one of a, b and c, then (3) has only the solution $(x,y,z) = (2,2,2)$.*

The proof of our theorem depends of the following lemma.

Lemma. *Let m, t be positive integers, and let p be a prime such that $p \mid m$ and $p \mid t$. If $p^\alpha \| t$ and $p^\beta \| m$ with $\alpha > 1$, then we have*

$$\binom{m}{k+1} t^k \equiv 0 \pmod{p^{\beta+1}}, \quad k = 1, \dots, m-1.$$

Proof: For $k = 1, \dots, m-1$, let $p^{\gamma_k} \| k+1$. Then we have

$$\gamma_k \leqslant \left[\frac{\log(k+1)}{\log p}\right] \leqslant k, \quad k = 1, \dots, m-1. \tag{4}$$

Since $p^2 \mid t$, we get

$$\binom{m}{k+1} t^k = m \binom{m-1}{k} \frac{t^k}{k+1} \equiv 0 \pmod{p^{\beta+1}}, \quad k = 1, \dots, m-1,$$

by (4). The lemma is proved.

Proof of Theorem: Let (x,y,z) be a solution of (3) with $(x,y,z) \neq (2,2,2)$. By [**1**, Lemma 2], we may assume that $z < \max(x,y)$. We now eliminate the following three cases.

Case I. $x > y$ and $y = z$. Then from (3) we get

$$a^x n^{x-y} = c^y - b^y. \tag{5}$$

Since $c + b \mid a^2$ by (1), if $2 \nmid y$, then from (5) we get $c^y - b^y \equiv -2b^y \equiv 0 \pmod{c+b}$. But, by (1), this is impossible. So we have $2 \mid y$ and

$$a^{x-2} n^{x-y} = \frac{c^y - b^y}{c^2 - b^2} - \sum_{i=0}^{y/2-1} \binom{y/2}{i+1} a^{2i} b^{y-2i-2},$$

https://doi.org/10.1017/S0004972700033177 Published online by Cambridge University Press

by (5). Let p be a prime factor of a. Since $gcd(a,b)=1$, we see from (6) that $p \mid y/2$. Further let $p^{\alpha}\|a$ and $p^{\beta}\|y/2$. By the Lemma, we obtain

$$\binom{y/2}{i+1}a^{2i}b^{y-2i-2}\equiv 0 \pmod{p^{\beta+1}}, \quad i=1,\dots,y/2-1. \tag{7}$$

This implies that

$$p^{\beta}\Big\| \sum_{i=0}^{y/2-1}\binom{y/2}{i+1}a^{2i}b^{y-2i-2}. \tag{8}$$

The combination of (6) and (8) yields

$$\alpha(x-2)\leqslant\beta. \tag{9}$$

Let p run through all distinct prime factors of a. Then, by (9), we get $a^{x-2}\mid y/2$ and

$$y\geqslant 2a^{x-2}. \tag{10}$$

However, since $x>y$ and $a>1$, (10) is impossible.

Case II. $y>x$ and $x=z$. Then we have

$$b^{y}n^{y-x}=c^{x}-a^{x}. \tag{11}$$

Since $c+a\mid b^2$, if $2\nmid x$, then from (11) we get $c^x-z^x\equiv 2c^x\equiv 0 \pmod{c+a}$, a contradiction. So we have $2\mid x$ and

$$b^{y-2}n^{y-x}=\frac{c^x-a^x}{c^2-a^2}=\sum_{i=0}^{x/2-1}\binom{x/2}{i+1}a^{x-2i-2}b^{2i}, \tag{12}$$

by (11). Using the same method as in Case I, we can prove that (12) is impossible.

By using the same arguments, we can prove that (3) has no solution (x,y,z) satisfying the folowing condition:

Case III. $x=y$ and $y>z$.

If $x>y>z$, then from (3) we get

$$a^{x}n^{x-y}+b^{y}=\frac{c^{z}}{n^{y-z}}, \tag{13}$$

where c^z/n^{y-z} is an integer with $c^z/n^{y-z}>1$. This implies that $P(n)\mid c$. Further, if $P(n)=P(c)$, then there exists a prime p such that $p\mid c^z/n^{y-z}$ and $p\mid n$. But, since $gcd(b,c)=1$, this is impossible by (13). So we have $P(n)\mid c$ and $P(n)<P(c)$. A similar result can be proved for $y>x>z$. Therefore, we get the condition (i).

If $x>z>y$, then we have

$$a^{x}n^{x-z}+\frac{b^{y}}{n^{z-y}}=c^{z}. \tag{14}$$

We see from (14) that $P(n)\mid b$. The condition (ii) is proved. By using the same arguments, we can obtain the condition (iii) if $y>z>x$. The proof is complete. ∎

References

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