ORBITAL STABILITY IN THE ELLIPTIC RESTRICTED THREE BODY PROBLEM

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ABSTRACT

Based on the concept of orbital stability introduced by G. W. Hill, a method is presented to facilitate the determination of the orbital stability of solutions to the planar elliptic restricted problem of three bodies. The invariant relation introduced by Szebehely and Giacaglia (1964) contains an integral which is expanded here about a Keplerian solution to the problem. If the expansion converges, it can be used to determine the conditions for Hill stability. With it one can also define stability in a periodic sense.

Szebehely and Giacaglia (1964) give the equations of motion of the elliptic restricted three body problem in a coordinate system rotating and pulsating with respect to an inertial system. The coordinate system is chosen in such a way that as a consequence of the rotation and pulsation the two primary masses are located at fixed points on the horizontal ξ axis. In two dimensions, the equations of motion of this system are

$$\begin{aligned} \xi'' - 2\eta' &= \omega_\xi \\ \eta'' + 2\xi' &= \omega_\eta \end{aligned} \tag{1}$$

The independent variable is the true anomaly f of the primaries. A prime indicates differentiation with respect to f. The potential function is defined by

$$\omega = (1 + \bar{e} \cos f)^{-1}\, \Omega, \tag{2}$$

where $\Omega = \tfrac{1}{2}(1-\mu)\rho_1^2 + \tfrac{1}{2}\mu\rho_2^2 + \dfrac{1-\mu}{\rho_1} + \dfrac{\mu}{\rho_2}$,

/. Szebehely (ed.), *Dynamics of Planets and Satellites and Theories of Their Motion*, 333-337.
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with $\rho_1^2 = (\xi - \mu)^2 + \eta^2$ and $\rho_2^2 = (\xi - \mu + 1)^2 + \eta^2$.

The eccentricity of the primary orbit is $\bar{e}$, the mass of the primary at the point $(\mu,0)$ is $1-\mu$ and the mass of the other primary is μ. ρ_1 and ρ_2 are the distances of the particle from the masses $1-\mu$ and μ respectively.

Since this system is nonconservative, the equations (1) do not furnish a Jacobi integral as do the equations of motion in the circular restricted problem. (The latter can be derived from Eqs. (1) and (2) by putting $\bar{e} = 0$, $f = t$.) However, the following invariant relation can be formally derived in the same manner as can the Jacobi integral in the circular problem:

$$(\xi')^2 + (\eta')^2 = 2\omega - C - 2\int_{f_o}^{f} \frac{\partial\omega}{\partial f}\, df. \tag{3}$$

Since in the circular case $\partial\omega/\partial t \equiv 0$, Eq. (3) would in that case represent an integral. Szebehely (1967) remarks that Eq. (3) may be used to obtain time varying curves of zero velocity in the case that $\bar{e} \neq 0$.

In order to discuss orbital stability, we need to describe the time variation of the curves of zero velocity. If we can establish that a particular solution of Eq. (1) has closed curves of zero velocity for some domain $f_1 \leq f \leq f_2$, then we will have established orbital stability for the system in the same sense as that defined by G. W. Hill (1905) for the lunar theory. Hill stability excludes the possibility of escape from a primary (i.e., ρ_1 and ρ_2 cannot approach infinity), although collisions may still occur. If we can establish Hill stability for $f_2 \to \infty$, we will have established orbital stability in a non-trivial sense.

In his above quoted book (p. 596), Szebehely suggests an expansion of the invariant relation Eq. (3) with respect to powers of $\bar{e}$. The integral term is included in this expansion. To do this, we note that

$$\frac{\partial\omega}{\partial f} = \frac{\Omega \quad \bar{e}\sin f}{(1 + \bar{e}\cos f)^2} \quad . \tag{4}$$

Ω in Eq. (4) is to be evaluated along a solution of Eq. (1); the integral is therefore a function of f.

$$\text{Define} \quad I(f) = \int_{f_o}^{f} \frac{\Omega(f)\,\sin f}{(1 + \bar{e}\cos f)^2}\, df \quad . \tag{5}$$

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The curves of zero velocity, if they exist, can be obtained from the equation

$$2\Omega(\xi,\eta) = \left[C + 2\bar{e}\ I(f)\right] \ (1 + \bar{e} \cos f) \ \equiv K(f). \tag{6}$$

As the particle follows its orbit, we can define a continuous set of zero velocity curves for the motion if the equation $2\Omega = K$ has a real locus. In order to evaluate I, it is assumed that the motion of the particle can be represented by perturbations of Keplerian motion. This will be true in three cases: Case I, ρ_1 is very small (C is large); Case II, ρ_2 is very small (C is large); Case III, ρ_1 and ρ_2 are not small but C is very large.

Cases I and II describe the motion of a particle moving close to one of the primaries and perturbed by the other.

Case III applies to the system in which a negligibly small outer planet moves in the gravitational field of the Sun and one larger planet whose orbit is completely inside the small planet's orbit.

The first step in the evaluation of I is to expand the various terms in Ω into series in the mean anomaly of the particle's orbit. In this paper, only Case II was considered; however, Case I can be obtained from Case II by interchanging μ and $1-\mu$.

ρ_2 is related to the radius vector of the particle's orbit about mass μ. To illustrate the method, consider the expansion of the term μ/ρ_2 in the function Ω. The expansion of a/r, where r is the radius vector in elliptic motion, is well known. In this problem, the radius vector r must be transformed, since the coordinate system defined by the axes ξ and η is rotating and pulsating. The rotation leaves r invariant, but because of the pulsation

$$\rho_2 = r(1 + \bar{e} \cos f) / (1 - \bar{e}^2) = r\,\bar{a}/\bar{r}, \text{ where } \bar{a} \equiv 1. \tag{7}$$

Therefore, we have the relation $\mu/\rho_2 = (\mu/a)\ (a/r)\ (\bar{r}/\bar{a})$, where the expansion of a/r is a Poisson series in the particle's mean anomaly and eccentricity $\bar{e}$. The form of the expansion of this term will be

$$\frac{\mu}{\rho_2} = \frac{\mu}{a} \sum_{j_1=0}^{\infty} \sum_{j_2=-\infty}^{+\infty} A_{j_1 j_2} \cos\,(j_1\ell + j_2\bar{\ell}) \quad ,$$

where $A_{j_1 j_2}$ is a function of e and $\bar{e}$. A similar method is applied to the expansion of the other terms in Ω.

After the expansion is performed, it is necessary to include the expansion of $\sin f(1 \pm \bar{e} \cos f)^{-2}$ before integrating Eq. (4). It will be necessary to adopt $\bar{\ell}$ as the independent variable instead of f, which

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requires one more step in the expansion process. Fully expanded, the integrand in Eq. (4) has the form

$$\frac{1}{\bar{e}} \frac{\partial \omega}{\partial f} = \sum_{j_1=0}^{\infty} \sum_{j_2, j_3 = -\infty}^{\infty} A_{j_1 j_2 j_3} \sin(j_1 \ell + j_2 \bar{\ell} + j_3 w) \quad (8)$$

where A is a function of μ, a, e, and $\bar{e}$; and w is the argument of pericenter of the particle's orbit. One may set $\bar{w} = 0$ without any loss of generality. The integration of (8) is performed, assuming a, e, and w are constants. This will give a first order approximation to I. For a small perturbation this will give sufficient accuracy for the determination of the range of I(f).

For Case II, $\mu/\rho_2 \simeq \mu/a$. If $a \leq \mu$, this will be the dominant term in the expansion. Another large term is $(1-\mu)(\rho_1^2/2+1/\rho_1) \simeq 3(1-\mu)/2$. Both of these lead to the largest term in I and have the frequency $\bar{n}$. (In dimensionless variables, $\bar{n} = 1$.) μ/ρ_2 contributes another large term with frequency $2\bar{n}$. The remaining terms are factored by powers of $\bar{e}$, a^2, e, or μ and are considerably smaller. The frequency of a term in the expansion is given by $pn + q\bar{n}$ where p and q are integers. If a resonance condition exists, i.e., if $n/\bar{n} = -q/p$, the integral will not converge and this representation is not applicable. If we exclude a resonance condition and if the orbit is quasi-periodic, then the Poisson series representation of the integral should be a good one. Using Eq. (8), we can write

$$I = \sum_{j_1, j_2, j_3} - \frac{A_{j_1 j_2 j_3}}{j_1 n + j_2 \bar{n}} \cos(j_1 \ell + j_2 \bar{\ell} + j_3 w) \Bigg|_{t_o}^{t} \quad (9)$$

If this series converges, then

$$|I| \leq M, \text{ where} \quad (10)$$

$$M = 2 \sum_{j_1 j_2 j_3} \left| \frac{A_{j_1 j_2 j_3}}{j_1 n + j_2 \bar{n}} \right|$$

If Eq. (9) holds, then we can substitute these results into Eq. (6) to obtain $(1 - \bar{e})(C - 2\bar{e}M) \leq K(f) \leq (1 + \bar{e})(C + 2\bar{e}M)$ if $C - 2\bar{e}M \geq 0$. The lower limit will be $(1 + \bar{e})(C - 2\bar{e}M)$ if $C - 2\bar{e}M < 0$. Considering Eq. (2) for Ω, we can derive the well known result that curves of zero velocity exist when $2\Omega \geq 3$. Whether or not these curves are closed depends on the value of μ. In the notation of Szebehely and Williams (1964), the curves of zero velocity are closed when $2\Omega > C_2$, for Case II.

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This implies that the motion of the particle is restricted to the vicinity of the mass μ. We will have Hill stability for all time if

$$(1 - \bar{e})\ (C - 2\bar{e}M) \geq C_2$$

It is also possible to define a slightly different type of stability using this method. Assume that there exists at least one value of f, say f*, where the integral in Eq. (10) takes on a value I* such that

$$(C + 2\bar{e}I^*)\ (1 + \bar{e}\cos f^*) = K(f^*) > C_2 .$$

Then, if the integral given by Eq. (10) is quasi-periodic, there will be another value of f, say $f^{**} > f^*$, where again $(K(f^{**}) > C_2$. This condition will occur periodically if Eq. (10) converges. One could say that regardless of the motion of the particle, the particle returns periodically to the vicinity of the mass μ for all time. The existence of this type of stability depends entirely on the behavior of the integral given by Eqn. (10) and the initial conditions of the system.

This theory can be applied to the motion of satellites of planets for which the solar perturbations are dominant. If the satellite is so close to the planet that the perturbations are very small, then these results are almost trivial. A more interesting problem would be to study the motion of satellites farther from the planet where the possibility exists that $K(f) < C_2$ for some value of f. In this case, however, a representation of the motion different from a Keplerian one, may be necessary.

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