Bull. Austral. Math. Soc. **78** (2008), 171–176
doi:10.1017/S0004972708000610

CONTINUITY OF MEASURABLE HOMOMORPHISMS

JANUSZ BRZDĘK

(Received 22 October 2007)

Abstract

We give some general results concerning continuity of measurable homomorphisms of topological groups. As a consequence we show that a Christensen measurable homomorphism of a Polish abelian group into a locally compact topological group is continuous. We also obtain similar results for the universally measurable homomorphisms and the homomorphisms that have the Baire property.

2000 *Mathematics subject classification*: 22A05, 39B52, 54H11.

Keywords and phrases: Christensen measurability, Baire property, universal measurability, Haar measurability, measurable homomorphism of groups.

Kleppner [8, Theorem 1, 9] (see also [16] and [10]) proves that every Haar measurable homomorphism between locally compact topological groups is continuous. This is interesting because there are no explicit global countability restrictions on the groups (though the local compactness yields some implicit conditions, as is shown in Remark 4).

We present a short and quite simple proof that results analogous to that of Kleppner are true in more general settings. In particular (see Corollary 7), we show that every Christensen measurable homomorphism mapping a Polish abelian group X into a locally compact topological group G is continuous (for details on Christensen measurability we refer to [6]). Further, our Theorem 6(i) provides a natural generalization of [6, Theorem 1] (the assumption on X is necessary in view of the fact that Christensen measurability is defined so far only for Polish abelian groups). We also generalize [8, Theorem 1] and obtain some results on universally measurable homomorphisms (thus generalizing to some extent [4, Theorem 2]) and homomorphisms with the Baire property.

Throughout the paper $\mathbb{N}$ denotes, as usual, the set of all positive integers. For subsets A, B of a group $(G, \cdot)$ we write $A^{-1} := \{a^{-1} \mid a \in A\}$ and $A \cdot B := \{a \cdot b \mid a \in A, b \in B\}$. Let us also recall that a group $(X, \circ)$, endowed with a topology, is semitopological provided that the mappings $x \to x \circ y$ and $x \to y \circ x$ are continuous for every $y \in X$ (see, for example, [7]).

To simplify the proofs we introduce the following two definitions.

DEFINITION 1. Let X be a nonempty set, Y be a topological space and $\mathcal{M} \subset 2^X$. We say that a function $f : X \to Y$ is $\mathcal{M}$-measurable provided that $f^{-1}(U) \in \mathcal{M}$ for every open set $U \subset Y$.

DEFINITION 2. Let X be a nonempty set. We say that $\Im \subset 2^X$ is a σ-ideal (in X) provided that $2^A \subset \Im$ for $A \in \Im$ and, for each $\{A_n\}_{n\in\mathbb{N}} \subset \Im$, $\bigcup_{n\in\mathbb{N}} A_n \in \Im$.

Given a group $(X, \cdot)$, with neutral element e and endowed with a topology, we use the following hypothesis concerning families $\mathcal{M} \subset 2^X$.

($\mathcal{H}$) There is a σ-ideal $\Im \subset 2^X$ such that:
(a) $\operatorname{int}(B) = \emptyset$ for $B \in \Im$;
(b) $B \cdot x, x \cdot B \in \Im$ for $x \in X$, $B \in \Im$;
(c) $e \in \operatorname{int}(A \cdot A^{-1})$ for $A \in \mathcal{M} \setminus \Im$.

REMARK 1. Actually in what follows condition (a) of ($\mathcal{H}$) can be replaced by the following weaker one.
(a′) If B is a subgroup of X and $B \in \Im$, then $\operatorname{int}(B) = \emptyset$.

REMARK 2. Hypothesis ($\mathcal{H}$) holds, for instance, in the case where $\mathcal{M}$ is the σ-algebra of Christensen measurable subsets of a Polish abelian group $(X, +)$ with $\Im = \{A \subset X \mid A$ is a Christensen zero set$\}$ (see [5, Theorems 2, 6]), or $\mathcal{M}$ is the σ-algebra of universally measurable subsets of a semitopological group $(X, +)$ that is abelian (for some results concerning the nonabelian case we refer to [17]) and metrizable with a complete metric (with $\Im$ being the σ-ideal generated by the family $\{A \subset X \mid A$ is universally measurable and $\operatorname{int}(A - A) = \emptyset\}$). ('Universally measurable' means measurable with respect to the universal completion of the Borel field in X.) The latter results from the following simple generalization of [4, Theorem 1].

PROPOSITION 1. *Let $(S, +)$ be an abelian semigroup with neutral element* 0*. Suppose that S is endowed with a topology, generated by a complete metric d, such that all translations $\tau_a : x \to a + x$ are continuous. Let $\{A_i\}_{i\in\mathbb{N}} \subset 2^S$ and* $\operatorname{int}(\bigcup_{i\in\mathbb{N}} A_i) \neq \emptyset$*. Then there is $k \in \mathbb{N}$ such that, for every universally measurable set $U \supset A_k$, the set $U - U := \{x \in S \mid (x + U) \cap U \neq \emptyset\}$ is a neighbourhood of* 0*.*

PROOF. For a proof by contradiction, suppose that for every $i \in \mathbb{N}$ there is a universally measurable set $D_i \subset S$ such that $A_i \subset D_i$ and $D_i - D_i$ is not a neighbourhood of 0. Let $z \in V := \operatorname{int}(\bigcup_{i\in\mathbb{N}} A_i)$ and $V_0 \subset S$ be a neighbourhood of 0 with $z + V_0 \subset V$. Write $B_i := \tau_z^{-1}(D_i)$ for $i \in \mathbb{N}$. Then $\{B_i \mid i \in \mathbb{N}\}$ is a family of universally measurable sets, $V_0 \subset \bigcup_{i\in\mathbb{N}} B_i$, and, for every $i \in \mathbb{N}$, $B_i - B_i$ is not a neighbourhood of 0 (because $B_i - B_i \subset D_i - D_i$).

Let $K = \{0, 1\}^{\mathbb{N}}$ be endowed with the product topology and the usual group structure. Define continuous $\varphi : K \to S$ as in the proof of [4, Theorem 1]. Then

https://doi.org/10.1017/S0004972708000610 Published online by Cambridge University Press

$\varphi^{-1}(V_0)$ is a nonempty open set in K and $\{\varphi^{-1}(B_i) \mid i \in \mathbb{N}\}$ is a covering of $\varphi^{-1}(V_0)$. Since K is compact, $\varphi^{-1}(B_m)$ has nonzero Haar measure for some $m \in \mathbb{N}$. The rest of the proof is the same as in [4, p. 125]. □

REMARK 3. Two other examples of families $\mathcal{M}$ satisfying $(\mathcal{H})$ are provided by the σ-algebra of Haar measurable subsets of a locally compact topological group $(X, \cdot)$ (with $\mathfrak{I} = \{A \subset X \mid A$ is locally of the Haar measure zero$\}$) (see [1] or [18]) and the σ-algebra of Baire subsets (that is, having the Baire property (see, for example, [11])) of a semitopological group $(X, \cdot)$ of second category of Baire (with $\mathfrak{I} = \{A \subset X \mid A$ is of first category$\}$); the latter follows, for example, from the Banach category theorem (see, for example, [11, Theorem 16.1]) and [3, Proposition 1] (see also [2, Proposition 1], which actually is stated for topological groups, though its proof is valid for semitopological groups as well). For further abstract generalizations of the last example refer to [15, Satz 4.14] and [13] (see also [7, 14]).

Now, let us recall [8, Lemma 2], which will be useful in the sequel.

LEMMA 2. *The following six conditions on a subgroup N of a group $(G, \cdot)$ are equivalent.*

(1) *For all $x \in G$, $[N \mid xNx^{-1} \cap N] \leq \infty$ (∞ denotes a countable infinity).*
(2) *Each double coset NxN is a union of countably many left N cosets.*
(3) *Each double coset NxN is a union of countably many right N cosets.*
(4) *For each $x \in G$ there is a countable set D such that $Nx \subset DN$.*
(5) *For each $x \in G$ there is a countable set D' such that $xN \subset ND'$.*
(6) *If C is a countable subset of G and M is the subgroup generated by $N \cup C$, then $[M : N] \leq \infty$.*

DEFINITION 3 (see [8, p. 394]). We say that a subgroup N of a group $(G, \cdot)$ is asoo provided it satisfies any of conditions (1)–(6) of Lemma 2.

The next lemma concerns the existence of some nonmeasurable sets.

LEMMA 3. *Let $(X, \cdot)$ be a group with neutral element e, endowed with a topology, $\mathcal{M} \subset 2^X$ satisfy $(\mathcal{H})$, and $H \in \mathfrak{I}$ be an asoo subgroup of X. Suppose that there exists a countable base $\{U_n\}_{n\in\mathbb{N}}$ of neighbourhoods of e. Then:*

($\star$) *there is a nonempty $S \in 2^X \setminus \mathcal{M}$ with $S = S \cdot H$.*

PROOF. Note that, by conditions (a) and (b) of $(\mathcal{H})$,

$$\operatorname{int}\left(\bigcup_{i=1}^{n} H \cdot y_i\right) = \emptyset \quad \text{for } n \in \mathbb{N},\ y_1, \ldots, y_n \in X.$$

Hence, there exists a set $\{x_n \mid n \in \mathbb{N}\} \subset X$ with $x_1 = e$ and

$$x_{n+1} \in U_{n+1} \Big\backslash \bigcup_{i=1}^{n} H \cdot x_i \quad \text{for } n \in \mathbb{N}.$$

Write $C := \{x_n \mid n \in \mathbb{N}\}$.

https://doi.org/10.1017/S0004972708000610 Published online by Cambridge University Press

Let G be the subgroup of X generated by $H \cup C$, Y_G be a set of right coset representatives of G in X (that is, $(G \cdot y_1) \cap (G \cdot y_2) = \emptyset$ for $y_1, y_2 \in Y_G$, $y_1 \neq y_2$, and $X = G \cdot Y_G$), and $S := H \cdot Y_G$. Suppose that $S \in \mathcal{M}$ and let Y_H be a set of left coset representatives of H in G. Then Y_H is countable (see Lemma 2(6)) and

$$Y_H \cdot S = Y_H \cdot H \cdot Y_G = G \cdot Y_G = X \notin \Im.$$

Thus $y_0 \cdot S \notin \Im$ for some $y_0 \in Y_H$, $S \notin \Im$, and consequently, in view of (c), $e \in \operatorname{int}(S \cdot S^{-1})$.

Clearly $x_n \in \operatorname{int}(S \cdot S^{-1})$ for some $n \in \mathbb{N}$. So there exist $z_1, z_2 \in H$ and $y_1, y_2 \in Y_G$ with

$$G \cdot y_2 \supset x_n \cdot H \cdot y_2 \ni x_n \cdot z_2 \cdot y_2 = z_1 \cdot y_1 \in H \cdot y_1 \subset G \cdot y_1,$$

which means that $(G \cdot y_1) \cap (G \cdot y_2) \neq \emptyset$. Consequently $y_1 = y_2$, thus $x_n = z_1 \cdot z_2^{-1} \in H$, and this contradicts the choice of x_n.

Thus we have shown that $S \notin \mathcal{M}$. □

In what follows we say that a subset G_0 of a topological group $(G, \cdot)$ is σ-right-bounded provided that, for every open set $U \subset G$, there exists a countable set $D \subset G$ with $G_0 \subset U \cdot D$. If G is abelian, then the notion of σ-right-boundedness coincides with that of σ-boundedness in the sense of Pettis [12] (see also [4, pp. 125–6]) and then we simply say σ-bounded instead of σ-right-bounded.

REMARK 4. In Proposition 4 and Corollary 5 below, we assume that a topological group possesses a σ-right-bounded open subgroup. Note that every σ-compact subset of a topological group is σ-right-bounded. Thus every locally compact topological group has an open σ-right-bounded subgroup, because it has open σ-compact subgroups.

Clearly, every separable topological group is σ-right-bounded.

Also, if a commutative topological group $(G, +)$ has a σ-bounded neighbourhood U of 0, then it possesses a σ-bounded open subgroup G_0. In fact, let

$$U^1 := U \cap (-U), \quad U^{n+1} := U^1 + U^n \quad \text{for } n \in \mathbb{N},$$

and $G_0 := \bigcup_{n \in \mathbb{N}} U^n$. It is easy to see that G_0 is an open subgroup of G and U^1 is σ-bounded. Fix $n \in \mathbb{N}$ and suppose that U^n is σ-bounded. Take an open neighbourhood $V \subset G$ of 0. There exist an open neighbourhood $W \subset G$ of 0 and a countable set $D \subset G$ with $W + W \subset V$ and $U^1 \subset U^n \subset W + D$. Hence,

$$U^{n+1} = U^1 + U^n \subset W + D + W + D \subset V + (D + D).$$

Thus we have proved that U^n is σ-bounded for every $n \in \mathbb{N}$, which means that so is G_0.

PROPOSITION 4. *Let $(X, \cdot)$ be a semitopological group, $\mathcal{M} \subset 2^X$ satisfy $(\mathcal{H})$, $(G, \cdot)$ be a topological group possessing an open σ-right-bounded subgroup G_0, and $h : X \to G$ be an $\mathcal{M}$-measurable homomorphism. Suppose that, for every asoo subgroup $H \in \Im$ of X, condition $(\star)$ of Lemma 3 holds. Then h is continuous.*

PROOF. Let e_X denote the neutral element in X. It suffices to show that h is continuous at e_X. Let V be a neighbourhood of the neutral element e_G in G. There exists an open neighbourhood $W \subset G$ of e_G with $W \cdot W^{-1} \subset V$. Suppose that $U := h^{-1}(W) \in \Im$. Note that G_0 is contained in a union of countably many right translates of W and consequently $H := h^{-1}(G_0)$ is contained in a union of countably many right translates of U, which means that $H \in \Im$. Next, for every $x \in G$, $x^{-1} \cdot G_0$ is open, which means that there is a countable set $D \subset G$ with $G_0 \subset x^{-1} \cdot G_0 \cdot D$. Consequently G_0 is asoo (see Lemma 2(5)) and so is H (see [8, p. 394]). Hence, by ($\star$), there exists a nonempty set $S \subset X$ such that $S \notin \mathcal{M}$ and $S \cdot H = S$. Setting $T = h(S)$ we get

$$S \subset h^{-1}(T \cdot G_0) = h^{-1}(T) \cdot h^{-1}(G_0) \subset (S \cdot H) \cdot H = S.$$

This is a contradiction, because $T \cdot G_0$ is open and therefore $h^{-1}(T \cdot G_0) \in \mathcal{M}$.

Thus we have proved that $U \in \mathcal{M} \setminus \Im$, which yields $e_X \in \operatorname{int}(U \cdot U^{-1})$. Now it is enough to note that $h(U \cdot U^{-1}) \subset W \cdot W^{-1} \subset V$. □

From Proposition 4 and Lemma 3 we easily obtain the following corollary.

COROLLARY 5. *Let $(X, \cdot)$ be a first countable semitopological group with neutral element e, $\mathcal{M} \subset 2^X$ satisfy $(\mathcal{H})$, $(G, \cdot)$ be a topological group possessing an open σ-right-bounded subgroup, and $h : X \to G$ be an $\mathcal{M}$-measurable homomorphism. Then h is continuous.*

Finally we are in a position to present the main result of the paper.

THEOREM 6. *Let $(G, \cdot)$ be a topological group possessing an open σ-right-bounded subgroup, $(X, \cdot)$ be a semitopological group and $h : X \to G$ be a homomorphism. Suppose that one of the following four conditions is valid.*

(i) *X is a Polish abelian group and h is Christensen measurable.*
(ii) *X is locally compact and h is Haar measurable.*
(iii) *X is first countable and of second category of Baire and h has the Baire property (that is, $h^{-1}(U)$ has the Baire property for every open set $U \subset G$).*
(iv) *X is abelian and metrizable with a complete metric and h is universally measurable (that is, the set $h^{-1}(U)$ is universally measurable for every open set $U \subset G$).*

Then h is continuous.

PROOF. If (ii) holds, then without loss of generality we may assume that X is compactly generated (see [8, p. 395]) and then use Proposition 4 and [8, Lemma 3]. In the remaining cases we use Corollary 5 and Remarks 2 and 3. □

In view of Remark 4, Theorem 6 generalizes [4, Theorem 2], [6, Theorem 1] and [8, Theorem 1], and, in particular, implies the following corollary.

COROLLARY 7. *Let $(G, \cdot)$ be a locally compact topological group and X and h be as in Theorem 6. Then h is continuous.*

https://doi.org/10.1017/S0004972708000610 Published online by Cambridge University Press

References

[1] A. Beck, H. H. Corson and A. B. Simon, 'The interior points of the product of two sets of a locally compact group', *Proc. Amer. Math. Soc.* **9** (1958), 648–652.

[2] J. Brzdęk, 'On the Cauchy difference on normed spaces', *Abh. Math. Sem. Univ. Hamburg* **66** (1996), 143–150.

[3] ———, 'On almost additive functions', *Bull. Austral. Math. Soc.* **54** (1996), 281–290.

[4] J. P. R. Christensen, 'Borel structures in groups and semigroups', *Math. Scand.* **28** (1971), 124–128.

[5] ———, 'On sets of Haar measure zero in abelian Polish groups', *Israel J. Math.* **13** (1972), 255–260.

[6] P. Fisher and Z. Słodkowski, 'Christensen zero sets and measurable convex functions', *Proc. Amer. Math. Soc.* **79** (1980), 449–453.

[7] Z. Kominek and M. Kuczma, 'Theorems of Bernstein-Doetsch, Piccard and Mehdi, and semilinear topology', *Arch. Math. (Basel)* **52** (1989), 595–602.

[8] A. Kleppner, 'Measurable homomorphisms of locally compact groups', *Proc. Amer. Math. Soc.* **106** (1989), 391–395.

[9] ———, 'Correction to "Measurable homomorphisms of locally compact groups" 106 (1989), 391–395', *Proc. Amer. Math. Soc.* **111** (1991), 1199.

[10] D. Noll, 'Souslin measurable homomorphisms of topological groups', *Arch. Math. (Basel)* **59** (1992), 294–301.

[11] J. C. Oxtoby, *Measure and Category*, Graduate Texts in Mathematics (Springer, Berlin, 1971).

[12] B. J. Pettis, 'On continuity and openness of homomorphisms in topological groups', *Ann. of Math.* (2) **52** (1950), 293–308.

[13] W. Sander, 'Verallgemeinerungen eines Satzes von S. Piccard', *Manuscripta Math.* **16** (1975), 11–25.

[14] ———, 'A generalization of a theorem of S. Piccard', *Proc. Amer. Math. Soc.* **73** (1979), 281–282.

[15] ———, 'Ein Beitrag zur Baire-Kategorie-Theorie', *Manuscripta Math.* **34** (1981), 71–83.

[16] Z. Sasvári, 'On measurable local homomorphisms', *Proc. Amer. Math. Soc.* **112** (1991), 603–604.

[17] S. Solecki, 'Amenability, free subgroups, and Haar null sets in non-locally compact groups', *Proc. London Math. Soc.* (3) **93** (2006), 693–722.

[18] K. Stromberg, 'An elementary proof of Steinhaus's theorem', *Proc. Amer. Math. Soc.* **36** (1972), 308.

JANUSZ BRZDĘK, Department of Mathematics, Pedagogical University,
Podchorążych 2, 30-084 Kraków, Poland
e-mail: jbrzdek@ap.krakow.pl

https://doi.org/10.1017/S0004972708000610 Published online by Cambridge University Press