# 2

# Linear Algebra and Bundle Theory

In this chapter we discuss the linear algebra of symplectic vector spaces and symplectic vector bundles. To prepare the ground for the discussion of Künneth structures on manifolds in later chapters we introduce linear Künneth structures on vector bundles, and we work out consequences of the existence of Künneth structures in terms of characteristic classes.

The earlier parts of this chapter contain standard material that some readers may be able to skip. There is a substantial overlap, for example, with Chapter 2 of the book of McDuff–Salamon [McS-95]. The later parts contain some important results that are used throughout the book. While not original, these results clarify some of the folklore around symplectic vector bundles and their Lagrangian subbundles. Our reference for the theory of characteristic classes is Milnor–Stasheff [MS-74].

## 2.1 Linear Algebra

### 2.1.1 Linear Symplectic Forms

Here is the most basic definition, which is the beginning of all of symplectic mathematics.

**Definition 2.1** Let $V$ be a finite-dimensional (real) vector space. A *symplectic form* on $V$ is a 2-form $\omega \in \Lambda^2 V^*$ that is *non-degenerate* in the sense that

$$\omega(v, u) = 0 \quad \text{for all } u \in V$$

implies $v = 0$. Equivalently,

$$\omega(v, V) \equiv 0 \Rightarrow v = 0$$



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or

$$i_v\omega = \omega(v, -) \equiv 0 \Rightarrow v = 0 \; .$$

We call $(V, \omega)$ a *symplectic vector space*.

One can characterise non-degenerate two-forms in the following way.

**Lemma 2.2** *A two-form $\omega \in \Lambda^2 V^*$ is non-degenerate if and only if the map*

$$\begin{aligned} \phi_\omega \colon V &\longrightarrow V^* \\ v &\longmapsto i_v\omega = \omega(v, -) \end{aligned}$$

*is an isomorphism.*

*Proof* In one direction, since $V$ and $V^*$ have the same dimension, the linear map $\phi_\omega$ is an isomorphism if and only if it is injective. Conversely, injectivity of $\phi_\omega$ is equivalent to the non-degeneracy of $\omega$. □

*Example* 2.3 Let $V = \mathbb{R}^{2n}$ with basis

$$(e_1, \ldots, e_n, f_1, \ldots, f_n) \; .$$

We denote the dual basis of $V^*$ by

$$(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) \; .$$

Then

$$\omega = \sum_{i=1}^{n} \alpha_i \wedge \beta_i$$

is a symplectic form on $V$. It is uniquely characterised by

$$\begin{aligned} \omega(e_i, f_j) &= \delta_{ij} \quad \text{(Kronecker delta)} \; , \\ \omega(e_i, e_j) &= 0 = \omega(f_i, f_j) \; . \end{aligned}$$

The map $\phi_\omega$ is given by

$$\begin{aligned} \phi_\omega(e_i) &= \beta_i \; , \\ \phi_\omega(f_i) &= -\alpha_i \; , \quad \forall i = 1, \ldots, n, \end{aligned}$$

showing that it is indeed an isomorphism between $V$ and $V^*$.

Just as in the case of scalar products, there is a notion of a symplectic orthogonal for a subspace.

**Definition 2.4** Let $(V, \omega)$ be a symplectic vector space and $U \subset V$ a linear subspace. The *symplectic orthogonal* of $U$ is defined as

$$\begin{aligned} U^{\perp\omega} &= \{v \in V \mid \omega(v, U) \equiv 0\} \\ &= \{v \in V \mid \omega(v, u) = 0 \quad \forall u \in U\} . \end{aligned}$$

In other words, if $i\colon U \hookrightarrow V$ is the injection, then $U^{\perp\omega}$ is the kernel of the linear map

$$V \xrightarrow{\phi_\omega} V^* \xrightarrow{i^*} U^* .$$

We will now prove that, up to a choice of basis, every linear symplectic form has the form given in Example 2.3. This is often called the *Linear Darboux Theorem*, because it is the infinitesimal version of the Darboux Theorem for symplectic forms on manifolds, to be proved later.

**Theorem 2.5** (Linear Darboux Theorem) *Let $\omega$ be a symplectic form on a vector space $V$. Then there exists a basis*

$$(e_1, \ldots, e_n, f_1, \ldots, f_n)$$

*of $V$ with dual basis*

$$(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$$

*of $V^*$ such that $\omega$ is given by*

$$\omega = \sum_{i=1}^{n} \alpha_i \wedge \beta_i .$$

Such a basis of $V$ (or $V^*$) is called a *symplectic basis* with respect to $\omega$.

*Proof* Since $\omega$ is non-degenerate, it is not identically zero, so there exist vectors $e_1, f_1 \in V$ with $\omega(e_1, f_1) = 1$. We set

$$V_1 = \operatorname{span}\{e_1, f_1\} .$$

Since $\omega$ is non-degenerate on $V_1$, the symplectic orthogonal $V_1^{\perp\omega}$ intersects $V_1$ only in the zero vector, and is a complement to $V_1$; compare Lemma 2.10 below. We claim that the restriction $\omega|_{V_1^{\perp\omega}}$ is non-degenerate. For the proof suppose there exists a vector $v$ in $V_1^{\perp\omega}$ with

$$\omega\left(v, V_1^{\perp\omega}\right) = 0 .$$

Since we also have

$$\omega(v, V_1) = 0 ,$$

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and $V_1$ and $V_1^{\perp\omega}$ are complementary, this would give

$$\omega(v, V) = 0 .$$

By the non-degeneracy of $\omega$ on $V$ we conclude $v = 0$.

We can now find a symplectic basis for $V$ by induction on the dimension, replacing $V$ by $V_1^{\perp\omega}$ in the inductive step. □

**Corollary 2.6** *If $\omega$ is a symplectic form on a real vector space $V$, then the dimension of $V$ is even,* $\dim V = 2n$.

**Corollary 2.7** *A two-form $\omega$ on a vector space $V$ of dimension $2n$ is symplectic if and only if*

$$\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n} \in \Lambda^{2n} V^*$$

*is non-zero, i.e. a volume form on $V$. In particular, every symplectic vector space has a canonical orientation defined by $\omega^n$.*

*Proof* If $\omega$ is symplectic, we can choose a symplectic basis for $V$ and calculate $\omega^n$. We then see that $\omega^n \neq 0$. Conversely, assume that $\omega$ is not symplectic, so that there exists a non-zero vector $v \in V$ with $i_v\omega = 0$. Then also $i_v(\omega^n) = 0$, and $\omega^n$ is not a volume form. □

Structure-preserving maps of symplectic vector spaces are called symplectomorphisms.

**Definition 2.8** Let $(V, \omega_V)$ and $(W, \omega_W)$ be symplectic vector spaces. A linear isomorphism $f\colon V \to W$ is called a *symplectomorphism* if

$$f^*\omega_W = \omega_V .$$

If such an $f$ exists, then $(V, \omega_V)$ and $(W, \omega_W)$ are called *symplectomorphic*.

We can rephrase the Linear Darboux Theorem (Theorem 2.5) to state that all symplectic vector spaces of the same dimension are symplectomorphic to one another.

### 2.1.2 Subspaces in Symplectic Vector Spaces

Let $(V, \omega)$ be a symplectic vector space of dimension $2n$. We are interested in linear subspaces of $V$ that are in a special position with respect to $\omega$.

**Definition 2.9** Let $U \subset V$ be a linear subspace.

(i) We call $U$ *symplectic* if the restriction $\omega|_U$ is symplectic.

(ii) We call $U$ *isotropic* if the restriction $\omega|_U$ vanishes identically.

(iii) We call $U$ *Lagrangian* if it is isotropic and

$$\dim U = \frac{1}{2}\dim V = n.$$

In terms of the symplectic orthogonal, one has the following.

**Lemma 2.10** *Let $U \subset V$ be a linear subspace. The following hold:*

(i) $\dim U + \dim U^{\perp\omega} = \dim V$,

(ii) $(U^{\perp\omega})^{\perp\omega} = U$,

(iii) *$U$ is symplectic if and only if $U \cap U^{\perp\omega} = 0$,*

(iv) *$U$ is isotropic if and only if $U \subset U^{\perp\omega}$,*

(v) *if $U$ is isotropic, then* $\dim U \leq \frac{1}{2}\dim V$.

*Proof* Since $U^{\perp\omega}$ is the kernel of

$$V \xrightarrow{\phi_\omega} V^* \xrightarrow{i^*} U^* ,$$

$\phi_\omega$ is an isomorphism and $i^*$ is an epimorphism, we have

$$\dim U^{\perp\omega} = \dim V - \dim U^* .$$

This proves the first claim. For the second, we first prove $U \subset (U^{\perp\omega})^{\perp\omega}$. Fix $u \in U$. Then

$$\omega(u, v) = 0 \quad \forall v \in U^{\perp\omega} ,$$

hence $u \in (U^{\perp\omega})^{\perp\omega}$. Using part (i), this inclusion cannot be strict, and thus (ii) holds.

For the third claim, we have $U \cap U^{\perp\omega} \neq 0$ if and only if there exists a $v \in U$ such that $\omega(v, U) = 0$. But this happens if and only if $\omega|_U$ is not symplectic.

For the fourth claim, we have $U \subset U^{\perp\omega}$ if and only if $\omega(U, U) \equiv 0$, i.e. if and only if $U$ is isotropic.

Finally, if $U$ is isotropic, then by (i) and (iv) we have

$$\dim V = \dim U + \dim U^{\perp\omega} \geq 2\dim U ,$$

proving the fifth claim. □

**Corollary 2.11** *A linear subspace $U$ in a symplectic vector space $(V, \omega)$ is Lagrangian if and only if $U = U^{\perp\omega}$. A Lagrangian subspace is an isotropic subspace of maximal dimension.*

*Example* 2.12 Let $(V, \omega)$ be a symplectic vector space of dimension $2n$. Choose a symplectic basis

$$(e_1, \ldots, e_n, f_1, \ldots, f_n)$$

for $V$. Then

$$L = \mathrm{span}(e_1, \ldots, e_n)$$

and

$$L' = \mathrm{span}(f_1, \ldots, f_n)$$

are Lagrangian subspaces. Note that these subspaces are complementary, i.e. $L \oplus L' = V$.

Conversely we have the following result.

**Proposition 2.13** *Let $(V, \omega)$ be a symplectic vector space of dimension $2n$ and $L \subset V$ a Lagrangian subspace. Any basis $(e_1, \ldots, e_n)$ for $L$ can be completed to a symplectic basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ of $V$.*

*Proof* Consider the span $U = \mathrm{span}\{e_2, \ldots, e_n\}$. The symplectic orthogonal $U^{\perp\omega}$ has dimension $n+1$ and contains $L$, which has dimension $n$. Hence there exists a vector $f_1 \in U^{\perp\omega}$ that is not an element of $L$. It satisfies

$$\omega(e_i, f_1) = 0 \quad \forall i = 2, \ldots, n$$

but

$$\omega(e_1, f_1) \neq 0 \,,$$

since $f_1$ is not an element of $L$ and $L$ is maximally isotropic. Normalising $f_1$ we can assume that

$$\omega(e_1, f_1) = 1 \,.$$

As in the proof of Theorem 2.5, let

$$V_1 = \mathrm{span}\{e_1, f_1\} \,.$$

Then $\omega$ restricts to a symplectic form on the orthogonal $V_1^{\perp\omega}$ and $U$ is a Lagrangian subspace in this symplectic vector space. By induction, we construct a symplectic basis

$$(e_1, \ldots, e_n, f_1, \ldots, f_n)$$

for $V$. □

**Corollary 2.14** *Let $L \subset V$ be a Lagrangian subspace in a symplectic vector space. Then there exists a complementary Lagrangian subspace $L'$ with $V = L \oplus L'$.*

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We also have the following Linear Darboux Theorem for a pair $(V, L)$ consisting of a symplectic vector space and a Lagrangian subspace.

**Corollary 2.15** *Let $(V, \omega_V)$ and $(W, \omega_W)$ be symplectic vector spaces of the same dimension and assume that $L_V \subset V$ and $L_W \subset W$ are Lagrangian subspaces. Then there exists a symplectomorphism $f\colon V \to W$ taking $L_V$ onto $L_W$.*

We can adapt the proof of Proposition 2.13 to show the following:

**Proposition 2.16** *Let $(V, \omega)$ be a symplectic vector space of dimension $2n$ and $L$, $L' \subset V$ two complementary Lagrangian subspaces. For any basis $(e_1, \dots, e_n)$ of $L$ there exists a* unique *basis $(f_1, \dots, f_n)$ of $L'$, so that*

$$(e_1, \dots, e_n, f_1, \dots, f_n)$$

*is a symplectic basis of $V$.*

*Proof* In the proof of Proposition 2.13 the subspace $U^{\perp\omega}$ intersects $L'$ in a one-dimensional subspace. Hence there is a unique vector $f_1$ in this intersection with

$$\omega(e_1, f_1) = 1.$$

By induction, this implies the claim. □

### 2.1.3 The Space of Lagrangian Complements

Let $(V, \omega)$ be a symplectic vector space. In Corollary 2.14 we showed that every Lagrangian subspace $L \subset V$ has a Lagrangian complement $L'$, so that $L \oplus L' = V$. However, the Lagrangian complement $L'$ is not unique. For the applications to Lagrangian distributions on manifolds, it is important to understand the space of all Lagrangian complements to a given Lagrangian subspace $L$.

We will approach this issue using the following construction of a tautological symplectic vector space. Let $W$ be an arbitrary real vector space of dimension $n$, and $V = W \oplus W^*$. Elements of $V$ consist of pairs $(w, \lambda) \in W \oplus W^*$. We define a map

$$\omega_0\colon V \times V \longrightarrow \mathbb{R}$$

by

$$\omega_0((w_1, \lambda_1), (w_2, \lambda_2)) = \lambda_1(w_2) - \lambda_2(w_1)\,.$$

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It is clear that $\omega_0$ is a skew-symmetric two-form on $V$. Moreover, it is non-degenerate and therefore symplectic. To see this, note that the map

$$\phi_{\omega_0}\colon W \oplus W^* \longrightarrow W^* \oplus W$$

is given by

$$(w, \lambda) \longmapsto (\lambda, -w) ,$$

which is an isomorphism.

It is clear that $W$ and $W^*$ are Lagrangian subspaces of $(V, \omega_0)$. Corollary 2.15 tells us that, for any pair consisting of a symplectic vector space $V$ of dimension $2n$ and a Lagrangian subspace $W$, there exists a symplectomorphism onto the pair $(V, W)$. To understand the space of Lagrangian complements for a Lagrangian in a general symplectic vector space it therefore suffices to understand the space of Lagrangian complements to $W$ in $(V, \omega_0)$.

**Lemma 2.17** *A vector subspace $A \subset V$ is complementary to $W$ if and only if $\pi_2|_A\colon A \to W^*$ is an isomorphism. If $A$ is complementary to $W$, then $A$ is the graph of a uniquely determined linear map $\alpha\colon W^* \to W$.*

*Proof* The subspace $A$ is complementary to $W$ if and only if $A \cap W = 0$ and $A$ has dimension $n$. The first fact is equivalent to $\pi_2|_A$ being injective. This implies the first claim.

Suppose $A$ is complementary to $W$. Then the inverse of $\pi_2|_A$ is of the form

$$\begin{aligned}(\pi_2|_A)^{-1}\colon W^* &\longrightarrow A \subset W \oplus W^* \\ \lambda &\longmapsto \quad (\alpha(\lambda), \lambda) .\end{aligned}$$

Since the inverse of $\pi_2|_A$ is linear, the map $\alpha$ itself is linear. This proves the second claim. □

We want to understand when a complementary subspace $A$ is Lagrangian. For this we use the natural pairing

$$\begin{aligned}\langle \cdot , \cdot \rangle\colon W \times W^* &\longrightarrow \mathbb{R} \\ (w, \lambda) &\longmapsto \lambda(w) .\end{aligned}$$

A linear map $\alpha\colon W^* \to W$ is *self-adjoint* with respect to this pairing if

$$\langle \alpha(\mu), \lambda \rangle = \langle \alpha(\lambda), \mu \rangle \quad \forall \mu, \lambda \in W^* .$$

We then have the following result.

**Proposition 2.18** *Let $A \subset V = W \oplus W^*$ be a linear subspace complementary to $W$, defined as the graph of a linear map $\alpha\colon W^* \to W$. Then $A$ is Lagrangian with respect to $\omega_0$ if and only if $\alpha$ is self-adjoint.*

*Proof* The subspace $A$ is Lagrangian if and only if $\omega_0$ vanishes on all pairs of vectors $(\alpha(\lambda), \lambda)$ and $(\alpha(\mu), \mu)$ in $V$. For the evaluation we have

$$\lambda(\alpha(\mu)) - \mu(\alpha(\lambda)) = \langle \alpha(\mu), \lambda \rangle - \langle \alpha(\lambda), \mu \rangle \, .$$

This vanishes if and only if $\alpha$ is self-adjoint. □

Since there is a one-to-one correspondence between complements $A$ and linear maps $\alpha \colon W^* \to W$, the space of all Lagrangian complements to $W$ can be identified with the vector space of symmetric, real $(n \times n)$-matrices.

We summarise this discussion in the following result.

**Theorem 2.19** *Let $(V, \omega)$ be an arbitrary symplectic vector space of dimension $2n$ and $L \subset V$ a Lagrangian subspace. Then the space $\mathcal{L}(V, \omega, L)$ of all Lagrangian subspaces $L' \subset V$ complementary to $L$ is a real vector space of dimension $\frac{1}{2}n(n+1)$.*

### 2.1.4 Compatible Complex Structures

Recall that a complex structure on a real vector space $V$ is an isomorphism $J \colon V \to V$ such that $J^2 = -\mathrm{Id}_V$. This makes $V$ into a complex vector space by declaring scalar multiplication by $i \in \mathbb{C}$ to be the application of $J$.

If $V$ is a symplectic vector space, there are compatibility conditions one can impose on the symplectic and complex structures.

**Definition 2.20** Let $(V, \omega)$ be a symplectic vector space. A complex structure $J \colon V \to V$ is *tamed* by the symplectic form if

$$\omega(v, Jv) > 0 \quad \forall v \neq 0 \in V \, .$$

A complex structure $J$ on $V$ is called *compatible* with the symplectic form $\omega$ if

$$g_J(v, w) = \omega(v, Jw)$$

defines a positive-definite scalar product $g_J$ on $V$.

We denote by $\mathcal{J}(V, \omega)$ the space of all complex structures on $V$ compatible with the given symplectic form $\omega$.

**Lemma 2.21** *A complex structure $J$ on $V$ is compatible with a symplectic form $\omega$ if and only if it is tamed by $\omega$ and $\omega$ is $J$-invariant in the sense that*

$$\omega(Jv, Jw) = \omega(v, w) \quad \forall v, w \in V \, .$$

*Proof* The complex structure is compatible with the symplectic form $\omega$ if and only if $g_J$ is symmetric and positive definite, i.e.

$$g_J(v,w) = g_J(w,v) \quad \forall v,w \in V ,$$
$$g_J(v,v) > 0 \quad \forall v \neq 0 \in V .$$

The second condition is equivalent to $J$ being tamed by $\omega$ and the first condition is equivalent to

$$\omega(Jv,Jw) = \omega(v,w) \quad \forall v,w \in V .$$

□

**Proposition 2.22** *Let $(V,\omega)$ be a symplectic vector space. Then there exists a compatible complex structure $J$.*

*Proof* According to the Linear Darboux Theorem (Theorem 2.5) there exists a symplectic basis

$$(e_1,\dots,e_n,f_1,\dots,f_n)$$

for $V$. Then $J\colon V \to V$, defined by

$$Je_i = f_i ,$$
$$Jf_i = -e_i ,$$

is a compatible complex structure. □

We want to understand the space $\mathcal{J}(V,\omega)$ of all compatible complex structures. In particular, we want to show that this space is contractible. There are several ways to prove this; the way we do it here involves Lagrangian subspaces.

**Lemma 2.23** *Let $(V,\omega)$ be a symplectic vector space of dimension $2n$ with a compatible complex structure $J$. If $L \subset V$ is a Lagrangian subspace then $JL$ is a complementary $g_J$-orthogonal Lagrangian subspace to $L$.*

*Proof* Since $J$ is an isomorphism, the subspace $JL$ has dimension $n$. It is Lagrangian, because

$$\omega(Jv,Jw) = \omega(v,w) = 0 \quad \forall v,w \in L .$$

Furthermore, it is $g_J$-orthogonal to $L$, for if $v \in L$ and $w \in JL$, then $Jw \in L$ and

$$g_J(v,w) = \omega(v,Jw) = 0 .$$

□

**Proposition 2.24** *Let $(V, \omega)$ be a symplectic vector space and $L \subset V$ a Lagrangian subspace. For every positive-definite scalar product $h$ on $L$ and every Lagrangian complement $L'$ there exists a unique, $\omega$-compatible complex structure $J$ on $V$ such that $L' = JL$ and $g_J|_L = h$.*

*Proof* We first prove the existence of $J$. To do so we choose an $h$-orthonormal basis $(e_1, \ldots, e_n)$ for $L$ and let $(f_1, \ldots, f_n)$ be the unique basis for $L'$ so that both bases together form a symplectic basis for $V$; see Proposition 2.16. We then define a complex structure $J$ on $V$ by

$$\begin{aligned} Je_i &= f_i \,, \\ Jf_i &= -e_i \,. \end{aligned}$$

By definition, we have $JL = L'$. It is clear that $J$ is compatible with $\omega$, because the basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ is symplectic. In addition, $(e_1, \ldots, e_n)$ is an orthonormal basis of $L$ for both $h$ and $g_J|_L$, hence $g_J|_L = h$.

To show uniqueness, suppose that $J$ and $J'$ are two complex structures that satisfy the condition in the proposition. Choose again an $h$-orthonormal basis $(e_1, \ldots, e_n)$ for $L$, and let

$$\begin{aligned} f_j &= Je_j \,, \\ f_j' &= J'e_j \,. \end{aligned}$$

The vectors $\{f_j\}$ and $\{f_j'\}$ each form a basis for $L'$. It suffices to show that

$$f_j = f_j'$$

for all indices $j$, since then $J \equiv J'$ on all of $V$.

By the uniqueness statement of Proposition 2.16 it suffices to show that

$$(e_1, \ldots, e_n, f_1, \ldots, f_n)$$

and

$$(e_1, \ldots, e_n, f_1', \ldots, f_n')$$

are symplectic bases for $V$. We show this for the first basis; the argument is the same for the second basis. We have

$$\omega(e_i, e_j) = 0 = \omega(f_i, f_j) \,,$$

since $L$ and $L'$ are Lagrangian. Furthermore,

$$\omega(e_i, f_j) = \omega(e_i, Je_j) = g_J(e_i, e_j) = h(e_i, e_j) = \delta_{ij} \,,$$

since $J$ is $\omega$-compatible. This proves the claim. □

We can now describe the space of compatible complex structures.

**Theorem 2.25** *Let* $(V, \omega)$ *be a symplectic vector space. Fix some Lagrangian subspace* $L$ *and denote by* $\mathrm{Met}(L)$ *the space of all positive-definite scalar products on* $L$*. Then the map*

$$\begin{aligned} F\colon \mathcal{J}(V,\omega) &\longrightarrow \mathcal{L}(V,\omega,L)\times\mathrm{Met}(L)\\ J &\longmapsto \qquad (JL, g_J|_L) \end{aligned}$$

*is a bijection, and a homeomorphism with respect to the natural topologies.*

*In particular, the space* $\mathcal{J}(V,\omega)$ *of complex structures compatible with* $\omega$ *is contractible.*

*Proof* The map $F$ is well defined and continuous. In Proposition 2.24 we constructed the continuous inverse.

By Theorem 2.19 the space $\mathcal{L}(V,\omega,L)$ is a vector space, and therefore is contractible. The space of metrics $\mathrm{Met}(L)$ is not a vector space, but is convex, and therefore contractible as well. □

To end this subsection, we need to discuss the relationship between orientations and Lagrangian splittings of symplectic vector spaces. Recall that a symplectic vector space $(V,\omega)$ of dimension $2n$ has a canonical orientation defined by $\omega^n$. This can also be thought of as the orientation defined by a compatible complex structure $J$.

Suppose that we are given a decomposition $V = L \oplus L'$ into complementary Lagrangian subspaces. Then it is possible to choose $J$ so that it maps $L$ isomorphically to $L'$, and so any Lagrangian splitting has the form $V = L \oplus L$.

**Lemma 2.26** *Let* $(V,\omega)$ *be a symplectic vector space of dimension* $2n$ *and* $V = L \oplus L$ *a splitting into complementary Lagrangian subspaces. Choose an orientation of* $L$*. Then the product orientation on* $L \oplus L$ *differs from the symplectic orientation of* $V$ *by the sign*

$$\epsilon(n) = (-1)^{\frac{n(n-1)}{2}}\ .$$

*Proof* We can choose a symplectic basis

$$(e_1,\dots,e_n,f_1,\dots,f_n)$$

for $V$, where the $e_i$ are an oriented basis for $L$, and the $f_i$ are an oriented basis for $L' = L$. The symplectic orientation of $V$ corresponds to

$$e_1 \wedge f_1 \wedge \ldots \wedge e_n \wedge f_n$$

and we need

$$1 + 2 + 3 + 4 + \cdots + (n-1) = \frac{1}{2}n(n-1)$$

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many transpositions to change

$$e_1 \wedge e_2 \wedge \ldots \wedge e_n \wedge f_1 \wedge f_2 \wedge \ldots \wedge f_n$$

to the symplectic orientation. This proves the claim. □

## 2.2 Symplectic and Complex Vector Bundles

Let $\pi\colon E \longrightarrow M$ be a smooth real vector bundle over a smooth manifold $M$.

**Definition 2.27** A *symplectic structure* on the vector bundle $E$ is a smooth section $\omega \in \Gamma(\Lambda^2 E^*)$ with the property that, for all $x \in M$, the form $\omega(x)$ is symplectic on the fibre $E_x$.

In other words, $\omega$ is a fibrewise symplectic form on the fibres of $E$ varying smoothly in the neighbourhood of any point. We will sometimes refer to the pair $(E, \omega)$ as a symplectic vector bundle.

Since symplectic vector spaces are even-dimensional and canonically oriented by the top power of the symplectic form, every symplectic vector bundle is oriented and of even rank $2n$.

**Definition 2.28** A *complex structure* on the vector bundle $E$ is a smooth section $J \in \Gamma(\mathrm{End}(E))$ with the property that, for all $x \in M$, the endomorphism $J(x)$ is a complex structure on the fibre $E_x$.

Such a $J$ makes $E$ into a complex vector bundle, with scalar multiplication by $i$ being given by the application of $J$.

It turns out that every symplectic vector bundle is a complex vector bundle in an essentially unique way. To discuss this we use compatibility between symplectic forms and complex structures, which can be formulated for bundles in the same way as we did for single vector spaces.

**Theorem 2.29** *Let $(E, \omega)$ be a symplectic vector bundle. Then there exists a compatible complex structure $J$ on $E$. Moreover, this complex structure is unique up to homotopy.*

*Proof* Let $\mathcal{J}(E, \omega) \to M$ be the locally trivial fibre bundle associated to $E$ whose fibre over $x$ consists of the space $\mathcal{J}(E_x, \omega(x))$ of compatible complex structures. According to Theorem 2.25, the fibres of this bundle are contractible. Therefore the bundle admits a section, and, moreover, any two sections are homotopic through sections. □

Conversely, if we are given a complex structure $J$ on an arbitrary real vector bundle $E$, we can always choose a $J$-invariant positive-definite fibre-wise scalar product $g$. Then

$$\omega(v, w) = g(Jv, w)$$

is skew-symmetric. Since $J$ is invertible and $g$ is non-degenerate, it follows that $\omega$ is non-degenerate and therefore a symplectic structure on $E$. Moreover, $J$ is compatible with this symplectic structure. The space of possible scalar products $g$ in this construction is convex, and so up to homotopy $\omega$ is independent of $g$ and depends only on $J$.

To summarise, symplectic and complex vector bundles are really the same (up to suitable notions of equivalence). We want to extend this equivalence to include Lagrangian subbundles on the symplectic side.

**Definition 2.30** Let $(E, \omega)$ be a symplectic vector bundle over a smooth manifold $M$. A subbundle $L \subset E$ is called *Lagrangian* if the fibre $L_x$ is Lagrangian in the symplectic vector space $(E_x, \omega(x))$, for all $x \in M$.

**Proposition 2.31** *Let $(E, \omega)$ be a symplectic vector bundle over a smooth manifold $M$, and $L \subset E$ a Lagrangian subbundle. Then there exists a complementary Lagrangian subbundle $L'$, so that $L \oplus L' = E$.*

*Proof* Let $\mathcal{L}(E, \omega, L) \to M$ be the smooth fibre bundle associated to $E$ and $L$ whose fibre over $x \in M$ consists of the space $\mathcal{L}(E_x, \omega(x), L_x)$ of Lagrangian subspaces in $E_x$ complementary to $L_x$. Since the fibres of this bundle are contractible according to Theorem 2.19, the bundle has a global section $L'$ over $M$. □

With Proposition 2.24 we obtain the following.

**Theorem 2.32** *Let $(E, \omega)$ be a symplectic vector bundle of rank $2n$ over a smooth manifold $M$. Suppose $E = L \oplus L'$ is a splitting of $E$ into two complementary Lagrangian subbundles. Choose an arbitrary, positive-definite bundle metric $h$ on $L$. Then there exists a unique complex structure $J$ on $E$, compatible with $\omega$, such that $L' = JL$ and $g_J|_L = h$.*

Recall that a totally real subbundle $F \subset E$ in a complex vector bundle $(E, J)$ is a subbundle with the property that $J(F_x) \cap F_x = 0$ for all $x \in M$. By the following result, totally real subbundles of maximal rank correspond to Lagrangian subbundles in symplectic vector bundles.

**Theorem 2.33** *A symplectic vector bundle $(E, \omega)$ of rank $2n$ admits a Lagrangian subbundle $L \subset E$ if and only if the corresponding complex vector bundle $(E, J)$ admits a totally real subbundle $F$ of rank $n$.*

*Proof* Given a Lagrangian subbundle $L$, the previous theorem gives a complex structure $J$ for which $L$ is totally real. Conversely, given a totally real subbundle $F$ in $(E, J)$, of maximal rank $n$, we can choose a $J$-invariant metric $g$ so that $F$ and $J(F)$ are $g$-orthogonal. Then the symplectic structure $\omega$ defined by $\omega(v, w) = g(Jv, w)$ on $E$ has $F$ as a Lagrangian subbundle. □

Since a complex structure $J\colon E \to E$ is an orientation-preserving bundle isomorphism, Lemma 2.26 implies the following.

**Corollary 2.34** *Let $(E, \omega)$ be a symplectic vector bundle of rank $2n$ over a smooth manifold $M$. Suppose $E = L \oplus L'$ is a splitting of $E$ into two complementary Lagrangian subbundles.*

(i) *The vector bundles $L$ and $L'$ are isomorphic as real,* unoriented *vector bundles over $M$.*
(ii) *If $L$ is orientable and we fix an orientation, then the product orientation on $L \oplus L$ differs from the symplectic orientation of $E$ by the sign*

$$\epsilon(n) = (-1)^{\frac{n(n-1)}{2}} .$$

## 2.3 Künneth Vector Bundles

We can now define linear Künneth or bi-Lagrangian structures on vector bundles.

**Definition 2.35** A *Künneth vector bundle* is a symplectic vector bundle $(E, \omega)$ together with a splitting $E = L \oplus L'$ into complementary Lagrangian subbundles.

Note that an $\omega$-compatible $J$ can be chosen so that it gives an isomorphism between $L$ and $L'$, and so the Lagrangian splitting always has the form $L \oplus L$. We will sometimes refer to the triple $(E, \omega, L)$ as a (linear) *Künneth structure*.

The definition of a Künneth structure can be reformulated in several ways.

**Proposition 2.36** *Let $(E, \omega)$ be a symplectic vector bundle of rank $2n$. The following conditions are equivalent:*

(i) $(E, \omega)$ *admits a Künneth structure,*
(ii) $(E, \omega)$ *admits a Lagrangian subbundle $L \subset E$,*
(iii) *the corresponding complex vector bundle $(E, J)$ admits a totally real subbundle of rank $n$,*

(iv) *the corresponding complex vector bundle* $(E, J)$ *is isomorphic to the complexification* $L \otimes_{\mathbb{R}} \mathbb{C}$ *of a real vector bundle* $L$ *of rank* $n$.

*Proof* Clearly the first condition implies the second. Moreover, the existence of Lagrangian complements in Proposition 2.31 gives the converse.

The second and third conditions are equivalent by Theorem 2.33.

The third condition implies the fourth since if $F \subset E$ is totally real for $J$, then $(E, J)$ is $\mathbb{C}$-linearly isomorphic to $F \otimes_{\mathbb{R}} \mathbb{C}$. Conversely, if $(E, J)$ is isomorphic to $L \otimes_{\mathbb{R}} \mathbb{C}$, then $E = L \oplus iL$, and both summands are totally real. □

A Künneth vector bundle is in particular symplectic and therefore orientable and oriented. However, the Lagrangian subbundle $L$ may very well be non-orientable. This motivates the following definition.

**Definition 2.37** A Künneth structure $(E, \omega, L)$ is *orientable* if $L$ is an orientable vector bundle.

The existence of a Künneth structure on a vector bundle will impose restrictions on its characteristic classes. As usual, we will call Chern classes of a symplectic vector bundle $(E, \omega)$ the Chern classes of the corresponding complex vector bundle $(E, J)$. Since $J$ is unique up to homotopy, the Chern classes are independent of the exact choice we make for $J$.

**Theorem 2.38** *Let* $(E, \omega)$ *be a symplectic vector bundle admitting a Lagrangian subbundle* $L \subset E$. *Then the odd-degree Chern classes* $c_{2i+1}(E) \in H^{4i+2}(M; \mathbb{Z})$ *are two-torsion classes. If the Lagrangian subbundle* $L$ *is orientable, then* $c_1(E) = 0$.

*Proof* Under the assumption of the theorem, the complex vector bundle $(E, J)$ is the complexification of $L$, and its underlying real bundle is isomorphic to $L \oplus L$. It follows that

$$c_{2i+1}(E) = \beta(w_{2i}(L) \cup w_{2i+1}(L)) \,, \tag{2.1}$$

where the $w_j$ denote the Stiefel–Whitney classes and

$$\beta\colon H^{4i+1}(M; \mathbb{Z}_2) \longrightarrow H^{4i+2}(M; \mathbb{Z})$$

is the Bockstein homomorphism associated to multiplication by 2 in the coefficients. See for example Problem 15-D in [MS-74].

If $L$ is orientable, so that $w_1(L) = 0$, then we obtain

$$c_1(E) = \beta(w_1(L)) = 0 \in H^2(M; \mathbb{Z}) \,. \tag{2.2}$$

□

This is of course a strong obstruction to the existence of orientable Künneth structures, because most real vector bundles do not admit a complex structure with vanishing first Chern class.

Another obstruction comes from the Euler class:

**Corollary 2.39** *Let $E$ be a Künneth vector bundle of real rank $4k + 2$. Then the Euler class $e(E)$ is a two-torsion class.*

*Proof* For any complex vector bundle, the Euler class of the underlying oriented real vector bundle equals the top Chern class. By (2.1) this is a 2-torsion class, since we assumed that the complex rank of $E$ was odd. □

If $(E, \omega)$ is a Künneth bundle with Lagrangian subbundle $L$, then, as a real unoriented vector bundle, $E$ is isomorphic to $L \oplus L$. If $L$ is orientable and oriented, then $L \oplus L$ has a product orientation induced from that of $L$. By Corollary 2.34 this agrees with the canonical, symplectic or complex, orientation of $E$ after multiplication by

$$\epsilon(n) = (-1)^{\frac{n(n-1)}{2}} ,$$

where $n$ is the complex rank of $E$, which is the real rank of $L$. Thus the isomorphism $E \cong L \oplus L$ is orientation-preserving if $\epsilon(n) = 1$, and orientation-reversing otherwise. If $\epsilon(n) = -1$, we can think of the oriented bundle $E$ as $L \oplus \overline{L}$, where $\overline{L}$ denotes $L$ endowed with the reversed orientation.

Now, by the Whitney sum formula for the Euler class, we have

$$e(E) = e(L) \cup \epsilon(n)e(L) = \epsilon(n)e(L)^2 . \tag{2.3}$$

If $n$ is odd, then $e(L)$ is a two-torsion class, and so is $e(E)$, which is what we saw above. However, when $n$ is even, $e(L)^2$ may well be non-torsion.

## Notes for Chapter 2

1. Using a different terminology, Künneth vector bundles were considered by Bejan in [Bej-93].
2. Dazord [Daz-81] claimed that the (real) Euler class of any Künneth bundle vanishes. His argument was that the Chern–Weil integrand vanishes identically if one chooses an orthogonal connection adapted to the splitting $E = L \oplus L$. A moment's thought about permutations shows that this argument requires the same dimension assumption as Proposition 2.39, when the claim reduces to (2.1). In the Erratum [Daz-85], Dazord mentions the dimension assumption, and then goes on to claim that $e(E_{\mathbb{R}}) = e(L)^2$, missing the sign in (2.3). This sign will be crucial in our considerations of tangent bundles of four-manifolds in Chapter 10.