ON A CONJECTURE OF HANNA NEUMANN

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1. Introduction. In what follows, groups are written additively, and commutators are denoted by brackets:

$$[x, y] = -x - y + x + y.$$

A group is *metabelian* if it satisfies the law

$$[[w, x], [y, z]] = 0.$$

It is conceivable, though not plausible, that this law is equivalent to a law, or a set of laws, in only three variables, or even two. The present note shows that this is not the case.

To describe the situation more precisely, let us call a group *n-metabelian* if every subgroup with n (or fewer) generators is metabelian. Every group is, trivially, 1-metabelian; and every 4-metabelian group is evidently metabelian. Then " to be 3-metabelian " is on the face of it a less restrictive property than " to be metabelian ", and " to be 2-metabelian " appears less restrictive still. If $\mathfrak{M}_n$ denotes the class † of all n-metabelian groups, then we have the following relations:

$$\mathfrak{M}_1 \supset \mathfrak{M}_2 \supseteq \mathfrak{M}_3 \supseteq \mathfrak{M}_4 = \mathfrak{M}_5 = \dots .$$

Hanna Neumann, in a study † of laws in groups, has conjectured that the inclusions are proper, i.e. that

$$\mathfrak{M}_2 \neq \mathfrak{M}_3 \neq \mathfrak{M}_4; \qquad \text{(1)}$$

differently put, she has conjectured the existence of 2-metabelian groups which are not 3-metabelian, and of 3-metabelian groups which are not metabelian. We here confirm this conjecture by constructing examples of such groups. Our examples have been chosen as finite groups of comparatively small orders, the orders being, moreover, powers of 2; as a consequence our groups belong to other, smaller, varieties as well. Thus, for example, our groups will be seen to have exponent 8; if $\mathfrak{B}_8$ denotes the class (" Burnside variety ") of all groups of exponent 8, then our examples show even more than (1), namely

$$\mathfrak{B}_8 \cap \mathfrak{M}_2 \neq \mathfrak{B}_8 \cap \mathfrak{M}_3 \neq \mathfrak{B}_8 \cap \mathfrak{M}_4;$$

and other, related, inequalities can also be derived. For a fuller discussion of the significance of these facts the reader is referred to Hanna Neumann's paper.‡

2. The first example. Our first example is indeed a familiar group, namely the 2-Sylow subgroup of the symmetric group of degree 8. Let us denote it by G. It can be generated by three involutory permutations, namely

$$a_1 = (1\ 2),$$
$$a_2 = (1\ 3)(2\ 4),$$
$$a_3 = (1\ 5)(2\ 6)(3\ 7)(4\ 8).$$

† This is in fact a *variety* of groups in the sense of Philip Hall.

‡ An account of her results will be found in " On varieties of groups and their associated near-rings ", *Math. Z.* **65** (1956), 36–39.

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Put
$$\begin{aligned} b_{12} &= [a_1, a_2] = (1\,2)(3\,4), \\ b_{13} &= [a_1, a_3] = (1\,2)(5\,6), \\ b_{23} &= [a_2, a_3] = (1\,3)(2\,4)(5\,7)(6\,8), \\ c &= [b_{12}, a_3] = (1\,2)(3\,4)(5\,6)(7\,8). \end{aligned}$$
The derived group $G' = [G, G]$ is generated by b_{12}, b_{13}, b_{23} and c, and its order is
$$|G'| = 2^4.$$
The factor group G/G' is elementary abelian of order
$$|G/G'| = 2^3,$$
and can not be generated by fewer than 3 elements. Hence G itself can not be generated by fewer than 3 elements. One easily verifies that
$$[b_{13}, b_{23}] = c\,;$$
hence G' is not abelian, and G is not metabelian. In fact
$$[G', G'] = G'' = \{c\},$$
and this is also the centre of G, and the fourth lower central group ${}^4G$ (defined inductively by ${}^1G = G$ and ${}^{n+1}G = [{}^nG, G]$). It is, moreover, the unique minimal (non-trivial) normal subgroup of G; to see this we write an arbitrary element $g \in G$ in the form
$$g = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \beta_{12} b_{12} + \beta_{13} b_{13} + \beta_{23} b_{23} + \gamma c,$$
the coefficients being taken modulo 2, and we consider the normal closure $\{g\}^G$ of g in G. Now
$$[b_{12}, g] = \alpha_3 c\,;$$
thus $c \in \{g\}^G$ if $\alpha_3 \neq 0$. Next, if $\alpha_3 = 0$,
$$\begin{aligned} [b_{13}, g] &= \alpha_2 b_{12} + \beta_{23} c, \\ [b_{23}, g] &= \alpha_1 b_{12} + \beta_{13} c. \end{aligned}$$
If $\alpha_2 = 0$ but $\beta_{23} \neq 0$, or if $\alpha_1 = 0$ but $\beta_{13} \neq 0$, then $c \in \{g\}^G$; if $\alpha_2 \neq 0$ or $\alpha_1 \neq 0$, then b_{12} or $b_{12} + c$, and hence also
$$[b_{12}, a_3] = [b_{12} + c, a_3] = c,$$
lies in $\{g\}^G$. Thus there remains only the case that $\alpha_1 = \alpha_2 = \alpha_3 = \beta_{13} = \beta_{23} = 0$. If $\beta_{12} \neq 0$, then the last argument again shows that $c \in \{g\}^G$. If $\beta_{12} = 0$ and $\gamma = 1$, then $g = c \in g^G$. If finally $\gamma = 0$ as well, then g is zero element and $\{g\}^G$ is trivial. This shows, then, that every non-trivial normal subgroup of G contains c, and using our remark that
$$\{c\} = G'' = {}^4G,$$
we have the following result:

LEMMA 1. *Every proper factor group of G is metabelian and nilpotent of class at most* 3.

In the parlance used elsewhere,† this means that G is "just soluble of length 3" and "just nilpotent of class 4". This is, however, only an incidental result: our real aim is information on the subgroups rather than the factor groups of G.

Let H be a maximal subgroup of G. Then $G' \subseteq H$, and $|G : H| = 2$. As H/G' is isomorphic to the four-group, we can generate H by the elements of G' and two further elements, h and k, say. The derived group H' is then generated, modulo commutators of weight 3, that is,

† "Ascending derived series", to be published in *Compositio Math.*

modulo ${}^3G$, by $[h, k]$ and its conjugates in H. Now the conjugates of any commutator are congruent to that commutator modulo ${}^3G$; hence H' is generated by $[h, k]$ modulo ${}^3G$, and

$$H' \subseteq \{{}^3G, [h, k]\}.$$

Next we notice that all commutators of weight 5 or more vanish; therefore ${}^3G$ lies in the centre of G' and thus also in the centre of $\{{}^3G, [h, k]\}$; but then this group must be abelian. It follows that H' is abelian, and that H is metabelian. As every proper subgroup of G is contained in some maximal subgroup H, we have the following result.

LEMMA 2. *Every proper subgroup of G is metabelian.*

Every two-generator subgroup of G is proper, because G requires three generators; hence we have the desired result:

COROLLARY 3. *The 2-Sylow subgroup of the symmetric group of degree 8 is 2-metabelian but not 3-metabelian.*

As already stated in the introduction, a sharper result can be obtained from the example by using the fact that it also belongs to other varieties. If $\mathfrak{V}$ denotes the least variety of groups that contains G, that is the variety defined by *all* the laws valid in G, then

$$\mathfrak{V} \subseteq \mathfrak{M}_2, \quad \mathfrak{V} \not\subseteq \mathfrak{M}_3. \qquad \text{(2)}$$

But it may be laborious to determine $\mathfrak{V}$; we may therefore also content ourselves with the result we obtain when we remark that G has exponent 8 and nilpotent class 4: denoting by $\mathfrak{V}_0$ the variety of the groups of exponent 8 and nilpotent class 4, we then have

COROLLARY 4. $\mathfrak{V}_0 \cap \mathfrak{M}_2 \neq \mathfrak{V}_0 \cap \mathfrak{M}_3$.

3. The second example. As our second example we take the group, again denoted by G, which is generated by the 14 elements

$$a_1, a_2, a_3, a_4, b_{12}, b_{13}, b_{14}, b_{23}, b_{24}, b_{34}, c_1, c_2, c_3, d, \qquad \text{(3)}$$

subject to the following 60 defining relations:

$$\begin{aligned}
&[a_i, a_j] = b_{ij} \quad (1 \leqslant i < j \leqslant 4);\\
&[a_i, b_{ij}] = [a_j, b_{ij}] = 0 \quad (1 \leqslant i < j \leqslant 4);\\
&[a_2, b_{34}] = [a_4, b_{23}] = c_1; \ [a_3, b_{24}] = 0;\\
&[a_3, b_{14}] = [a_4, b_{13}] = c_2; \ [a_1, b_{34}] = 0;\\
&[a_1, b_{24}] = [a_4, b_{12}] = c_3; \ [a_2, b_{14}] = 0;\\
&[a_1, b_{23}] = [a_2, b_{13}] = [a_3, b_{12}] = 0;\\
&[a_i, c_k] = 0 \quad (i \neq k, 1 \leqslant i \leqslant 4, 1 \leqslant k \leqslant 3);\\
&[a_1, c_1] = [a_2, c_2] = [a_3, c_3] = d;\\
&[a_i, d] = 0 \quad (1 \leqslant i \leqslant 4);\\
&2a_i = 2b_{ij} = 2c_k = 2d = 0,
\end{aligned}$$

for all the generators (3). The following facts can be readily verified from these equations.

(i) G is generated by a_1, a_2, a_3, a_4;

(ii) ${}^2G = G'$ is generated by the b_{ij}, c_k, and d;

(iii) ${}^3G$ is generated by the c_k and d;

(iv) ${}^4G$ is generated by d;

(v) d lies in the centre of G; hence ${}^5G = 0$;

(vi) G/G' is elementary abelian of order 2^4; hence

(vii) G cannot be generated by fewer than 4 elements.

Every element of G can be written in the form

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$$g=\sum_i \alpha_i a_i+\sum_{i<j}\beta_{ij}b_{ij}+\sum_k \gamma_k c_k+\delta d, \quad\text{.................................(4)}$$

where the coefficients α_i, β_{ij}, γ_k, δ are integers modulo 2, and where the generators are arranged as in (3) ; hence the order of G is at most 2^{14}. To verify that this is in fact the order of G we associate with each element h of G a mapping $\rho(h)$ of the set of formal sums (4) into itself, carrying g into

$$g\rho(h)=\sum_i(\alpha_i+\kappa_i)a_i+\sum_{i<j}(\beta_{ij}+\lambda_{ij})b_{ij}+\sum_k(\gamma_k+\mu_k)c_k+(\delta+\nu)d.$$

The following table gives $\rho(h)$ for the first 7 generators (3).

	$\rho(a_1)$	$\rho(a_2)$	$\rho(a_3)$	$\rho(a_4)$	$\rho(b_{12})$	$\rho(b_{13})$	$\rho(b_{14})$
κ_1	1	0	0	0	0	0	0
κ_2	0	1	0	0	0	0	0
κ_3	0	0	1	0	0	0	0
κ_4	0	0	0	1	0	0	0
λ_{12}	α_2	0	0	0	1	0	0
λ_{13}	α_3	0	0	0	0	1	0
λ_{14}	α_4	0	0	0	0	0	1
λ_{23}	0	α_3	0	0	0	0	0
λ_{24}	0	α_4	0	0	0	0	0
λ_{34}	0	0	α_4	0	0	0	0
μ_1	0	$\alpha_3\alpha_4+\beta_{34}$	0	β_{23}	0	0	0
μ_2	$\alpha_3\alpha_4$	0	β_{14}	β_{13}	0	0	0
μ_3	$\alpha_2\beta_4+\beta_{24}$	0	0	β_{12}	0	0	0
ν	γ_1	$\alpha_3\beta_{14}+\alpha_4\beta_{13}+\gamma_2$	$\alpha_4\beta_{12}+\gamma_3$	0	β_{34}	β_{24}	β_{23}

For the remaining generators, all entries but one are 0, the only 1 occurring in the place corresponding to the generator, that is, at λ_{23}, λ_{24}, λ_{34}, μ_1, μ_2, μ_3, ν for $\rho(b_{23})$, $\rho(b_{24})$, $\rho(b_{34})$, $\rho(c_1)$, $\rho(c_2)$, $\rho(c_3)$, $\rho(d)$ respectively. As already stated, the coefficients are calculated modulo 2. If we interpret the formal sums (4) as 14-dimensional vectors over $GF(2)$, the mappings are not linear ; but they are permutations of the system of vectors, because each of them is easily seen to be involutory. A straightforward but tedious verification shows that the mappings associated with the generators (3) satisfy all the defining relations of G. Also the permutation group they generate is transitive on the system of 2^{14} vectors : it is not difficult to see that the null vector is carried into the vector (4) by the product of mappings corresponding to the generators that appear with coefficient 1 in (4), taken always in the order (3). Thus the mappings generate a group of order 2^{14} at least which is, on the other hand, a factor group of G and thus of order 2^{14} at most. It follows that the order of G is 2^{14} precisely, and that the group of mappings we have defined is isomorphic to G. It is in fact the regular permutation representation of G by right additions.

Either from the defining relations, or from the representation of the group, one readily verifies that

$$[b_{12}, b_{34}]=[b_{13}, b_{24}]=[b_{14}, b_{23}]=d\ ;$$

thus, for example, one computes

$$\begin{aligned}[b_{12}, b_{34}]=[a_1+a_2+a_1+a_2, b_{34}]&=[a_1+a_2, b_{34}]+[[a_1+a_2, b_{34}], a_1+a_2]+[a_1+a_2, b_{34}]\\ &=c_1+[c_1, a_1+a_2]+c_1=c_1+d+c_1=d.\end{aligned}$$

It follows that G' is not abelian, and G is not metabelian.

As in our first example, the centre of G is cyclic of order 2, and is the only minimal normal subgroup, and thus contained in every non-trivial normal subgroup ; it is the group

$$\{d\} = G'' = {}^4G,$$

and thus again *every proper factor group of G is metabelian and nilpotent of class at most* 3. This fact is not required, however, and we omit the verification.

We now investigate what subgroups of G are not metabelian. Let H be such a subgroup; then H contains elements h_1, h_2, h_3, h_4 such that

$$[[h_1, h_2], [h_3, h_4]] \neq 0.$$

Let
$$h_i = \sum_i \alpha_{ij} a_j \qquad (\text{mod } G').$$

Then
$$[h_1, h_2] = \sum_{i<j} (\alpha_{1i}\alpha_{2j} - \alpha_{1j}\alpha_{2i}) b_{ij} \qquad (\text{mod } {}^3G),$$

and similarly
$$[h_3, h_4] = \sum_{i<j} (\alpha_{3i}\alpha_{4j} - \alpha_{3j}\alpha_{4i}) b_{ij} \qquad (\text{mod } {}^3G).$$

The coefficients are, as we know, taken modulo 2, so there is no real distinction between plus and minus signs: the minus signs are used here to set the determinant character of the coefficients into evidence. Next, modulo ${}^5G$,

$$[[h_1, h_2], [h_3, h_4]] = \sum_{\substack{i<j \\ p<q}} (\alpha_{1i}\alpha_{2j} - \alpha_{1j}\alpha_{2i})(\alpha_{3p}\alpha_{4q} - \alpha_{3q}\alpha_{4p})[b_{ij}, b_{pq}]$$
$$= \sum (\alpha_{1i}\alpha_{2j} - \alpha_{1j}\alpha_{2i})(\alpha_{3p}\alpha_{4q} - \alpha_{3q}\alpha_{4p})\, d = \Delta d,$$

say, where the coefficient of d is the sum extended over $i<j$, $p<q$, and i, j, p, q all different. As ${}^5G = 0$, these congruences modulo ${}^5G$ are in fact equations. The coefficient of d here is, by the Theorem of Laplace, the 4×4 determinant

$$\Delta = \| \alpha_{ij} \|,$$

and by our assumption on the h_i this does not vanish. This means that h_1, h_2, h_3, h_4 are linearly independent modulo G', and thus generate the whole of G modulo G'. Differently put, H and G' together generate G; but G' is the Frattini subgroup of G, and thus can be omitted from any generating set of G. Hence H by itself generates G, that is, it coincides with G. Thus we have shown that the only subgroup of G which is not metabelian is G itself. Hence we have the following result.

LEMMA 5. *Every proper subgroup of G is metabelian.*

Every three-generator subgroup of G is proper, because G requires four generators; hence we have the desired result:

COROLLARY 6. *The group G here constructed is* 3-*metabelian but not* 4-*metabelian* (*i.e. not metabelian*).

As before we can sharpen this result by considering the least variety of groups to which G belongs. But again, as it may be laborious to determine this variety, we may content ourselves with the result we obtain when we remark that G here again has exponent 8 and nilpotent class 4: that the class is 4 we have already seen; the exponent can not exceed 8 because G/G' and G'/G'' and G'' all have exponent 2; the exponent is not less than 8 because the element $a_1 + a_2 + a_3 + a_4$ is in fact of order 8. If $\mathfrak{V}_0$ again denotes the variety of the groups of exponent 8 and nilpotent class 4, then we have finally:

COROLLARY 7. $\mathfrak{V}_0 \cap \mathfrak{M}_3 \neq \mathfrak{V}_0 \cap \mathfrak{M}_4$.

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