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PROJECTIVE CHARACTERS WITH PRIME POWER DEGREES

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Abstract

We consider the relationship between structural information of a finite group G and $\mathrm{cd}_\alpha(G)$, the set of all irreducible projective character degrees of G with factor set α. We show that for nontrivial α, if all numbers in $\mathrm{cd}_\alpha(G)$ are prime powers, then G is solvable. Our result is proved by classical character theory using the bijection between irreducible projective representations and irreducible constituents of induced representations in its representation group.

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1. Introduction

Throughout this paper, G will be a finite group. Let N be a normal subgroup of G and $\lambda \in \mathrm{Irr}(N)$ be G-invariant. Define $\mathrm{Irr}(G \mid \lambda) = \{\chi \in \mathrm{Irr}(G) \mid [\chi_N, \lambda] \neq 0\}$ and $\mathrm{cd}'(G \mid \lambda) = \{\chi(1)/\lambda(1) \mid \chi \in \mathrm{Irr}(G \mid \lambda)\}$. A great deal of information about the quotient group G/N is encoded in the set $\mathrm{cd}'(G|\lambda)$. In [2, 3], Gluck and Wolf proved the following theorem, now called the Gluck–Wolf theorem: *if G is a p-solvable group and no number in $\mathrm{cd}'(G \mid \lambda)$ is divisible by p, then G/N has an abelian Sylow p-subgroup.* They gave a proof of the Brauer height zero conjecture for p-solvable groups based on this result. To prove the Brauer height zero conjecture for general groups, a generalised Gluck–Wolf theorem is needed. Recently, Navarro and Tiep [13] proved the generalised Gluck–Wolf theorem: *if no number in $\mathrm{cd}'(G \mid \lambda)$ is divisible by p, then G/N has an abelian Sylow p-subgroup.* Finite groups G for which $\mathrm{cd}'(G \mid \lambda)$ has other special forms have also been studied. Higgs [5] proved that *if all numbers of $\mathrm{cd}'(G \mid \lambda)$ are powers of p for a fixed prime p, then G/N is solvable.* In [6] Higgs conjectured that if all numbers of $\mathrm{cd}'(G|\lambda)$ are odd, then G/N is solvable. Higgs' conjecture was proved by Moretó [10]. In this paper, we consider finite groups such that all numbers of $\mathrm{cd}'(G \mid \lambda)$ are prime powers.

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Theorem 1.1. *Let G be a finite group, $L \trianglelefteq G$ and $\lambda \in \mathrm{Irr}(L)$. If λ does not extend to G and $\chi(1)/\lambda(1)$ is a prime power for any $\chi \in \mathrm{Irr}(G \mid \lambda)$, then G/L is solvable.*

Projective representation theory is an important part of group representation theory. It is well known that λ does not always extend to G (as an ordinary character), but it may extend to an irreducible projective character of G. (A projective character is a character of the projective representation; see [7, Ch. 11] for more details.) Furthermore, there is a bijection $\eta \mapsto \chi_\eta$ from $\mathrm{Irr}(\mathbb{C}_\alpha G/N)$ to $\mathrm{Irr}(G \mid \lambda)$ such that $\chi_\eta(1)/\lambda(1) = \eta(1)$, where $\mathbb{C}_\alpha G/N$ is a twisted group algebra of G/N over $\mathbb{C}$ with factor set α (see [11, Ch. 3, Theorem 5.6]). Set $\mathrm{cd}_\alpha(G) = \{\eta(1) \mid \eta \in \mathrm{Irr}(\mathbb{C}_\alpha G)\}$. Thus $\mathrm{cd}_\alpha(G)$ is the set of degrees of irreducible projective representations of G with factor set α. Compared with the many results about the relationship between $\mathrm{cd}(G)$ and the structure of G, much less is known about the influence of $\mathrm{cd}_\alpha(G)$ on the structure of G. However, as Navarro remarked in [12]: 'it would be a mistake to underestimate the importance of these projective degrees in character theory and all questions about $\mathrm{cd}(G)$ should have a twisted version'. Based on Theorem 1.1, we can easily obtain the following result, which can be viewed as a criterion for solvability based on projective character degrees.

Theorem 1.2. *Let G be a finite group. If all numbers of $\mathrm{cd}_\alpha(G)$ are prime powers for a nontrivial factor set α, then G is solvable.*

2. Preliminaries

In this section, we give several important results that will be used in the proof of Theorem 1.1.

First, we give a classification of simple groups which have a solvable subgroup with prime-power index (see [14, Proposition 5.2] or [4, Theorem 1]).

Lemma 2.1. *Let G be a nonabelian simple group which has a solvable subgroup $H < G$ with $|G : H| = p^n$, where p is a prime. Then one of the following holds:*

(a) *$G = A_5$ and $H \cong A_4$ has index 5;*
(b) *$G = \mathrm{PSL}_3(2)$ and H is a subgroup of index 7;*
(c) *$G = \mathrm{PSL}_3(3)$ and H is a subgroup of index 13;*
(d) *$G = \mathrm{PSL}_2(\mathrm{q})$ and H is a subgroup of index $q + 1$, where $q + 1$ is a prime power.*

The next result was first proved by Higgs [5] and there is a short proof in [10].

Lemma 2.2. *Let N be a normal subgroup of a finite group G and let $\phi \in \mathrm{Irr}(N)$ be G-invariant. Assume that $\chi(1)/\phi(1)$ is a power of a fixed prime p for every $\chi \in \mathrm{Irr}(G \mid \phi)$. Then G/N is solvable.*

Finally, we present a result about finite groups with prime-power character degrees from [9] and [15, Proposition B].

LEMMA 2.3. *Let G be a finite group such that $cd(G)$ consists of prime powers. If G is solvable, then $|\rho(G)| \leq 2$. If G is nonsolvable, then $G \cong A_5 \times A$ or $G \cong \mathrm{PSL}_2(8) \times A$, where A is an abelian group.*

3. Proof of Theorem 1.1

PROOF OF THEOREM 1.1. We argue by induction on $|G : L|$. We assume that G/L is not solvable and seek a contradiction. Let N/L be the largest normal solvable group of G/L and choose $\beta \in \mathrm{Irr}(N \,|\, \lambda)$.

Step 1. β cannot be extended to G.

Otherwise, we may assume that $\chi \in \mathrm{Irr}(G)$ is an extension of β. It follows that $\mathrm{Irr}(G \,|\, \beta) = \{\chi\phi \mid \phi \in \mathrm{Irr}(G/N)\}$. So every irreducible character G/N has prime power degree and $G/N \cong A_5$ or $\mathrm{PSL}_2(8)$ by Lemma 2.3. If $G/N \cong A_5$, there exist two irreducible characters in $\mathrm{Irr}(G \,|\, \beta)$ with degrees $3\beta(1)$ and $4\beta(1)$, respectively. Then $3\beta(1)/\lambda(1)$ and $4\beta(1)/\lambda(1)$ are prime powers. So $\beta(1) = \lambda(1)$, that is, λ extends to β. Therefore χ is an extension of λ, which is a contradiction. If $G/N \cong \mathrm{PSL}_2(8)$, then there exist two irreducible characters in $\mathrm{Irr}(G \,|\, \beta)$ with degrees $7\beta(1)$ and $8\beta(1)$, respectively, and we obtain a contradiction in the same way.

Step 2. G/N is nonabelian simple group.

If there is a proper normal subgroup M such that M/N is a minimal normal subgroup of G/N, then M/N is nonsolvable. For any $\eta \in \mathrm{Irr}(M \,|\, \beta)$, $\eta(1)/\beta(1)$ is a prime power. So β extends to M by induction and every irreducible character of M/N has prime power degree. By Lemma 2.3, $M/N \cong A_5$ or $\mathrm{PSL}_2(8)$.

Next we show that M/N is the unique minimal normal subgroup of G/N. Otherwise K/N is another minimal normal subgroup of G/N and, as before, $K/N \cong A_5$ or $\mathrm{PSL}_2(8)$. Choose $\gamma \in \mathrm{Irr}(M \,|\, \beta)$ with $\gamma(1) = 4\beta(1)$ (respectively, $8\beta(1)$) according as $M/N \cong A_5$ (respectively, $\mathrm{PSL}_2(8)$). Then γ is the unique irreducible constituent in $\mathrm{Irr}(M \,|\, \beta)$ with degree $\gamma(1)$, that is, γ is MK-invariant. For any $\theta \in \mathrm{Irr}(MK \,|\, \gamma)$, $\theta(1)/\gamma(1)$ is a 2-power because $\theta(1)/\lambda(1) = \theta(1)/\gamma(1) \cdot \gamma(1)/\beta(1) \cdot \beta(1)/\lambda(1)$ is a prime power. Then $MK/K \cong K/N$ is solvable by Lemma 2.2, which is a contradiction. Therefore M/N is the unique minimal normal subgroup of G/N and $C_{G/N}(M/N) = 1$, that is, $G/N \leq \mathrm{Aut}(M/N)$. Since $|\mathrm{Out}(A_5)| = 2$ and $|\mathrm{Out}(\mathrm{PSL}_2(8))| = 3$, it follows that $G/N = \mathrm{Aut}(M/N)$. Denote by $\chi \in \mathrm{Irr}(M)$ an extension of β. By the theory of character extensions and information about the character degrees of A_5 or $\mathrm{PSL}_2(8)$), χ is the unique irreducible constituent of β^M with degree $\beta(1)$. Thus χ is G-invariant and extends to G because G/M is cyclic. So β extends to G in contradiction to Step 1.

Step 3. $I_G(\beta) = G$.

Set $T = I_G(\beta)$. If $T < G$, then $|G : T|$ is a p-power and so $\gamma(1)/\beta(1)$ is a p-power for any $\gamma \in \mathrm{Irr}(T \,|\, \beta)$ and T/N is solvable by Lemma 2.2. Thus the simple group G/N is one of the cases listed in Lemma 2.1.

We claim that $(|G : T|, |T/N|) = 1$. This is trivial for the first three cases and so we may assume that $G/N \cong \mathrm{PSL}_2(q)$. If q is even, then $|G/N| = q(q-1)(q+1)$ and

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$|G:T| = q+1$, so $(|G:T|, |T/N|) = (q+1, q(q-1)) = 1$. If q is odd, then $q+1$ is a 2-power and $q+1 = |\mathrm{PSL}_2(q)|_2$. So the claim is true and $\gamma(1) = \beta(1)$ for any $\gamma \in \mathrm{Irr}(T \mid \beta)$. Therefore T/N is abelian and we obtain a contradiction.

By the theory of isomorphic character triples, there exists a finite central extension (Γ, M, ϕ) of G/N having the projective lifting property (that is, $M \leq Z(\Gamma)$, $G/N \cong \Gamma/M$ and there is a bijection τ of $\mathrm{Irr}(G \mid \beta)$ onto $\mathrm{Irr}(\Gamma \mid \phi)$ such that $\chi(1)/\beta(1) = \tau(\chi)(1)/\phi(1)$ for any $\chi \in \mathrm{Irr}(G \mid \beta)$; for more details, see [7, Theorem 11.28]). So we may assume that N is cyclic and central and that β is faithful. Then $N = Z(G)$ and $G = G'N$ for $N \subseteq Z(G)$ and G/N is a nonabelian simple group. Let $M = N \cap G'$. Then $M = Z(G')$ and $G'/M \cong G/N$ is a simple group, so G' is a quasisimple group.

Next we show that $\mathrm{Irr}(G' \mid \beta_M) = \bigcup_{\chi \in \mathrm{Irr}(G \mid \beta)} \mathrm{Irr}(G' \mid \chi_{G'})$. To see that the left side is included in the right side, choose $\eta \in \mathrm{Irr}(G' \mid \beta_M)$. Then $0 \neq [\eta_M, \beta_M] = [(\eta_M)^N, \beta] = [(\eta^G)_N, \beta]$ by Frobenius reciprocity and the Mackey formula. Hence there exists $\chi \in \mathrm{Irr}(G \mid \eta)$ such that $\chi \in \mathrm{Irr}(G \mid \beta)$. Next we prove the reverse inclusion. For any $\chi \in \mathrm{Irr}(G \mid \beta)$ and $\gamma \in \mathrm{Irr}(G' \mid \chi_{G'})$,

$$[\gamma_M, \beta_M] = [(\gamma_M)^N, \beta] = [(\gamma^G)_N, \beta] \geq [\chi_N, \beta] > 0.$$

So all the irreducible characters of G' lying over β_M have prime power degrees. By Lemma 2.2, $|\mathrm{cd}(G' \mid \beta_M)| > 1$. Because G is a central product of N and G', β_M cannot extend to G'. By [8, Theorem 1.1] and the result in [1], G' is one of the following cases: $\mathrm{SL}_2(q)$ for some q, $2.\mathrm{Sp}_6(2)$, $3.\Omega_7(3)$, $3.G_2(3)$, $4_1.\mathrm{L}_3(4)$, $2.\Omega_8^+(2)$, or $2.A_n$ where $n = 2^{m+1} + 2$. Take the group $2.\mathrm{Sp}_6(2)$ as an example. Suppose that $\chi \in \mathrm{Irr}(2.\mathrm{Sp}_6(2))$ with prime power degree p^n. Then $\chi(1) \in \{8, 64, 2^9\}$ and $p = 2$ by [8, Theorem 1.1]. Similarly, we can check that every number in $\mathrm{cd}(G' \mid \beta_M)$ is a power of a fixed prime for the other groups. Thus G'/M is solvable by Lemma 2.2, which is a contradiction. □

For a set X of positive integers, define $\rho(X) = \{p \text{ prime} : p|n \text{ for some } n \in X\}$.

Corollary 3.1. *Let G be a finite group, $L \trianglelefteq G$ and $\lambda \in \mathrm{Irr}(L)$. If $\chi(1)/\lambda(1)$ is a prime power for any $\chi \in \mathrm{Irr}(G \mid \lambda)$, then $|\rho(\mathrm{cd}'(G \mid \lambda))| \leq 3$.*

Proof. Let $T = I_G(\lambda)$. If $T < G$, then $\mathrm{cd}(G \mid \lambda) = \{|G:T|\beta(1) \mid \beta \in \mathrm{Irr}(G \mid \lambda)\}$. So $|\rho(\mathrm{cd}'(G \mid \lambda))| = 1$. Next we assume that $T = G$, that is, λ is G-invariant. If λ extends to G, then $\mathrm{cd}'(G \mid \lambda^G) = \mathrm{cd}(G/L)$ and the corollary follows from Lemma 2.3. If λ cannot be extended to G, then G/L is solvable by Theorem 1.1. By the theory of isomorphic character triples [7, Ch. 11], we may assume that L is a central group and $\lambda(1) = 1$. So $\mathrm{cd}(G \mid \lambda) = \mathrm{cd}'(G \mid \lambda)$ consists of prime powers. Denote by K the maximal normal subgroup which has an extension of λ. Then $\mathrm{cd}(K \mid \lambda) = \mathrm{cd}(K/L)$ and $|\rho(\mathrm{cd}(K \mid \lambda))| \leq 2$ by Lemma 2.3. Choose a chief factor M/K with order p^n for prime p. For any prime $q \neq p$ with $q \in \rho(\mathrm{cd}(G \mid \lambda))$, choose $\gamma \in \mathrm{Irr}(G \mid \lambda)$ with q-power degree and $\beta \in \mathrm{Irr}(M)$ such that $[\beta, \gamma|_M] \neq 0$. By the choice of K, it follows that β has nontrivial q-power degree and β_K is irreducible for $(|M/K|, q) = 1$. So $q \in \rho(\mathrm{cd}(K \mid \lambda))$, that is, $|\rho(\mathrm{cd}(G \mid \lambda)) - \{p\}| \leq 2$, and so $|\rho(\mathrm{cd}(G \mid \lambda))| \leq 3$. □

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