A REMARK ON THE NILPOTENCY INDEX OF THE RADICAL OF A GROUP ALGEBRA OF A p-SOLVABLE GROUP

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(Received 6th June 1980)

Let $K$ be a field of characteristic $p > 0$, $G$ a finite $p$-solvable group, $P$ a $p$-Sylow subgroup of $G$ of order $p^n$, $KG$ the group algebra of $G$ over $K$, and $J(KG)$ the Jacobson radical of $KG$. In the present paper we study the nilpotency index $t(G)$ of $J(KG)$, which is the least positive integer $t$ with $J(KG)^t = 0$. Since $J(EG) = E \otimes_K J(KG)$ for any extension field $E$ of $K$ (cf. [7, Proposition 12.11]), we may assume that $K$ is algebraically closed.

D. A. R. Wallace [12] proved that

$$t(G) \geq a(p - 1) + 1.$$ 

There is a problem to determine the structure of $G$ with the property $t(G) = a(p - 1) + 1$. When $G$ is of $p$-length 1, by the results of S. A. Jennings [6] and K. Morita [8], $t(G) = a(p - 1) + 1$ if and only if $P$ is elementary abelian (cf. [10, Corollary 1]). But for $p$-solvable groups $G$ of $p$-length $\geq 2$ the assertion does not hold in general. Indeed, K. Motose and Y. Ninomiya [10] showed that when $p = 2$ and $G = S_4$ (which denotes the symmetric group of degree 4), $t(G) = 4$ though $P$ is dihedral of order 8. Recently, K. Motose [9] proved that if $p = 2$, $P$ is metacyclic and $G/O_2(G) = S_4$, then $t(G) = a + 1$ if and only if $P$ is elementary abelian. The purpose of this paper is to consider the proposition for the case where $p$ is odd. If $p$ is odd and $P$ is metacyclic, then $P$ is a regular $p$-group (cf. [5, III 10.2 Satz (c)]). Y. Tsushima [11] claimed that when $P$ is regular, $t(G) = a(p - 1) + 1$ if and only if $P$ is elementary abelian. At line 11 of page 37 in [11], he says that since $P$ has exponent $p$, $G$ is of $p$-length 1 from [4, Theorem A (ii)]. However, Tsushima's assertion is not correct. There exists an example (to be given later) of a $p$-solvable group $G$ of $p$-length $\geq 2$ such that $P$ has exponent $p$ and so that $P$ is regular. Our main result can be stated as follows: If $p$ is odd and $P$ is metacyclic, then $t(G) = a(p - 1) + 1$ if and only if $P$ is elementary abelian.

Throughout this paper we use the following notation. We write $O_p'(G)$ and $O_p(G)$ for the maximal normal subgroup of $G$ of order prime to $p$ and the maximal normal $p$-subgroup of $G$, respectively. We define $O_{p', p}(G)$ by $O_p(G/O_p(G)) = O_{p', p}(G)/O_{p'}(G)$. We write $H < G$ if $H$ is a normal subgroup of $G$. For a finite group $Y$, $|Y|$ and Aut($Y$) denote the order of $Y$ and the group of all automorphisms of $Y$, respectively. When $X$ is a subgroup of $G$, we write $N_G(X)$, $C_G(X)$ and $|G : X|$ for the normaliser of $X$ in $G$, the centraliser of $X$ in $G$ and the index of $X$ in $G$, respectively. If $x_1, \ldots, x_n$ are in $G$, we write $\langle x_1, \ldots, x_n \rangle$ for the subgroup of $G$ generated by $\{x_1, \ldots, x_n\}$. When $H$ is a
subgroup of \(G\) and \(g \in G\), let \([H, g] = \langle h^{-1}g^{-1}hg \mid h \in H \rangle\) and \([H, g, g] = [[[H, g], g], g]\). We write \(GL(2, p)\) and \(SL(2, p)\) for the general linear group and the special linear group, respectively (cf. [3, p. 40]).

For an odd prime \(p\), we say that \(G\) is \(p\)-stable in the sense of [1, p. 1104 Definition 2.3].

We write \(Qd(p)\) for the group defined in [1, p. 1104] and [2, p. 32]. Then \(Qd(p)\) is the semi-direct product of \(R\) by \(SL(2, p)\) with respect to the identity map \(SL(2, p) \to GL(2, p) = Aut(R)\), where \(R\) is an elementary abelian group of order \(p^2\). It is noted that if \(p\) is odd then the \(p\)-Sylow subgroup of \(Qd(p)\) is nonabelian of order \(p^3\) of exponent \(p\) (cf. [2, p. 32 and p. 33 Example 11.4]).

To begin with, we state the next two lemmas which are useful for our aim.

**Lemma 1.** Let \(G\) be a finite group and \(p\) an odd prime. If the \(p\)-Sylow subgroup of \(G\) is of order \(p^3\) with exponent \(p^2\), then \(G\) is \(p\)-stable.

**Proof.** By [1, Lemma 6.3], it suffices to show that \(X/Y \neq Qd(p)\) for any subgroup \(X\) of \(G\) and any \(Y < X\) (see [1, p. 1103] for the term "involved"). Assume that \(X/Y = Qd(p)\) for some subgroup \(X\) of \(G\) and some \(Y < X\). Since the order of the \(p\)-Sylow subgroup of \(Qd(p)\) is \(p^3\) by [2, p. 32], \(p \mid |G: X|\) and \(p \notmid |Y|\). Let \(P\) be a \(p\)-Sylow subgroup of \(X\). Then \(P\) is a \(p\)-Sylow subgroup of \(G\), so that \(P\) has exponent \(p^2\). On the other hand, \((PY)/Y\) is a \(p\)-Sylow subgroup of \(X/Y\). Hence \((PY)/Y\) has exponent \(p^2\) from [2, p. 33 Example 11.4]. This is a contradiction since \((PY)/Y \approx P/(P \cap Y) \approx P\). This completes the proof.

**Lemma 2** [3, Theorem 8.1.3]. Let \(p\) be an odd prime, and let \(G\) be a finite group with a \(p\)-Sylow subgroup \(P\) such that \(O_p(G) \neq 1\) and \(G\) is \(p\)-stable and \(p\)-solvable. If \(A\) is an abelian normal subgroup of \(P\), then \(A \leq O_{p', p}(G)\).

**Proof.** Let \(H = O_{p', p}(G)\), \(Q = P \cap H\), \(N = N_G(Q)\) and \(C = C_G(Q)\). Then \(O_{p', p}(G) \cdot Q = H \triangleleft G\). Take any \(x \in A\). Clearly \(x \in N\). Since \(A < P \leq Q\), \([Q, x] \leq A\). Since \(A\) is abelian, \([Q, x, x] \leq [A, x] = 1\), so that \([Q, x, x] = 1\). Since \(G\) is \(p\)-stable, \(xC \in O_p(N/C)\). This shows \((AC)/C \subseteq O_p(N/C)\). Since \(G\) is \(p\)-solvable, \(C \leq H\) by [3, Theorem 6.3.3], so that \(C \subseteq H \cap N\). By the Frattini argument [3, Theorem 1.3.7], \(G = HN\). Then we have the following epimorphism

\[
\begin{array}{ccc}
N/C & \xrightarrow{f} & N/(H \cap N) \\
\psi & & \psi \\
yC & \xrightarrow{g} & y(H \cap N)
\end{array}
\]

Since \(H = O_{p', p}(G)\), \(O_p(G/H) = 1\), so that \(O_p(N/(H \cap N)) = 1\). Since \(f\) is an epimorphism, \(f(O_p(N/C)) \subseteq O_p(N/(H \cap N))\). This implies \(f((AC)/C) = 1\), so that \(A \leq H \cap N\).

Using these lemmas we can prove the next main result of this paper.

**Theorem.** Let \(p\) be an odd prime, and let \(G\) be a finite \(p\)-solvable group with a metacyclic \(p\)-Sylow subgroup \(P\) of order \(p^a\). Then \(t(G) = a(p - 1) + 1\) if and only if \(P\) is elementary abelian.
NILPOTENCY INDEX OF THE RADICAL OF A GROUP ALGEBRA 33

Proof. Assume that $P$ is elementary abelian. By [3, Theorem 6.3.3], $P \subseteq O'_{p',p}(G)$. This implies that $G$ is of $p$-length 1. So that $t(G) = a(p-1)+1$ by [10, Corollary 1].

Suppose $t(G) = a(p-1)+1$. We use induction on $|G|$. Assume $G \neq 1$. Let $H = O_{p}(G)$. By [12, Theorems 2.2 and 3.3], $a(p-1)+1 \leq t(G/H) \leq t(G) = a(p-1)+1$. Hence we may assume $H = 1$ by induction. Let $R = O_{p}(G)$ and $|R| = p^{b}$, so that $1 \leq b \leq a$. Then $a(p-1)+1 = t(G) = t(R) + t(G/R) - 1$ by [12, Theorem 2.4]. Since $t(R) \geq b(p-1)+1$ and $t(G/R) \geq (a-b)(p-1)+1$ by [12, Theorem 3.3], we have $t(R) = b(p-1)+1$. So that $R$ is elementary abelian from [10, Theorem 1]. Since $P$ is metacyclic, $R$ is cyclic of order $p$ or is elementary abelian of order $p^{2}$. Then $C_{G}(R) = R$ by [3, Theorem 6.3.3], so that

$$G/R = N_{C}(R)/C_{G}(R) \cong \text{Aut}(R).$$

(*)

If $R$ is cyclic of order $p$, then $p | |G/R|$ by (*), so that $P$ is cyclic of order $p$. Hence we may assume that $R$ is elementary abelian of order $p^{2}$. By [10, Corollary 1], it suffices to show that $G$ is of $p$-length 1. Suppose that $G$ is of $p$-length $\leq 2$. Since $|\text{Aut}(R)| = |GL(2, p)| = p(p-1)^{2}(p+1)$ by [3, Theorem 2.8.1], $|P/R| = 1$ or $p$ from (*). This shows that $|P| = p^{2}$ or $p^{3}$. Since $G$ is of $p$-length $\leq 2$, $P$ is nonabelian from [3, Theorem 6.3.3]. Hence $|P| = p^{3}$. Since $P$ is metacyclic, we can write

$$P = M_{3}(p) = \langle x, y \mid x^{p} = y^{p^{2}} = 1, x^{-1}yx = y^{p+1} \rangle$$

by [3, Theorem 5.5.1]. Then $G$ is $p$-stable by Lemma 1. Since $\langle x, y^{p} \rangle \triangleleft P$ and $\langle y \rangle \triangleleft P$ and since $R \neq 1$, we have that $\langle x, y \rangle \subseteq R$ and $\langle y \rangle \subseteq R$ by Lemma 2. Then $x, y \in R$, so that $P = R$. Hence $G$ is of $p$-length 1, a contradiction. This completes the proof.

Finally we give an example as mentioned in the introduction.

Example. Let $p = 3$, and let $R$ be an elementary abelian group of order 9. Let $G$ be the semi-direct product of $R$ by $SL(2, 3)$ with respect to the identity map $SL(2, 3) \rightarrow GL(2, 3) = \text{Aut}(R)$. Then $G = Qd(3)$ (cf. [1, p. 1104] and [2, p. 32]). Let $R = (b, c)$ and $S = SL(2, 3)$. For each $x = (u, v) \in S$, we can write that $x^{-1}bx = b'c'$ and $x^{-1}cx = b''c''$. Let $a = (\frac{1}{0}, \frac{1}{1}) \in S$, then $a$ is of order 3, so that we can write $P = \langle a, b, c \mid a^{3} = b^{3} = c^{3} = 1, a^{-1}ba = bc, a^{-1}ca = c, b^{-1}cb = c \rangle$, where $P$ is a 3-Sylow subgroup of $G$. Then $P$ has exponent 3 (cf. [2, pp. 32–33] and [3, p. 203]). Let $Q$ be a 2-Sylow subgroup of $S$. Since $Q \triangleleft S$ and since $Q$ is quaternion of order 8, $S$ has the unique involution $z = (\frac{-1}{0}, \frac{-1}{0})$ in $Q$. Let $H = O_{2}(G)$. Since $|G| = 2^{3} \cdot 3^{3} = 216$, $H = O_{2}(G)$. Since $Q$ is a 2-Sylow subgroup of $G$, $H \subseteq Q$. Evidently, $HR = H \times R$. If $H \neq 1$, then $z \in H$, so that $z \in C_{G}(R)$, a contradiction. Hence $H = 1$. On the other hand, $P$ is not normal in $G$. So that $G$ is of 3-length 2.

Acknowledgement. The author wishes to express his gratitude to the referee for his kind advice.

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