BINARY FORMATION AND INTERACTIONS WITH FIELD STARS

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ABSTRACT. Binaries provide an energy source in dense stellar systems. Exothermic gravitational interactions in star clusters can play a role similar to that of nuclear reactions in single stars. These gravitational interactions can be modeled in a laboratory setting, in the form of numerical binary-single star and binary-binary scattering experiments. Gravitational cross sections obtained this way can be applied to model star cluster evolution, just as nuclear cross sections are used as input data in stellar evolution calculations. References are given to detailed descriptions of gravitational cross sections, and a useful new example of an application is given: the rate at which hard binaries form in a homogeneous stellar background, as the solution of an integral equation describing the combined effects of creation, destruction, hardening and softening of binaries.

1. THE ROLE OF BINARIES IN STELLAR DYNAMICS

An interesting analogy can be made between binary reactions in star cluster evolution and nuclear reactions in stellar evolution. In both cases negative binding energy is accumulated in internal degrees of freedom, since exothermic reactions dominate statistically over endothermic reactions. Energy conservation produces positive energy in external degrees of freedom, in the form of excess kinetic energy in the particles remaining as the reaction products. After a few scatterings this excess energy is distributed over the thermal heat reservoir of the star or star cluster. In this way nuclear or binary reactions provide the balance in the energy budget, by replacing the energy lost at the surface by escaping photons or stars, for a star or star cluster, respectively. The analogy is only approximate, since the stars which manage to completely escape from an isolated globular cluster carry only a very small amount of excess energy to infinity. Therefore, the heat loss of a globular cluster core might be described as a local escape of stars from core to halo, or in more physical terms as an outwardly directed heat flow which fuels the expansion of the outer layers.

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This analogy is helpful in understanding the overall evolution of star clusters, although there are other important qualitative differences as well. For example, nuclear reaction rates increase at higher density, and the same is true for gravitational binary-single star and binary-binary reactions. However, nuclear reaction rates increase steeply with increasing temperature, whereas binary reactions show the opposite behavior, releasing less energy with higher dispersion velocities in the star system. Even without considering binaries, there are important differences between stellar-dynamical systems and the gaseous interior of a single non-degenerate star. For example, increasing the density at constant temperature will make a star more opaque, thus lowering the conductivity. Increasing the density in a stellar system, while holding the velocity dispersion fixed, will increase the rate of two-body relaxation effects, and thus increase the effective conduction of energy through the system (*cf.* Lynden-Bell and Eggleton, 1980).

With these cautions in mind, I will review our knowledge of binary dynamics on three levels. The first level is a microscopic description of gravitational scattering processes between binaries and single stars in Sect. 2, and between binaries and binaries in Sect. 3. On the second level we can determine local statistical quantities to describe the effects of binaries on their immediate surroundings, in a thermodynamic rather than statistical-mechanics approach, as detailed in Sect. 4. The third level concerns the behavior of binaries in a global and realistic, dynamically changing environment, and the influence of the binaries on the evolution of that environment, as discussed in Sect. 5. Many recent developments on all three levels will be mentioned below. For reviews of earlier work and general background, see Heggie (1975b, 1980), and for a one-page summary of the effects of binaries in star clusters see the allegoric description by Lynden-Bell (1975).

2. BINARY-SINGLE STAR SCATTERING

Let us start with an intuitive definition of a differential cross section $\frac{d\sigma}{dy}$ for gravitational binary-single star scattering in which the binary binding energy changes by a net amount y. This definition can be introduced by closely following the language of laboratory experiments in atomic (nuclear, particle,...) physics, even though our astrophysical laboratory is a purely numerical one modeled by a computer. For simplicity let us consider gravitational interactions between stars in the approximation of point particles, postponing a discussion of internal degrees of freedom and finite-size effects of the stars to Sect. 5.

First we have to set up a target "plate" filled with binary stars, each well separated from the others so that each binary can be treated as an unperturbed two-body system. Next we prepare a bundle of single stars which all move parallel to each other. Again we only consider low-intensity beams with interstellar distances large enough to guarantee sufficiently undisturbed rectilinear motions.

We can now perform an experiment by aiming a bundle of single stars at right angles to a target plate of binaries, and subsequently observing the characteristics of the debris of the reactions. Many different kind of experiments are possible, with different prescriptions for the parameters describing the incoming bundle and the target plate. Let us start with a simple case. Take a monochromatic bundle of stars, all of equal mass m_3 (*i.e.* all stars have the same energy and hence the same velocity). Prepare a target plate of binaries which all consist of two stars with masses m_1 and m_2. These two masses do not have to be equal but are the same for all binaries in the target. Furthermore, give all binaries the same binding energy x, or equivalently the same semi-major axis.

When performing this experiment, we can measure the energy with which the debris leaves the target. This debris will consist of single stars and binaries. Occasional triple systems might emerge (only if the energy of an incoming star is lower than the binding energy of a binary), but they are all unstable, and will sooner or later decay in a single star and a binary. The reason is the absence of dissipation or simply the time-reversibility of Newton's equations: stable hierarchical systems are stable in the future *and* in the past. Therefore a capture will lead to an escape in the generic case, with the exceptions occupying a lower-dimensional subspace in the space of initial conditions (Chazy, 1929) which implies a vanishing probability of occurrence for any finite number of experiments.

Let us first study those scattering events where the binaries are left intact, and the same incoming star escapes again (possibly after a temporary capture). In this case we define the differential cross section $\frac{d\sigma}{dy}(x,y;V)$ as follows: the expression

$$\frac{d\sigma}{dy}(x,y;V)dy$$

gives the total cross section for all scattering processes which start with a binary of energy x and an incoming star with velocity V (with respect to the center of mass of the binary), and end with the same binary having an energy between y and y + dy.

As in the familiar case of atomic physics, this total cross section is defined as follows: Take a target plate with a large number of replicas of the same binary, but with random orientation, phase, and eccentricity (according to a thermal distribution, *i.e* $f(e) \propto e$, *cf*. Heggie, 1975a). Now aim a beam of single stars at this target, and measure the fraction of stars causing a transition into the right energy range y,y + dy. Multiply the beam area by this fraction and divide by the number of target binaries, and *voilà*: this is the (effective) cross section.

Of course, rather than taking up cosmic billiards, it is simpler to perform computer experiments to determine three-body scattering cross sections. Instead of a dilute target plate of binaries, consider now a large number of replicas of the same binary, again with random phase, etc. Now shoot (*i.e.*, calculate numerically the orbit of) one single star at each of these binaries, with impact vector $\vec{p}$ chosen randomly within a fixed area A perpendicular to the incoming direction ($\vec{p}$ measures the amount and direction of the off-center aim at infinity, with respect to the center of mass of the binary; $p = |\vec{p}|$ is the impact parameter).

In the limit of an ensemble consisting of an infinite number of replica's of such a system, the *fraction* of the experiments resulting in a new binary binding energy between y and y + dy is by definition

$$\frac{1}{\|A\|}\frac{d\sigma}{dy}(x,V;y)dy$$

where $\| A \|$ is the total surface area of A, and with the *caveat* that only those area's A may be used which are large enough to intersect *all* orbits of incoming stars which can possibly cause a transition to a new binary binding energy in the range y,y + dy. This is the precise definition of a cross section as an "effective surface area." Of course, for indificual experiments the above expression describes only the *probability* for the stated outcome.

Here we have arrived at the central problem of numerical scattering experiments: how does one determine the maximum impact parameter? An underestimate will result in too small a measured value of the cross section, since some of the relevant scattering processes are left out. An overestimate, however, will waste computer time and can easily make the number of "hits" too small to be statistically significant. Unfortunately, there is an extra complication. It would be extremely inefficient to perform a whole new series of numerical experiments for any additional question one wants to ask. Instead, the only reasonable way to get simultaneous information about many different parameters of interest is to mimic a particle accelerator experiment: first run a series of continuous experiments, the outcome of each of which is stored on tape; then analyze the one set of data later in a number of different ways. And there is the problem: different questions asked afterwards require different values for the maximum impact parameter...

This problem can only be solved satisfactorily by a combination of two approaches. First, the density of numerical experiments can be made to fall off with increasing impact parameter, so that one has a better coverage of near-central collisions than of rather wide encounters. Secondly, an extended (and, alas, time-consuming) series of test runs can be carried out in *every* new and previously unexplored parameter domain (*e.g.* for different incoming velocity of the third star, different eccentricity of the binary, etc.).

Given these formidable difficulties, a systematic and unbiased survey of three-body scattering became possible only recently with decreasing cost of computer time. The pioneer in the field of gravitational three-body scattering is Hills, who was the first to accumulate scattering statistics, starting in the sixties. In the light of the central problem outlined above, Hills chose to perform the majority of his experiments for binary orbits with zero initial eccentricity and at zero impact parameter, *i.e.* most of his collisions were exactly head-on. In this way he obtained a wealth of information, and obtained the first numerical three-body scattering cross sections (Hills, 1975) by making estimates about the dependence of energy exchange as a function of impact parameter based on a smaller amount of off-axis scattering experiments. However, it was not possible to accurately estimate the systematic errors introduced by the zero-impact-parameter approach. Other early papers reporting three-body experiments (*e.g.* Saslaw *et al.*, 1974, followed by Valtonen, 1975) did not express the results in a form which could be translated into cross sections.

In the same period Heggie (1975a) published an extremely detailed and comprehensive paper which is still *the* standard reference on analytical approximations for the determination of three-body scattering cross sections. He used a large arsenal of different approximations in many different parameter regimes, such as the use of impulsive approximations, of adiabatic invariants, and of phase-space-volume methods where the memory of initial states was assumed to be lost. Monaghan (1976a,b; 1977) applied a slightly different technique to the analysis of stochastic decay of (temporarily) bound three-body systems.

An extensive series of numerical experiments without any *a priori* limitations on the initial conditions was carried out much later, when large amounts of computer time were available. Here several months of C.P.U. time on a VAX 11/780 computer enabled literally millions of scattering experiments to be performed (Hut and Bahcall, 1983). To the extent that the results of these experiments had been predicted by Heggie (1975a), they agreed remarkably well, generally within a factor two and often significantly better. Of course, only the numerical results could accurately determine the behavior in the transition regions between the domains of validity of Heggie's analytic estimates, but the details of the asymptotic behavior had been correctly predicted. Some additional analytic work was needed to describe and interpret some other details such as the dependence of differential cross sections on eccentricity of the original binary orbit, and the behavior of exchange scattering, and was published by Hut (1983a).

Another result of the more recent calculations answered a controversy which had existed in the literature for many years, concerning the average amount of energy increase of hard binaries in a field of single stars. Hills' (1975) numerical results fell below Heggie's (1975a) analytical prediction, yielding a value only 40% of that of Heggie's estimate, and it was not clear how this difference arose (*cf.* Spitzer and Mathieu, 1980, who decided to use a value 60% as large as Heggie's estimate in their model calculations of globular cluster evolution). Was Heggie's analytical assumption of phase space mixing for strong scattering not accurate enough or did Hills' choice of zero impact parameter for most of his numerical experiments introduce systematic effects? The answer turned out to depend somewhat on the degree of hardness of the binary. For only moderately hard binaries ($E_{bin} \sim 10kT$) an average binding energy increase of 60% of Heggie's estimate was measured, in closer agreement with Hills' results. For very hard binaries ($E_{bin} \sim 500kT$), however, the measured value rose to 80% of Heggie's estimate (Hut, 1983b, 1985). Indeed, Spitzer and Mathieu's (1980) inspired guess turned out to have been the most accurate overall, especially since they modeled the effects of moderately hard binaries.

As a guide to the literature of gravitational three-body scattering, the following list names those papers which describe more than a thousand numerical experiments each: Saslaw *et al.*, 1974; Hills, 1975, 1983a,b; Heggie, 1975a; Valtonen, 1975; Valtonen and Aarseth, 1977; Valtonen and Heggie, 1979; Hills and Fullerton, 1980; Fullerton and Hills, 1982; Hut and Bahcall, 1983; Hut, 1983b,c, 1984a,b,c; Hut and Paczynski, 1984; Bahcall *et al.*, 1985.

The most detailed information available at present is an atlas of equal-mass differential scattering cross sections, with nearly a hundred different graphs which show separately the contributions of all kind of different processes to the final differential cross sections for energy exchange between binary and single star in a three-body scattering experiment (Hut, 1984a).

All experiments mentioned above involved a Monte Carlo sampling of initial conditions, in order to obtain a physicist's description of gravitational three-body scattering in terms of cross sections. From the point of view of a mathematician interested in the three-body system as a dynamical system (*cf.* Alekseev, 1981), additional insight into the extremely rich microscopic structure of the space of orbits has been obtained from a series of experiments for a grid of initial conditions, determined by stepwise varying several parameters independently while keeping the other parameters fixed (Hut, 1983c).

Specific astrophysical applications of three-body scattering experiments have been discussed recently in a variety of papers. The effects of three-body scattering on the formation and evolution of X-ray sources in globular clusters are discussed by Hut and Verbunt (1983a,b); Krolik(1983, 1984); Krolik *et al.* (1984); Hut and Paczynski (1984), and earlier references which can be found in these papers. The effects of perturbations of passing field stars on wide binaries are discussed by Hut (1984b) and Bahcall *et al.* (1985).

The analogy between gravitational scattering and atomic physics can be extended to a classification of the intermediate and final states. For example: ionization, charge exchange and resonant scattering all have their gravitational counterpart (Heggie, 1975a; Hut and Bahcall, 1983). Also, detailed-balance relations between each scattering process and its reverse can be derived in similar fashion (Heggie, 1975a). An important and interesting practical difference is that in gravitational scattering classical mechanics causes computational difficulties where quantum mechanics makes life easier. In classical mechanics there do not exist simple highly symmetric low-lying energy levels (let alone a spherically symmetric ground state). However, also in atomic physics there are cases where classical approximations are called for, when studying scattering processes with atoms where the outer electron occupies a very highly excited state. The techniques used in some of these approximations (*cf.* Shakeshaft and Spruch, 1979 and references therein) have been adopted successfully for the gravitational scattering case (Hut, 1983a; Heggie and Hut, 1975).

3. BINARY-BINARY SCATTERING

The number of different types of outcome is much larger in the case of four-body scattering than for three-body scattering. Not only are there more dynamical degrees of freedom and therefore more possibilities to exchange energy (and to exchange stars themselves), there is also a qualitatively new type of outcome: the formation of hierarchical triples. In binary-single star scattering, a temporary triple system can be formed but will "always" decay (in the physical, not in the mathematical sense; see the previous section). A heuristic way to realize this is to consider time reversal: a hierarchical stable triple will generally be stable in both future and past, and can therefore not have been formed in a single binary-single star scattering event. However, it is possible to shoot a single star at a hierarchical triple in such a way that two binaries result which subsequently move away from each other. Because this process is not an exceptional one requiring fine-tuning, also the reverse will occur regularly: the formation of stable hierarchical triples in binary-binary scattering (Mikkola, 1983, finds a relative rate of 20% for this process among strong interactions between hard binaries).

The first four-body scattering experiments were reported by Harrington (1974) and Saslaw *et al.* (1974), who were mainly interested in the details of the decay modes and did not present their results in a form suitable to distill scattering cross sections. Only very recently has progress been made in this field, where the numerical difficulties are so much more formidable than in the three-body scattering case. Hoffer (1983) has reported the results of 40,000 binary-binary scattering experiments. His results mainly concern soft binary encounters, limited to binary orbits of zero eccentricity; his hard binary encounters were not followed long enough to determine the final outcome. Following Hills (1975), Hoffer carried out part of his scattering experiments at zero impact parameter, which introduces another uncertain element in the derivation of cross sections. His main results are a determination of the average amount of energy exchange between the binding energy of the binaries and the kinetic energy of the single stars and center-of-mass motion of the binaries emerging from the scattering process, for a variety of different mass combinations.

Mikkola (1983a,b; 1984a,b,c) has reported the results of ca. 20,000 binary-binary scattering experiments. He has limited himself to considering only stars of equal mass, but at the same time treating the scattering process to a very high degree of accuracy, using sophisticated regularization techniques, some of which of a type which has not been applied to N-body experiments before. Most interesting is the simultaneous four-body regularization proposed by Heggie (1974), where additional dimensions are introduced to simultaneously eliminate *all* possible two-body collision singularities, as a generalization of the Kustaanheimo-Stiefel regularization (*cf.* Stiefel and Scheifele, 1971; Stiefel, 1973). This is the best type of coordinate transformation and dimensional extension possible, since it has been shown that three-body collision singularities can not be regularized (*cf.* Siegel and Moser, 1971). However, the equations of motions are considerably more complicated in this formalism, where the relative motion of the four bodies are described by 49 first-order equations (!), as compared to 18 in the standard Newtonian formalism (where the three relative positions and velocities have three Cartesian components each).

Mikkola has obtained cross sections for a variety of processes, for soft as well as hard binaries. These processes can be divided in four categories, in which (1) two binaries emerge from the scattering experiments (containing either the original or exchanged binary members); (2) one star escapes and a hierarchical triple is formed; (3) two single stars escape leaving a binary behind; (4) all four stars escape in different directions. The final errors in his cross sections are a combination of statistical and systematic effects. Statistical limitations follow from the limited number of experiments for each choice of parameters, while systematic uncertainties are introduced because several percent of the experiments were halted before the outcome was determined because of limitations on computer time. Nonetheless, the resulting cross sections represent a major improvement in our knowledge of binary-binary scattering.

4. FORMATION AND EVOLUTION RATES OF BINARIES IN A STATIC STELLAR BACKGROUND

The scattering cross sections reviewed in the previous two sections contain an overwhelming amount of information which can be applied to the modeling of dynamically evolving star clusters, as is discussed in the next section. But before applying these cross sections to a complicated realistic system, it is illuminating to first consider a simplified application, by studying the behavior of binaries in a homogeneous, static background of field stars. This approach, although informative, is formally not really correct since in stellar dynamics as well as in gas dynamics no consistent solutions exist which allow a static homogeneous distribution of matter. Therefore this assumption has been called the "Jeans swindle," after Jeans (1902) who took up the same assumption of a static background to derive linear stability criteria against gravitational collapse of gas clouds.

Before applying the three-body reaction rates to a static backdrop of single stars, let us quickly classify the main types of binaries and their average behavior. The most important feature of binary--single star scattering is energy exchange between external and internal degrees of freedom. Hard binaries, with an orbital velocity much larger than typical field star velocities, behave differently from soft binaries, for which the orbital velocity is much lower than that of the field stars. A fundamental law of three-body stellar dynamics is: *hard binaries tend to become harder while soft binaries tend to become softer.* This can be described heuristically by the following equipartition argument (Gurevich and Levin, 1950; Heggie, 1975a).

A fast star moving past a slowly revolving binary will on average lose some energy to the binary. However, trying to speed up the binary members will put them in wider orbits with an actually *lower* velocity (loosely speaking a Kepler orbit seems to have a negative 'specific heat', a general phenomenon for gravitational interactions; *cf.* Lynden-Bell, 1973). Hard binaries, on the other hand, can capture a slowly incoming field star under formation of a bound triple system. After some time, generally orders of magnitude longer than the initial binary period, one of the stars is ejected more or less stochastically. The velocity with which it reaches infinity is typically of order of the internal binary velocities, and therefore much larger than the initial field star velocity. The binary has to increase its binding energy in order to give off this energy, thereby shrinking and increasing its orbital velocity. Of course, not all hard binaries harden during each encounter with a field star, nor do soft binaries loosen up monotonically; both processes take place for both types of binaries, but the net energy balance has a different sign.

We are now in a position to predict the formation and subsequent evolution of a binary population against a background of single stars. Starting without any binaries, very soft binaries will soon be formed in large numbers, simply because the phase space available increases rapidly with decreasing binding energy. The easiest way to see this is to start with the quantum mechanical description of the hydrogen atom, and to approach the classical limit by increasing the size of the atom. In this way, by taking the limit of very highly excited states, the correspondence principle tells us that we will arrive at the proper classical description (*cf.* Heggie, 1972). Another way of seeing this is by recalling the way in which quantization was first attempted by Bohr and Sommerfeld who basically counted volumes in phase space in units of Planck's constant h, using the classical action integrals

$$\oint p_i \, dq_i = n_i h$$

over a full period of the generalized coordinates q_i, with the quantization conditions of integer values for the n_i. Thereby they computed the degeneracy of each energy level as being proportional to the *classical* amount of phase space available at that energy. Seen in this light, mentioning the correspondence principle is a sophisticated way of expressing that Bohr and Sommerfeld did our classical homework already.

The argument runs as follows. The distribution function $f(E)$ which describes the population of energy levels of the hydrogen atom in the classical limit can be written as

$$f(E) = w(E)\rho(E)e^{-E/kT}$$

where the three factors at the right are the statistical weight function, which measures the amount of degeneracy within one energy value, the density of levels of different discrete energy values, and the thermal Boltzmann factor, respectively.

The density of energy levels can be found simply from the relation between the energy E and the principal quantum number $n \gg 1$:

$$E \propto \frac{1}{n^2}$$

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which implies

$$\rho\,(E) = \frac{dn}{dE} \propto E^{-3/2} \quad .$$

The degeneracy per energy level can be counted directly from the range of the quantum numbers $l = 0,1....,n-1$ and $m = -l,....+l$ as

$$w(n) = \sum_{0}^{n-1} (2l + 1) = n^2$$

and therefore

$$w(E) \propto E^{-1} \; .$$

Combining the equations above gives the correct classical distribution function for binary stars describing a state of thermal equilibrium:

$$f(E) \propto E^{-5/2} e^{-E/kT} \; .$$

It is immediately obvious that the equilibrium distribution found above is rather pathological: it diverges *both* for very hard and for very soft binaries! For soft binaries the divergence is not a serious problem. As mentioned above, we expect many very soft binaries to form in very wide orbits since so much phase space is available for them. These binaries form and break up at such high rates that most of them only traverse a tiny part of one orbital revolution before breaking up again, and for all practical purposes these short, formally momentarily bound encounters between stars play no role of any significance.

The divergence at the hard side, for binaries with $E >> kT$ is much more serious. In the limit of point masses, a binary can absorb an unlimited amount of (negative) binding energy, which automatically translates via the Boltzmann factor into an unlimited population of these low-lying energy levels. Here we have a clear indication that in practice a self-gravitating system can never attain true thermodynamic equilibrium, even though it perpetually strives to reach that state, as we will see below ("the great tragedy of self-gravitating systems," as Jeremy Goodman expressed it).

What will happen instead? Starting from a background of single stars only, at first many soft binaries will be created, most of which will be destroyed soon afterwards. After a while, some binaries will harden against the odds, and an occasional binary might be formed at an already somewhat higher binding energy. Those two effects will provide a slow but steady increase in the number density of hard binaries, in an effort to approach the equilibrium distribution. For hard binaries, at any given value for the binding energy $E_{bin} >> kT$ a balance will be reached between the incoming flux of binaries which were softer and are hardening to reach E_{bin} and the outgoing flux of binaries which already had an energy around E_{bin} and are presently moving towards even larger values of binding energy. The flux in the opposite direction is much smaller, since hard binaries predominantly tend to harden.

Heggie (1975) showed that for hard binaries the average energy increase per strong scattering encounter varies inversely proportional to the cross section for such an encounter. Therefore, even though the cross sections for strong encounters decrease with time, on average hard binaries harden at a constant rate. This implies a flat distribution of binaries, all traveling at the same speed towards larger values of binding energy, as modeled by Retterer (1980ab, 1984) who used the strong-scattering limit of Heggie's analytic expressions for the binary hardening rates. Goodman and Hut (1985) have followed up on Retterer's work using the results of detailed numerical experiments. These results were directly derived from the experimental data and were used in tabulated form; for practical reasons they are published in graphical form (Hut, 1983b, 1984a, 1985). Whereas Retterer calculated the time-dependent progressive hardening of binaries, Goodman and Hut solved an integral equation for the time-independent steady-state solution. This integral equation forces an exact balance at every value of the binary binding energy between the binaries arriving at that energy value (via creation, hardening or softening) and the binaries leaving that energy value (via ionization, hardening or softening). Numerically, this equation was solved by discretization and matrix inversion.

Fig. 1 shows both the equilibrium distribution (short-dashed line) and the steady-state solution (full line) as derived by Goodman and Hut (1985). Indeed, the steady-state solution approaches a constant value for large $E_{bin} \gg kT$ and thereby falls far below the exponentially rising equilibrium distribution. Fig. 1 agrees qualitatively with Fig. 3 of Retterer (1980a), with a quantitative difference of about a factor of two in the near-constant value of the distribution function for very hard binaries with $E_{bin} = 100$ kT: Retterer found a value of 0.7 whereas Goodman and Hut found a value of 0.3 (in Retterer's units).

The long-dashed line in Fig. 1 indicates the effects of introducing a finite escape velocity. For very hard binaries ($E_{bin} > 100$ kT) there is a significant chance that a binary will escape from the cluster after a strong encounter because of the large recoil velocity, which explains the rapid fall-off at large x-values. Introducing a finite escape velocity in a homogeneous background amounts to adopting a model of a star cluster in a finite-depth square-well potential. This computation (Goodman and Hut, 1985) forms the logical next step following Spitzer and Härm's (1958) derivation of the rate of escape of single stars due to two-body encounters in a finite-depth square-well potential: it gives the rate of escape of binaries due to three-body encounters, as discussed below. Figs. 2 and 3 show that the formation of full-grown hard binaries is not unlike that of most plants and simpler animals: many are created but while they grow up they are decimated at every stage. A quantitative measure is given in the figure captions of Figs. 2 and 3, showing that most hard binaries are born and have spent their youth in the dangerous areas where $E_{bin} \ll$ kT. Only those few which manage to grow harder than 2.9 kT finally face favorable odds for survival.

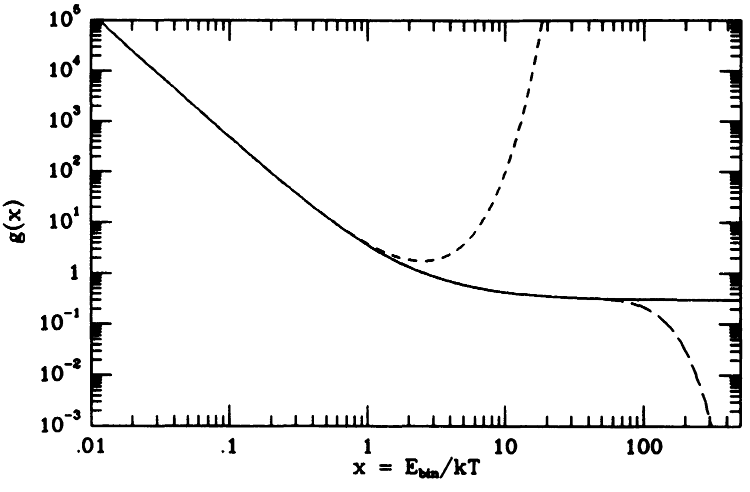

Fig. 1. The steady-state distribution of binaries g(x) in a homogeneous static stellar background. The binary binding energy x is given in units of $kT = m\sigma^2$, where σ is the one-dimensional velocity dispersion of the stars all of which have mass m. The short dashes follow the equilibrium distribution. The long dashes indicate the effects of a finite square-well potential, of depth $\phi = 6kT/m$.

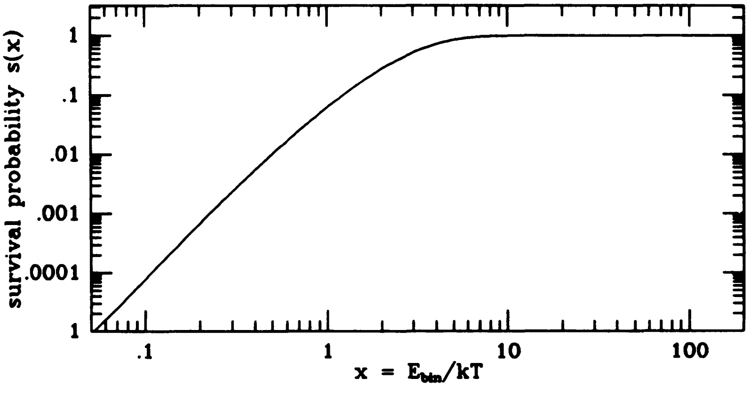

Fig. 2. The survival probability s(x) for binaries in a homogeneous static stellar background. A newly formed binary has a more than fifty-fifty chance to survive forever only if the initial energy $x > x_s = 2.9$.

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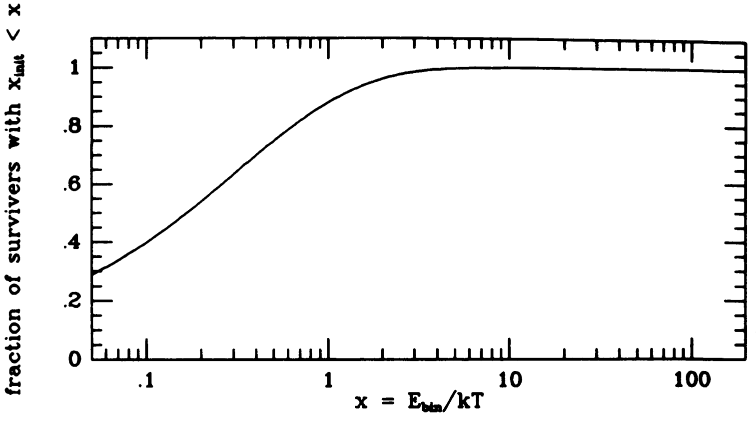

Fig. 3. The ordinate shows the fraction of permanent binaries which are born with an energy x_{init} less than x. The value $x = x_c = 0.17$ marks the median energy for the creation of binaries which survive forever.

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For the modeling of the energy generation in star clusters, the most important result of these new calculations presented in the figures is an accurate determination of the formation rate of permanent hard binaries. The value found by Goodman and Hut (1985) for the creation rate of these "immortal" binaries is

$$C = (0.90 \pm 0.05) \frac{n^3 G^5 m^5}{\sigma^9}$$

per unit volume and per unit time, where n denotes the density of field stars, each of which has a mass m, σ the one-dimensional velocity dispersion of the field stars, and G the gravitational constant. The fact that the coefficient is so close to unity is somewhat fortuitous; if we had chosen to express our result in terms of the three-dimensional velocity dispersion instead, the coefficient would have been two orders of magnitude larger. In astrophysical units the creation rate of permanent binaries reads

$$C = (1.36 \pm 0.08) \times 10^{-12} \left(\frac{n}{10^5 \mathrm{pc}^{-3}} \right)^3 \left(\frac{m}{M_\odot} \right)^5 \left(\frac{10\mathrm{km/s}}{\sigma} \right)^9 \mathrm{pc}^{-3}\mathrm{yr}^{-1}.$$

Heggie guessed a value of 1.3×10^{-12} for the coefficient in this expression (Heggie, 1980, eq. 19), adding that "the numerical coefficient was not well established." The fact that his guess turned out to lie within the tight error bars of the experimental value is especially remarkable because of the extreme sensitivity to the choice of natural units, given the high powers occurring in the equations above. Clearly, only at the last stages of core collapse can densities be reached which are high enough to dynamically form hard binaries, and then only in a limited volume with dimensions much less than a parsec (*cf.* Heggie's review article in these proceedings).

5. FORMATION AND EVOLUTION RATES OF BINARIES IN A DYNAMIC STELLAR BACKGROUND

In a realistic self-gravitating background of single stars, binary formation and subsequent hardening will be much more complicated than the simple picture sketched in the previous section for a static homogeneous background. Two major complications arise. Within the approximation of point particles the first problem of calculating the self-consistent interplay of binaries and single stars in the combined evolving potential is already rather formidable. However, a realistic description has to take into account the fact that stars can tidally capture each other, and can physically collide and sometimes even merge in the process. This second problem leads us outside stellar dynamics and introduces many uncertainties concerning mass loss, tidal effects, stellar evolution in (detached, semi-detached or contact) binaries, etc. Both problems are addressed in a variety of different approaches in a number of different contributions to the present symposium, and no attempt will be made here to review all these approaches, most of which are very recent and are still under development. Spitzer's review of pre-core-collapse models and Heggie's review of post-core-collapse models in the present proceedings give many references to the literature.

Nearly a quarter century ago Hénon (1961) published his Ph.D. thesis, in which he not only predicted the phenomenon of core collapse, but even went so far as to suggest the solution, at least on the level of point-particle approximations: the formation of binaries. It is encouraging that we now have begun to finally fill in the quantitative details of Hénon's suggestion, thereby solving a long-standing mathematical physics problem, so simple in its formulation: what is the long-term evolution of a system of self-gravitating point masses?

The non-gravitational effects which are expected to be crucially important in the later stages of the evolution of globular clusters are only beginning to be explored now (*cf.* Ostriker's review in the present proceedings). Possible complications such as *e.g.* repeated merging of stars leading to the formation of a relatively massive star are essentially unexplored as yet. Such a star might evolve on a time scale of order a million years before undergoing a supernova explosion which will cause a significant amount of mass to escape from the globular cluster core. These effects cannot yet be accurately estimated but they might turn out to have an importance comparable to that of stellar ejection by three-body encounters.

Now that our understanding of core collapse of globular clusters has improved dramatically over the last few years, the stage is set for further investigations as to the precise character of the central energy source. We know that stars can shine mainly because they utilize nuclear energy and in some cases energy from gravitational contraction. What maintains the central oven which enables globular clusters to boil off stars continuously at their surface: gravitational 'binary burning,' or stellar collisions and explosions? The answer is likely to contain elements from either type, and theoretical studies should explore these and other possibilities. The prospect of observational data with unprecedented resolution from Space Telescope (*cf.* Bahcall, these proceedings) forms an extra encouragement to study the complicated dynamics of the long-term evolution of globular clusters.

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DISCUSSION

KING: With Space Telescope we should be able to resolve, in nearby globular clusters, soft binaries with E/kt less than a few tenths. Is it reasonable to look for these binaries?

HUT: Unfortunately, this will not be a very interesting study from a dynamical point of view because the equilibrium distribution of wide binaries is nearly the same as that of a random distribution of uncorrelated single stars. The excess correlation is given by the Boltzmann factor exp $(E_{bin}/kT) << 1$ for soft binaries.

TERLEVICH: In order to apply your calculations to open clusters and very young globular clusters; have you thought about including a mass function and therefore mass loss from stellar evolution?

HUT: Yes, I have thought about that, but I have not yet found the time to include these effects.

SPITZER: For the similar problem of H-atom ionization, the infinite number of high-n atoms is eliminated by consideration of the perturbation experienced by atoms with very extended electron orbits. Have you considered similar effects which would eliminate the infinity in the computed population of soft binaries?

HUT: Yes, and I can quote a result recently derived by Scott Tremaine (private communication): In every self-gravitating system the number of binaries which survive long enough for the two stars to complete at least one revolution around each other, is of order unity ($\sim$6) independent of N, the total number of stars in the system. The much larger number of very soft binaries expected theoretically does exist only temporarily and is formed by random short-living encounters.

SHAPIRO: How is your steady-state solution modified if, still in the context of the Newtonian point-mass approximation, you included the dissipative (hardening) effect due to gravitational radiation in binaries? One might imagine systems which, though nonrelativistic ($v/c << 1$), decay due to gravitational radiation on timescales *shorter* than other relevant timescales (i.e., relaxation) in the problem. Have you considered this effect?

HUT: I have not yet included this effect. If one were to include tidal formation of very close binaries, this effect could be important, together with other effects such as magnetic braking. Very hard binaries in globular clusters are also in this respect analogous to cataclysmic variables (*cf.* Hut & Verbunt, 1983, *Nature* 301, 587).

LARSON: This is a remark related to Shapiro's question about possible effects of gravitational radiation. In my poster presentation I speculate that the cores of globular clusters are dominated by heavy remnants with a spectrum of masses possibly extending up to several thousands of solar masses. The two most massive objects are likely to form a central binary that absorbs a substantial fraction of the binding energy of the system. It was recently pointed out to me by Kip Thorne that such a binary consisting of two moderately massive black holes would decay by gravitational radiation in a time shorter than the Hubble time. So maybe gravitational radiation would have significant effects in a condensed core of heavy remnants.

GOODMAN: The question of energy loss by gravitational radiation was set as an exercise in Jerry Ostriker's dynamics class, and I performed it. For point masses, the amount of gravitational radiation lost per relaxation time is comparable to the binding energy of the cluster. However, the amount of radiation energy released has such a steep dependence on impact parameter that the effect goes away when one puts in a finite size (e.g., 10km) for the stars.

JERNIGAN: Soft binaries are not clearly observable. How many hard binaries will be formed in a real cluster? How does one define a hard binary in light of the range of field parameters?

HUT: In a real cluster at most a few dynamically formed hard binaries will be present in the core at any given time. A hard binary is defined as having a binding energy $E_{bin} \gg kT = m\sigma^2$ where the one-dimensional velocity dispersion σ is a *local* average.

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