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A STOCHASTIC ERGODIC THEOREM FOR GENERAL ADDITIVE PROCESSES

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ABSTRACT. In this article, we obtain the stochastic ergodic theorem for general additive processes. That is, we prove that there exists $\bar{f} \in L_1$, such that $\lim_n \mu(A \cap \{x : |f_n(x) - \bar{f}(x)| > \alpha\}) = 0$ whenever $\alpha > 0$ and A is a measurable set with $\mu(A) < \infty$, where $f_n = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} U_{(i,j)}f, f \in L_1$ and $\{U_{(i,j)}\}$ an arbitrary (two dimensional) semigroup of L_1-contractions. This result generalizes the stochastic ergodic theorem (SET) of U. Krengel and the SET of M. A. Akcoglu and L. Sucheston.

1. Introduction. In 1966 U. Krengel [5] introduced the concept of stochastic convergence of Cesaro averages and proved the stochastic ergodic theorem (SET) in the one dimensional case. M. A. Akcoglu and L. Sucheston [1] later gave a more elementary proof of this result using a truncation argument in a more general context, that is, for multidimensional superadditive processes. The technique they employed to prove the multidimensional SET for superadditive processes also yields to the multidimensional SET for general additive purposes (i.e., the additive processes with respect to a not necessarily positive semigroup of L_1- contractions $\{T_u\}$.) if $\{T_u\}$ is dominated by a semigroup of positive L_1-contractions [1, p. 344]. However, it is well known that if T and S are two L_1-contractions, then $|TS| = |T|\,|S|$ is not true in general [3], where $|T|$ denotes the linear modulus of T. Hence SET does not hold in the case at a general semigroup of L_1-contractions. Our aim in this paper is to obtain the SET for general additive processes without any further condition on $\{T_u\}$, using a technique due to A. Brunel [2].

We will prove the SET in the two dimensional case only, for the sake of simplicity, since the extension of the results to arbitrary n-dimensional case, $n \geqq 2$, is straightforward.

2. Preliminaries. Let (X, F, μ) be a σ-finite measure space and $L_1 = L_1(X, F, \mu)$ be the classical Banach space of real-valued integrable functions on X. All the relations below are defined modulo sets of measure zero. The characteristic functions of a set $A \in F$ will be denoted by χ_A, and the positive cone of L_1 will be denoted by L_1^+, and $L_1(P) = \{f \in L_1 : \text{support of } f \subset P, P \in F\}$.

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Let $\mathbf{R}^2 = \mathbf{R}\times\mathbf{R}$ be the usual two-dimensional real vector space, considered with all its usual structure. In particular, $\mathbf{R}^2$ is partially ordered in the usual way, i.e., for any $(u, v), (t, r) \in \mathbf{R}^2$, $(u, v) \leqq (t, r)$ if $u \leqq t$ and $v \leqq r$, $(u, v) < (t, r)$ if $(u, v) \leqq (t, r)$ and $(u, v) \neq (t, r)$. The positive cone of $\mathbf{R}^2$ is $\mathbf{R}^2_+$ and $\mathbf{N}$ will denote the set of non-negative integers and $\mathbf{K} = \mathbf{N}^2$. For any $t \in \mathbf{R}$, $\underline{t} = (t, t) \in \mathbf{R}^2$.

Consider a semigroup $U = \{U_{(m,n)}\}_{(m,n)\in\mathbf{K}}$ of L_1-contractions with $U_0 = I$, the identity operator on L_1. That is,

(2.1) $U_{(m,n)}$ is a linear operator on L_1 for each $(m, n) \in \mathbf{K}$.

(2.2) $\|U_{(m,n)}\|_1 \leqq 1$ for each $(m, n) \in \mathbf{K}$.

(2.3) $U_{(m,n)}U_{(t,r)} = U_{(m+t,n+r)}$ for each $(m.n), (t, r) \in \mathbf{K}$.

Given $f \in L_1$, the family $\{\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} U_{(i,j)}f\}_{(m,n)\in K}$ is called a U-additive process. We will let $A_{(n,m)}f = \frac{1}{nm}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} U_{(i,j)}f$. Notice that for each (m, n), $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} U_{(i,j)}f$ is in L_1.

Definition (2.4) *A sequence $\{f_n\} \subset L_1$ is said to converge stochastically to $f \in L_1$ if*

$$\lim_{n\to\infty} \mu(A \cap \{x : |f_n(x) - f(x)| > \alpha\}) = 0$$

whenever $\alpha > 0$ and $A \in F$ with $\mu(A) < \infty$.

This definition (2.4) is equivalent to [1]:

$$\lim_{n\to\infty} \||f_n - f| \wedge \emptyset\| = 0 \text{ whenever } \emptyset \in L_1^+. \tag{2.5}$$

Moreover, if μ is a finite measure, definition (2.4) is simply the convergence in measure.

The following decomposition theorem is due to M. A. Akcoglu and L. Sucheston [1], which is an essential tool in the proof. We adopt it here without proof.

Theorem (2.6) *Let $\{T_i\}_{i\in I}$ be an arbitrary family of positive L_1-contractions. Then there exists a unique (modulo sets of measure zero) partition of X into two sets P and N, called the positive and null parts of $\{T_i\}$, such that*

(i) *There is a $g \in L_1^+$ such that $T_i g = g$ for all $i \in I$ and such that $P = \operatorname{supp} g = \{x : g(x) > 0\}$.*

(ii) *if f_n is a bounded sequence in L_1^+ such that*

$$\lim_n \|f_n - T_i f_n\| = 0 \text{ for each } i \in I,$$

then $\chi_N f_n$ converges stochastically to zero.

If P and N are positive and null parts of a semigroup $\{T_{(u,v)}\}$ of positive L_1-contractions, then for each $(u, v) T_{(u,v)}$ maps $L_1(P)$ and the semigroup $\{T_{(u,v)}|_{L_1(P)}\}$ has the same properties as $\{T_{(u,v)}\}$.

Let $h \in L_1^+(\mu), h > 0$, be an invariant function for an $L_1(\mu)$-contraction T, i.e. $Th = h$. Then $m = h \cdot \mu$ defines a new measure which is finite and equivalent to μ. In

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this case $f \in L_1(m)$ if and only if $f \cdot h \in L_1(\mu)$. Hence $L_1(\mu)$ and $L_1(m)$ are isomorphic. For any $f \in L_1(m)$, define $T'f = h^{-1}T(fh)$. Then T' is an $L_1(m)$-contraction, and furthermore $T'1 = 1$, so T' is also an L_∞-contraction. Clearly $T'^n f = h^{-1}T^n(fh)$ for all $f \in L_1(m)$.

3. The main result. In order to obtain the result announced in the introduction, we will make use of the following result due to A. Brunel [2, Theorem 3.7] which will enable us to dominate a given (not-necessarily positive) semigroup by a positive one. We state its two dimensional version here without proof:

THEOREM (3.1) *Let $U = \{U_{(i,j)}\}_{(i,j)\in K}$ be a semigroup of L_1-contractions. Then there exists a constant $\delta > 0$ and a positive L_1-contraction T (called the barycentric operator) defined by $T = \sum_{(i,j)\in K} a_{ij}|U_{(i,0)}|\,|U_{(0,j)}|$, such that for any $f \in L_1$*

$$\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}|U_{(i,j)}f| \leqq \frac{\delta}{[\sqrt{n}+1]}\sum_{i=0}^{[\sqrt{n}+1]-1} T^i|f|, \tag{3.2}$$

where $[a]$ denotes the greatest integer less than a.

Also we need the following result:

THEOREM (3.3) *If $f \in L_1(P)$, then $\lim_{(m,n)\to(\infty,\infty)} A_{(m,n)}f$ exists in L_1-norm. (where $m \to \infty$, $n \to \infty$ independently), where P is the positive part of T in Theorem (3.1).*

PROOF. Let h be the T-invariant function obtained by Theorem (2.6) (i) with $P = \operatorname{supp} h$. Then $g = h \vee 4\delta h$ is also T-invariant where δ is the constant in Theorem (3.1). If $f \in L_1$ with $|f| \leqq g$, then using (3.2) we have $|U_{(1,0)}f| \leqq g$ and $|U_{(0,1)}f| \leqq g$. Now passing to a new reference measure $m = g \cdot \mu$ we can work on $L_1(P,m)$ and assume that $U_{(i,j)} : L_1(P,m) \to L_1(P,m)$ is a contraction as well as an L_∞-contraction for all $(i,j) \in K$. Then the result follows from Theorem VIII.6.9 [4].

Now we state and prove the main result.

THEOREM (3.4) *Let $U = \{U_{(m,n)}\}_{(m,n)\in K}$ be a semigroup of L_1-contractions. Then, for any $f \in L_1$, $\lim_{n\to\infty} A_{\underline{n}}f$ exists stochastically, i.e. there is an $\bar{f} \in L_1$ such that $\lim_{n\to\infty} \||A_{\underline{n}}f - \bar{f}| \wedge \emptyset\| = 0$ for all $\emptyset \in L_1^+$. Furthermore $\lim_n \chi_P A_{\underline{n}}f = \bar{f}$ in L_1 norm.*

PROOF. First, by using (3.1) one can dominate the process $\{A_{\underline{n}}f\}$ by a (one-dimensional) process $\{A_k|f|\}$ where $k = k(n) = [\sqrt{n}+1]$ in (3.2), that is

$$|A_{\underline{n}}f| \leqq \delta A_k|f| = \frac{\delta}{k}\sum_{i=0}^{k-1} T^i|f|.$$

Now to get the result, we will make use of an idea of M. A. Akcoglu and L. Sucheston [1]. Let P and N be the positive and null parts of T as given by theorem (2.6). Then $|\chi_N A_{\underline{n}}f| \leqq \delta\chi_N A_k|f|$. For any $m \geqq 1$, we have $\lim_{k\to\infty} \|A_k|f| - T^m A_k|f|\| = 0$. This, combined with the fact that $\{A_k|f|\}$ is a bounded sequence, gives that $\chi_N A_k|f| \longrightarrow 0$ stochastically by Theorem (2.6) (ii). Hence $\lim_n \chi_N A_{\underline{n}} f = 0$ stochastically.

By Theorem (3.3), $\lim_n A_{\underline{n}}(\chi_P A_{\underline{m}} f)$ exists in L_1-norm for each $m \geqq 1$. Let $k = [\sqrt{n}+1]$ and $r = [\sqrt{m}+1]$. Now

$$\begin{aligned}
\|\chi_P A_{\underline{n}} \chi_N A_{\underline{m}} f\| &\leqq \delta^2 \|\chi_P A_k \chi_N A_r |f|\| \\
&= \delta^2 \|\chi_P A_k A_r |f| - \chi_P A_k \chi_P A_r |f|\| \\
&\leqq \delta^2 \left[\|\chi_P A_k A_r |f|\| - \|\chi_P A_k \chi_P A_r |f|\|\right] \\
&= \delta^2 \left[\|\chi_P A_k A_r |f|\| - \|\chi_P A_r |f|\|\right] \\
&\leqq \delta^2 \left[\lim_k A_k |f|\| - \|\chi_P A_r |f|\|\right],
\end{aligned}$$

which goes to o as $m \longrightarrow \infty$, uniformly with respect to n.

Hence

$$\begin{aligned}
&\|A_{\underline{n}} \chi_P A_{\underline{m}} f - \chi_P A_{\underline{n}} A_{\underline{m}} f\| \\
&\quad = \|A_{\underline{n}} \chi_P A_{\underline{m}} f - \chi_P A_{\underline{n}} \chi_P A_{\underline{m}} f\| + \|\chi_P A_{\underline{n}} \chi_N A_{\underline{m}} f\| \\
&\quad = \|\chi_N A_{\underline{n}} \chi_P A_{\underline{m}} f\| + \|\chi_P A_{\underline{n}} \chi_N A_{\underline{m}} f\|.
\end{aligned}$$

Now, given $h \in L_1^+(P)$, consider the nfunction $h \wedge rg$, where g is the invariant function for T with $P = \operatorname{supp} g$ and r is a positive integer. Then

$$\begin{aligned}
0 \leqq \chi_N A_{\underline{n}}(h \wedge rg) &\leqq \delta \chi_N A_k (h \wedge rg) \text{ by (3.2)} \\
&\leqq \delta r \chi_N A_k g = \delta r \chi_N g = 0
\end{aligned}$$

by Theorem (2.6) (i). Then, the fact that $\lim_{r \longrightarrow \infty} \|h - h \wedge rg\|_1 = 0$ implies that $\chi_N A_{\underline{n}} h = 0$. So given $\epsilon > 0$, find m such that $\sup_n \|A_{\underline{n}} \chi_P A_{\underline{m}} f - \chi_P A_{\underline{n}} A_{\underline{m}} f\| < \epsilon$. Therefore

$$\begin{aligned}
&\limsup_{n \longrightarrow \infty} \|A_{\underline{n}} \chi_P A_{\underline{m}} f - \chi_P A_{\underline{n}} f\| \\
&\qquad \leqq \limsup_{n \longrightarrow \infty} \|A_{\underline{n}} \chi_P A_{\underline{m}} f - \chi_P A_{\underline{n}} A_{\underline{m}} f\| \\
&\qquad + \limsup_{n \longrightarrow \infty} \|A_{\underline{n}} A_{\underline{m}} f - A_{\underline{n}} f\| < \epsilon,
\end{aligned}$$

since $\|A_{\underline{n}} A_{\underline{m}} f - A_{\underline{n}} f\| \longrightarrow 0$ as $n \longrightarrow \infty$ for a fixed m. Thus, since $\lim_n A_{\underline{n}} \chi_P A_{\underline{m}} f$ exists in L_1-norm, we see that $\lim_n \chi_P A_{\underline{n}} f$ exists in L_1-norm. Hence there exists $\bar{f} \in L_1$ such that $\lim \|\chi p A_{\underline{n}} f - \bar{f}\| = 0$, and hence

$$\begin{aligned}
\lim_n \||A_{\underline{n}} f - \bar{f}| \wedge \emptyset\| &\leqq \lim_n \||\chi_P A_{\underline{n}} f - \bar{f}| \wedge \emptyset\| + \lim_n \||\chi_N A_{\underline{n}} f| \wedge \emptyset\| \\
&\leqq \lim_n \|\chi_P A_{\underline{n}} f - \bar{f}\| + \lim_n \||\chi_N A_{\underline{n}} f| \wedge \emptyset\| \\
&= 0 \text{ for all } \emptyset \in L_1^+.
\end{aligned}$$

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