Mathematical Structures in Computer Science (2021), 1–21
doi:10.1017/S096012952100030X

CAMBRIDGE UNIVERSITY PRESS

PAPER

Elementary fibrations of enriched groupoids[1]

Jacopo Emmenegger[1], Fabio Pasquali[2] and Giuseppe Rosolini[2*]

[1]School of Computer Science, University of Birmingham, Birmingham, UK, and [2]DIMA, Università di Genova, Genova, Italy
*Corresponding author. Email: rosolini@unige.it

(Received 31 July 2020; revised 13 September 2021; accepted 13 September 2021)

Abstract

The present paper aims at stressing the importance of the Hofmann–Streicher groupoid model for Martin Löf Type Theory as a link with the first-order equality and its semantics via adjunctions. The groupoid model was introduced by Martin Hofmann in his Ph.D. thesis and later analysed in collaboration with Thomas Streicher. In this paper, after describing an algebraic weak factorisation system (L, R) on the category $\mathcal{C}$-$\mathcal{Gpd}$ of $\mathcal{C}$-enriched groupoids, we prove that its fibration of algebras is elementary (in the sense of Lawvere) and use this fact to produce the factorisation of diagonals for (L, R) needed to interpret identity types.

Keywords: Elementary fibrations; algebraic weak factorisation system; groupoid model

1. Introduction

The work that Martin Hofmann produced in his Ph.D. thesis (Hofmann 1997) was revolutionary at the time. As for many relevant mathematical results, it would require time to be digested, appreciated in full and complemented by the scientific community. When it happened, the effect was stunning as the reader can appreciate browsing through Univalent Foundations Program (2013). The groupoid model which was presented in Hofmann's thesis and put to use in two papers in collaboration with Thomas Streicher (Hofmann and Streicher 1994; 1998) became immediately the benchmark for various independence results in type theory whose applications in automated theorem proving were being developed.

The proof-relevant character of dependently typed languages makes equality in that context a much subtler concept than equality in the first-order logic. While the latter has a robust and elegant algebraic description in terms of adjunctions in Lawvere (1970), there is nothing comparable to it for the semantics of equality in dependent type theories. The result of Hofmann and Streicher (1998) showed that it was possible to make sense of the proof-relevant character of equality in Martin-Löf type theory using the structure provided by groupoids.

As already mentioned, this result was pivotal in the recognition of structures from homotopy theory in the semantics of identity types and to the birth of Homotopy Type Theory. The importance of the Hofmann–Streicher groupoid model as the first step towards algebraic treatments of higher equalities is universally recognised.

The present paper relates the groupoid model with the first-order equality and its semantics via adjunctions. More specifically, we prove that the fibration of groupoids in the Hofmann–Streicher model is elementary (in the technical sense specified in Section 2). An analysis of this structure shows that, in the groupoid model, it gives rise to the Hoffman–Streicher interpretation of the identity types. In fact, it is possible to reconstruct the interpretation of identity types (including the elimination rule) from the elementary structure of the fibration.

For this reason, in Section 3, we exhibit a new class of examples with groupoids enriched in a category $\mathcal{C}$ with finite limits, which form a category $\mathcal{C}\text{-}\mathcal{G}pd$ also with finite limits. We begin describing an algebraic weak factorisation system (L, R) on $\mathcal{C}\text{-}\mathcal{G}pd$ that will serve as providing the interpretation of type dependency. This algebraic weak factorisation system is the enriched version of the algebraic weak factorisation system on $\mathcal{G}pd$ which is part of the canonical (or folk) algebraic Quillen model structure on $\mathcal{G}pd$. We then prove that the fibration $\mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathcal{C}\text{-}\mathcal{G}pd$ of algebras for the monad on the right functor R is elementary, generalising the standard ($\mathcal{S}et$-enriched) case.

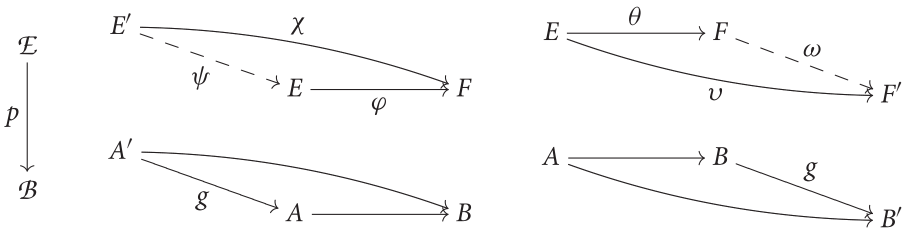
We then proceed to investigate the relationship between this elementary structure and the structure needed to interpret identity types in a fibration of algebras for an algebraic weak factorisation system which, as shown in Gambino and Larrea (2021), amounts to a suitable functorial factorisation of diagonals. In Section 4, we prove that the algebraic weak factorisation system on $\mathcal{C}\text{-}\mathcal{G}pd$ does have such a factorisation. As it clearly appears from the proof, the construction of such a factorisation makes heavy use of the elementary structure of the fibration $\mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathcal{C}\text{-}\mathcal{G}pd$. This observation indicates that it should be possible, given an algebraic weak factorisation system on a category $\mathcal{C}$ with weak finite limits, to isolate conditions that would ensure the existence of a suitable factorisation of diagonals from the assumption that the fibration of algebras is elementary. We leave this question to future investigations.

Section 5 concludes the paper with the observation how the above construction can be iterated. In the specific case of the enrichment in $\mathcal{G}pd$, the enrichment produces the categories of n-groupoids, together with forgetful functors. Each of these is equipped with an algebraic weak factorisation system whose fibration of algebras is elementary.

2. Elementary Fibrations

Let $p\colon \mathcal{E} \longrightarrow \mathcal{B}$ be a functor. An arrow φ in $\mathcal{E}$ is said to be ***over*** an arrow f in $\mathcal{B}$ when $p(\varphi) = f$. For A in $\mathcal{B}$, the fibre $\mathcal{E}_A$ is the subcategory of $\mathcal{E}$ of arrows over id_A. In particular, an object E in $\mathcal{E}$ is said to be ***over*** A when $p(E) = A$, and an arrow φ is ***vertical*** when $p(\varphi) = \mathrm{id}_A$.

Recall that an arrow $\varphi\colon E \longrightarrow F$ is ***cartesian*** if, for every $\chi\colon E' \longrightarrow F$ such that $p(\chi)$ factors through $p(\varphi)$ via an arrow $g\colon A' \longrightarrow A$, there is a unique $\psi\colon E' \longrightarrow E$ over g such that $\varphi\psi = \chi$, as in the left-hand diagram below. And an arrow $\theta\colon E \longrightarrow F$ is ***cocartesian*** if it satisfies the dual universal property of cartesian arrows depicted in the right-hand diagram below.



(1)

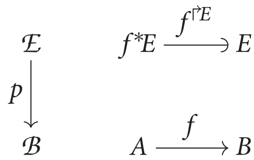
Once we fix an arrow $f\colon A \longrightarrow B$ in $\mathcal{B}$ and an object F in $\mathcal{E}_B$, a cartesian arrow $\varphi\colon E \longrightarrow F$ over f is uniquely determined up to isomorphism, that is, if $\varphi'\colon E' \longrightarrow F$ is cartesian over f, then there is a unique vertical isomorphism $\psi\colon E' \longrightarrow E$ such that $\varphi\psi = \varphi'$.

Clearly, every property of cartesian arrows applies dually to cocartesian arrows. So for an arrow $f\colon A \longrightarrow B$ in $\mathcal{B}$ and an object E in $\mathcal{E}_A$, a cocartesian arrow $\theta\colon E \longrightarrow F$ over f is uniquely determined up to vertical isomorphism.

In the following, we often write cartesian arrows as $\longrightarrow$ and vertical arrows as $\longrightarrow$.

https://doi.org/10.1017/S096012952100030X Published online by Cambridge University Press

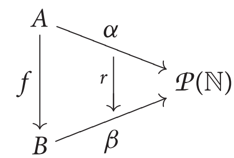
A functor $p\colon \mathcal{E} \longrightarrow \mathcal{B}$ is a ***fibration*** if, for every arrow $f\colon A \longrightarrow B$ in $\mathcal{B}$ and for every object E in $\mathcal{E}_B$, there is a ***cartesian lift*** of f into E, that is, an object f^*E and a cartesian arrow $f^{\urcorner E}\colon f^*E \longrightarrow E$ over f; so part of the diagram (1) becomes



(2)

A ***cleavage*** for the fibration p is a choice of a cartesian lift for each arrow $f\colon A \longrightarrow B$ in $\mathcal{B}$ and object F in $\mathcal{E}_B$, and a ***cloven fibration*** is a fibration equipped with a cleavage. In a cloven fibration, for every $f\colon A \longrightarrow B$ in $\mathcal{B}$, there is a functor $f^*\colon \mathcal{E}_B \longrightarrow \mathcal{E}_A$ called ***reindexing*** along f. Henceforth, we assume that fibrations come with a cleavage.

Example 2.1. Let $p\mathcal{A}sm$ denote the category whose objects are pairs (A, α) where A is a set and $\alpha\colon A \longrightarrow \mathcal{P}(\mathbb{N})$ a function, and whose arrows are pairs (f, r) as in the diagram:



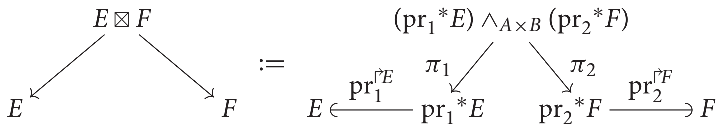
where $f\colon A \longrightarrow B$ is a function and $r\colon \bigcup_{a\in A} \alpha(a) \longrightarrow \bigcup_{b\in B} \beta(b)$ is a total function obtained as the restriction of a partial recursive function to the two subsets such that for all $a \in A$ and all $n \in \alpha(a)$ it holds that $r(n) \in \beta(f(a))$. Composition is $(f, r)(f', r') = (ff', rr')$ The functor $\mathsf{dom}\colon p\mathcal{A}sm \longrightarrow \mathcal{S}et$ acting as the first projection is a cloven fibration. Given a function $f\colon A \longrightarrow B$ and (B, β) over B, the arrow $(f, i)\colon (A, \beta f) \longrightarrow (B, \beta)$ with $i = \lambda x.x$ is over f. It is actually cartesian. Indeed for every $(g, \psi)\colon (C, \gamma) \longrightarrow (B, \beta)$ and $k\colon C \longrightarrow A$ with $fk = g$, consider $(k, \psi)\colon (C, \gamma) \longrightarrow (A, \beta f)$, it is $(f, \mathrm{id}_{\mathbb{N}})(k, \psi) = (fk, \psi))(g, \psi)$. For uniqueness, take (k, ψ'), from $(g, \psi) = (f, \mathrm{id}_{\mathbb{N}})(k, \psi') = (g, \psi')$, follows $\psi = \psi'$.

A fibration $p\colon \mathcal{E} \longrightarrow \mathcal{B}$ ***has finite products*** if the base $\mathcal{B}$ has finite products as well as each fibre $\mathcal{E}_A$, and each reindexing functor preserves products. Equivalently, both $\mathcal{B}$ and $\mathcal{E}$ have finite products and p preserves them, see (Hermida 1994, Corollary 3.7).

Notation 2.2. When we write 1 we refer to any terminal object in $\mathcal{B}$ and, similarly for objects A and B in $\mathcal{B}$, when we write $A \times B$, $\mathrm{pr}_1\colon A \times B \longrightarrow A$ and $\mathrm{pr}_2\colon A \times B \longrightarrow B$, we refer to any diagram of binary products in $\mathcal{B}$. Universal arrows into a product induced by lists of arrows shall be denoted as $\langle f_1, \ldots, f_n \rangle$, but lists of projections $\langle \mathrm{pr}_{i_1}, \ldots, \mathrm{pr}_{i_k} \rangle$ will always be abbreviated as $\mathrm{pr}_{i_1,\ldots,i_k}$. In particular, as an object A is a product of length 1, sometimes we find it convenient to denote the identity on A as pr_1, the diagonal $A \longrightarrow A \times A$ as $\mathrm{pr}_{1,1}$ and the unique $A \longrightarrow 1$ as pr_0. As the notation is ambiguous, we shall always indicate domain and codomain of lists of projections and sometimes we may distinguish projections decorating some of them with a prime symbol.

We shall employ a similar notation for terminal objects, binary products and projections in a fibre $\mathcal{E}_A$, as $\top_A$, $E \wedge_A F$, $\pi_1\colon E \wedge_A F \longrightarrow E$ and $\pi_2\colon E \wedge_A F \longrightarrow F$, dropping the subscript in $\top_A$ and $\wedge_A$ when it is clear from the context. Moreover, given objects E in $\mathcal{E}_A$ and F in $\mathcal{E}_B$, write

https://doi.org/10.1017/S096012952100030X Published online by Cambridge University Press



for a product diagram of E and F in $\mathcal{E}$, where we employed the notation introduced in (2) in the diagram on the right-hand side. Given a third object G and two arrows $\varphi_1 \colon G \longrightarrow E$ and $\varphi_2 \colon G \longrightarrow F$, we denote the induced arrow into $E \boxtimes F$ also as $\langle \varphi_1, \varphi_2 \rangle$.

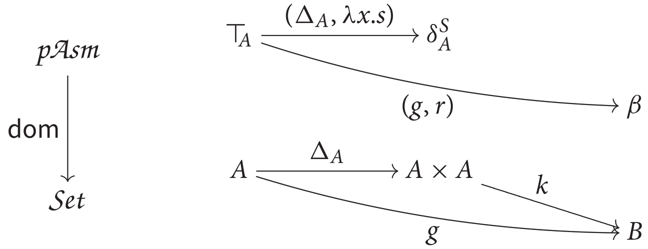
Example 2.3. The fibration $\mathsf{dom} \colon p\mathcal{A}sm \longrightarrow \mathcal{S}et$ introduced in 2.1 has finite products. Let $h \colon \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$ be a recursive bijection and denote its inverse by $k \colon \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$. The product $(A, \alpha) \wedge (A, \alpha') = (A, k[\alpha \cdot \alpha'])$ where $\alpha \cdot \alpha'(a) = \alpha(a) \times \alpha'(a) \subseteq \mathbb{N} \times \mathbb{N}$. The two projections are $\pi_1 = \mathrm{pr}_1 h$ and $\pi_2 = \mathrm{pr}_2 h$. To verify that this is a product over A take (id_A, r_1) and (id_A, r_2) from (A, γ) to the first and the second factor. The universal arrow is $k\langle r_1, r_2 \rangle$ as if f is such that $\pi_1 f = \mathrm{pr}_1 hf = r_1$ and $\pi_2 f = \mathrm{pr}_2 hf = r_2$, then $hf = \langle r_1, r_2 \rangle$, so $f = khf = k\langle r_1, r_2 \rangle$. The terminal object over A is $\top_A = (A, a \mapsto \{0\})$ introduced in Example 2.1.

Recall from Streicher (2020) that an arrow $\varphi \colon E \longrightarrow F$ is ***locally epic with respect to*** p if, for every pair $\psi, \psi' \colon F \longrightarrow F'$ such that $p(\psi) = p(\psi')$, whenever $\psi\varphi = \psi'\varphi$ it is already $\psi = \psi'$.

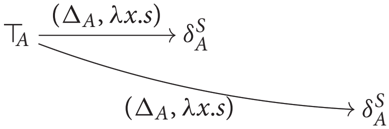
Example 2.4. Recall the fibration $\mathsf{dom} \colon p\mathcal{A}sm \longrightarrow \mathcal{S}et$ introduced in Example 2.1 and fix a non-empty subset $S \subseteq \mathbb{N}$. Let $\delta_A^S \colon A \times A \longrightarrow \mathcal{P}(\mathbb{N})$ be the function defined by the assignment:

$$\delta_A^S(a, a') = \begin{cases} S \text{ if } a = a' \\ \emptyset \text{ otherwise} \end{cases}$$

Take any $s \in S$. The arrow $(\Delta_A, \lambda x.s) \colon \top_A \longrightarrow \delta_A^S$ is weakly cocartesian over Δ_A in the sense that whenever one considers a situation as the one in the diagram below:



there is an arrow $(k, t) \colon \delta_A^S \longrightarrow \beta$ in $p\mathcal{A}sm$ that makes the triangle commute (take as t the constant function whose unique value is $r(0)$). The arrow $(\Delta_A, \lambda x.s)$ is locally epic with respect to dom if and only if $S = \{s\}$—a sort of Uniqueness of Identity Proofs. Indeed, suppose $S = \{s\}$ and let $(k, t') \colon \delta_A^S \longrightarrow \beta$ make the triangle above commute. Recall that $t = t'$ are equal if they agree on $\bigcup_{(a,a') \in A \times A} \delta_A^S(a, a') = \{s\}$ which is turn is the image of $\lambda x.s$. The claim follows from the equality $t'(\lambda x.s) = t(\lambda x.s)$. Conversely let $(\Delta_A, \lambda x.s)$ be cocartesian. The following diagram over Δ_A



can be filled only by one arrow, and this is necessarily the identity $(\mathrm{id}_{A \times A}, \lambda x.x)$. But also the arrow $(\mathrm{id}_{A \times A}, \lambda x.s)$ makes the triangle commute and $\lambda x.s = \lambda x.x$ implies $S = \{s\}$.

https://doi.org/10.1017/S096012952100030X Published online by Cambridge University Press

Fibrations are ubiquitous in mathematics and in this paper we are interested in those called elementary in Lawvere (1970), Jacobs (1999), and Maietti and Rosolini (2013*a*; 2015) that encode the notion of equality as particular fibred left adjoints.

Definition 2.5. *A fibration with products $p: \mathcal{E} \longrightarrow \mathcal{B}$ is* ***elementary*** *if for every pair of objects B and A in $\mathcal{B}$, reindexing along the parametrised diagonal $\mathrm{pr}_{1,2,2}: B \times A \longrightarrow B \times A \times A$ has a left adjoint $\exists_{B,A}: \mathcal{E}_{B\times A} \longrightarrow \mathcal{E}_{B\times A\times A}$, and these satisfy:*

Frobenius Reciprocity: *for every E over $B \times A$ and F over $B \times A \times A$, the canonical arrow:*

$$\exists_{B,A}({\mathrm{pr}_{1,2,2}}^{*}(F) \wedge E) \longrightarrow F \wedge \exists_{B,A} E$$

is an isomorphism, and

Beck–Chevalley Condition: *for every pullback square:*

$$\begin{array}{ccc} B \times A & \xrightarrow{\quad f \times A \quad} & Z \times A \\ {\scriptstyle \mathrm{pr}_{1,2,2}} \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_{1,2,2}} \\ B \times A \times A & \xrightarrow[\quad f \times A \times A \quad]{} & Z \times A \times A \end{array}$$

and every E over $Z \times A$, the canonical arrow:

$$\exists_{B,A}(f \times A)^{*} E \longrightarrow (f \times A \times A)^{*} \exists_{Z,A} E$$

is an isomorphism.

Example 2.6. Faithful elementary fibrations are the elementary doctrines of Maietti and Rosolini (2013*a*;b). They are complete with respect to the $(\wedge, \top, =)$-fragment of First Order Predicate Logic over a many sorted signature. Thus, if p is such a fibration, then any E in the fibre over $B \times A$ can be seen as a well-formed formula with free variables in a context $y: B, x: A$. In particular, the value of the left adjoint $\exists_{B,A}(E)$ is (isomorphic to) the formula $E \wedge x = x'$ in the context $y: B, x: A, x': A$.

Example 2.7. For the fibration $\mathsf{dom}: p\mathcal{A}sm \longrightarrow \mathcal{S}et$ in Example 2.1, the assignment $\exists_{B,A}(B \times A, \varphi)(y, x, x') = \varphi(y, x) \wedge \delta_A^{\{0\}}(x, x')$ shows that dom is elementary. The reader may find many other instances of elementary fibrations in Emmenegger et al. (2021).

Remark 2.8. It is well known that for $f: A \longrightarrow B$ an arrow in the base $\mathcal{B}$ of a fibration $p: \mathcal{E} \longrightarrow \mathcal{B}$, the reindexing functor $f^{*}: \mathcal{E}_B \longrightarrow \mathcal{E}_A$ is right adjoint if and only if the arrow f has cocartesian lifts at every $E \in \mathcal{E}_A$. We refer to Emmenegger et al. (2021, Section 4) for a reformulation of the Frobenius Reciprocity and the Beck–Chevalley condition in terms of cocartesian lifts.

In the following definitions, we introduce the notions required to state the characterisation of elementary fibrations.

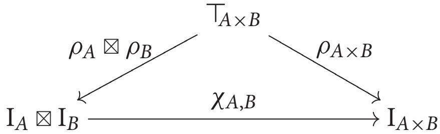
Definition 2.9. *Let $p: \mathcal{E} \longrightarrow \mathcal{B}$ be a fibration with finite products. We say that p has* ***strictly productive loops*** *if*

(i) *(Existence of loops) for every A in $\mathcal{B}$ there is a* ***loop*** *on A, that is, an object I_A over $A \times A$ and an arrow $\rho_A: \top_A \longrightarrow \mathrm{I}_A$ over $\mathrm{pr}_{1,1}: A \longrightarrow A \times A$;*

(ii) *(Loops are productive) for every A, B in $\mathcal{B}$, there is a vertical arrow $\chi_{A,B}: \mathrm{I}_A \boxtimes \mathrm{I}_B \longrightarrow \mathrm{I}_{A\times B}$;*

https://doi.org/10.1017/S096012952100030X Published online by Cambridge University Press

(iii) *(Loops are strictly productive) for every A, B in $\mathcal{B}$ the following diagram commutes*



Example 2.10. The fibration dom: $p\mathcal{A}sm \longrightarrow \mathcal{S}et$ as in 2.1 has strictly productive loops: let $\mathrm{I}_A = \delta_A^{\{0\}}$ and define $\rho_A = (\Delta_A, \lambda x.0)$ and $\chi_{A,B} = (\mathrm{id}_{A\times B\times A\times B}, \lambda xy.0)$.

Notation 2.11. Let $p: \mathcal{E} \longrightarrow \mathcal{B}$ be a fibration with loops. Given E over A, we find it convenient to write δ_E for the arrow $\langle \mathrm{pr}_{1,1}^{\ulcorner(\mathrm{pr}_1{}^*E)}, \rho_A !_E\rangle: E \longrightarrow (\mathrm{pr}_1{}^*E) \wedge \mathrm{I}_A$. We shall also need a parametric version of it. We write δ_E^B for the arrow $\langle \mathrm{pr}_{1,2,2}^{\ulcorner(\mathrm{pr}_{1,2}{}^*E)}, \theta\rangle: E \longrightarrow (\mathrm{pr}_{1,2}{}^*E) \wedge (\mathrm{pr}_{2,3}{}^*\mathrm{I}_A)$, where $\theta: E \longrightarrow \mathrm{pr}_{2,3}{}^*\mathrm{I}_A$ is the unique arrow over $\mathrm{pr}_{1,2,2}$ obtained by cartesianness of $\mathrm{pr}_{2,3}{}^*\mathrm{I}_A \longrightarrow \mathrm{I}_A$ from the composite:

$$E \xrightarrow{\ !_E\ } \top_{B\times A} \xrightarrow{\ \mathrm{pr}_2^{\ulcorner\top_A}\ } \top_A \xrightarrow{\ \rho_A\ } \mathrm{I}_A$$

and $\mathrm{pr}_{1,1}\mathrm{pr}_2 = \mathrm{pr}_{2,3}\mathrm{pr}_{1,2,2}: B \times A \longrightarrow A \times A$.

We write

$$\Delta_p := \left\{\mathrm{pr}_{1,2,2}: B \times A \longrightarrow B \times A \times A \;\middle|\; A, B \text{ objects in } \mathcal{B}\right\} \tag{3}$$

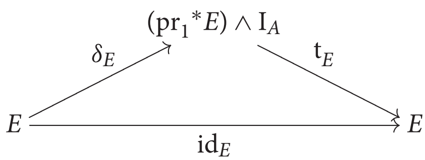
the class of arrows of the form $\mathrm{pr}_{1,2,2}: B \times A \longrightarrow B \times A \times A$, for B, A in $\mathcal{B}$, and

$$\Lambda_p := \left\{\delta_E^B: E \longrightarrow (\mathrm{pr}_{1,2}{}^*E) \wedge (\mathrm{pr}_{2,3}{}^*\mathrm{I}_A) \;\middle|\; X, B \text{ in } \mathcal{B}; E \text{ over } B \times A\right\} \tag{4}$$

the class of arrows of the form δ_E^B in $\mathcal{E}$, for A, B in $\mathcal{B}$ and E over $B \times A$.

Definition 2.12. *Let $p: \mathcal{E} \longrightarrow \mathcal{B}$ be a fibration with strictly productive loops. We say that p has* ***strictly productive transporters*** *if for every E over A*

(i) *(Carriers exist) there is a loop $\rho_A: \top_A \longrightarrow \mathrm{I}_A$ on A together with a* ***carrier for the loop*** *ρ_A* ***at*** *E, that is, an arrow $\mathrm{t}_E: (\mathrm{pr}_1{}^*E) \wedge \mathrm{I}_A \longrightarrow E$ over $\mathrm{pr}_2: A \times A \longrightarrow A$;*
(ii) *(Carriers are strict) the following diagram commute*



Example 2.13. Consider the fibration dom: $p\mathcal{A}sm \longrightarrow \mathcal{S}et$ and loops as as in 2.10. Let $h: \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$ and $k: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ be a recursive bijections as in 2.3. A carrier for $\delta_A^{\{0\}}$ at (A, φ) is $(\mathrm{pr}_2, \lambda x.\mathrm{pr}_1 hx)$.

Remark 2.14. The reader may have noticed that the condition of carriers is simply for objects in the fibre over A. But strict productivity of loops allows to generate carriers also in the fibre over $B \times A$ as follows. Given F an object in the fibre over $B \times A$, one first considers the carrier $\mathrm{t}_F: (\mathrm{pr}_{1,2}{}^*F) \wedge \mathrm{I}_{B\times A} \longrightarrow F$ for the loop $\rho_{B\times A}$ at F, which is an arrow in $\mathcal{E}$ over $\mathrm{pr}_{3,4}: B \times A \times B \times A \longrightarrow B \times A$. On the other hand, by strict productivity one obtains a composite arrow:

$$\begin{array}{ccc} \mathrm{pr}_{2,3}{}^*\mathrm{I}_A & \longrightarrow & \mathrm{I}_{B\times A} \\ \big\downarrow{\wr} & & \big\uparrow{\chi_{B,A}} \\ \top_B \boxtimes \mathrm{I}_A & \xrightarrow[\rho_B \boxtimes \mathrm{id}_{\mathrm{I}_A}]{} & \mathrm{I}_B \boxtimes \mathrm{I}_A \end{array}$$

over $\mathrm{pr}_{1,2,1,3}\colon B \times A \times A \longrightarrow B \times A \times B \times A$. Applying $(\mathrm{pr}_{1,2}{}^*F) \wedge -$ to it and post-composing with t_F gives us the desired arrow $\mathrm{pr}_{1,2}{}^*F \wedge \mathrm{pr}_{2,3}{}^*\mathrm{I}_A \longrightarrow F$.

Remark 2.15. Suppose $f\colon A \longrightarrow A'$ is an arrow in $\mathcal{B}$ and write K_f for $(f \times f)^*\mathrm{I}_{A'}$ an object over $A \times A$. Applying the previous Remark 2.14 to the object A with parameter A as well, and taking K as F, one obtains an arrow:

$$\mathrm{pr}_{1,2}{}^*K_f \wedge \mathrm{pr}_{2,3}{}^*\mathrm{I}_A \xrightarrow{\quad k \quad} K_f \xrightarrow{(f\times f)^{\urcorner\!\to}\mathrm{I}_A} \mathrm{I}_A$$

over $A \times A \times A \xrightarrow{\ \mathrm{pr}_{1,3}\ } A \times A$. Composing k with the reindexing of the loop $\rho_A\colon \top_A \longrightarrow \mathrm{I}_A$ along $f \times f$, one obtains an arrow $\widehat{f}\colon \mathrm{I}_A \longrightarrow \mathrm{I}_{A'}$.

Remark 2.16. With the notation of Remark 2.15, the product $\mathrm{I}_A \boxtimes \mathrm{I}_B$ is $K_{\mathrm{pr}_1} \wedge K_{\mathrm{pr}_2}$ for $\mathrm{pr}_1\colon A \times B \longrightarrow A$ and $\mathrm{pr}_2\colon A \times B \longrightarrow B$ the two projections. From this, one obtains a commutative diagram:

$$\begin{array}{ccccc} & & \top_{A\times B} & & \\ & {\scriptstyle \rho_A \boxtimes \rho_B}\swarrow & & \searrow{\scriptstyle \rho_{A\times B}} & \\ \mathrm{I}_A \boxtimes \mathrm{I}_B & & \xleftarrow{\langle \widehat{\mathrm{pr}_1}, \widehat{\mathrm{pr}_2} \rangle} & & \mathrm{I}_{A\times B} \end{array}$$

where $\widehat{\mathrm{pr}_1}\colon \mathrm{I}_{A\times B} \longrightarrow \mathrm{I}_A$ and $\widehat{\mathrm{pr}_2}\colon \mathrm{I}_{A\times B} \longrightarrow \mathrm{I}_B$ are as in Remark 2.15. When both $\rho_A \boxtimes \rho_B$ and $\rho_{A\times B}$ are locally epic, $\chi_{A,B}$ and $\langle \widehat{\mathrm{pr}_1}, \widehat{\mathrm{pr}_2} \rangle$ are inverse of each other.

The request of local epicity is necessary as one easily sees with the fibration in Example 2.4 taking loops with $\mathrm{card}\,(S) \geq 2$.

We are at last in a position to state the characterisation theorem that we shall use in the next section.

Theorem 2.17. *A fibration $p\colon \mathcal{E} \longrightarrow \mathcal{B}$ with products is elementary if and only if the following hold*

(i) *p has strict productive loops;*
(ii) *p has strict carriers for its loops;*
(iii) *the arrows in Λ_p are locally epic with respect to p.*

Proof. See Emmenegger et al. (2021, Theorem 4.8). □

Remark 2.18. Since faithful fibrations are equivalent to indexed posets, the equivalence in Theorem 2.17 gives Proposition 2.4 of Emmenegger et al. (2020).

Remark 2.19. Given a weak factorisation system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ with finite limits, there is a fibration $\mathcal{R} \longrightarrow \mathcal{C}$ and, in fact, a full comprehension category:

$$\begin{array}{ccc} \mathcal{R} & \hookrightarrow & \mathcal{C}^{\mathbf{2}} \\ & \searrow \ \swarrow & \\ & \mathcal{C} & \end{array}$$

https://doi.org/10.1017/S096012952100030X Published online by Cambridge University Press

The fibration $\mathcal{R} \longrightarrow \mathcal{C}$ always has strict productive transporters, but the loops are locally epic if and only if the left arrows r_A in a factorisation of the diagonal $\mathrm{pr}_{1,1}: A \longrightarrow A \times A$ have unique solutions to lifting problems. It follows that the fibration $\mathcal{R} \longrightarrow \mathcal{C}$ is not elementary in general but it is so when, for instance, the weak factorisation system $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system. We refer the interested reader to the examples and to Section 5 of Emmenegger et al. (2021).

3. Enriched Groupoids

Let $\mathcal{C}$ be a category with finite limits and let $\mathcal{C}\text{-}\mathcal{Gpd}$ be the category of $\mathcal{C}$-enriched groupoids and $\mathcal{C}$-enriched functors with respect to the symmetric monoidal structure of $\mathcal{C}$ given by finite products. There is an algebraic weak factorisation system (L, R) on $\mathcal{C}\text{-}\mathcal{Gpd}$ whose fibration of algebras for the monad on R is elementary and such that these algebras are the $\mathcal{C}$-enriched isofibrations with a splitting. In the following, we present that algebraic weak factorisation system and describe the enriched isofibrations.

Recall that a ***$\mathcal{C}$-enriched groupoid*** $\mathbb{A}$ consists of a set $|\mathbb{A}|$, a family $\big(\mathsf{hom}_{\mathbb{A}}(A, A')\big)_{A,A' \in |\mathbb{A}|}$ of objects of $\mathcal{C}$, and three families:

$$\Big(1 \xrightarrow{1_A} \mathsf{hom}_{\mathbb{A}}(A, A)\Big)_{A \in |\mathbb{A}|} \qquad \Big(\mathsf{hom}_{\mathbb{A}}(A_1, A_2) \xrightarrow{\mathsf{inv}_{A_1,A_2}} \mathsf{hom}_{\mathbb{A}}(A_2, A_1)\Big)_{A_1,A_2 \in |\mathbb{A}|}$$
$$\Big(\mathsf{hom}_{\mathbb{A}}(A_1, A_2) \times \mathsf{hom}_{\mathbb{A}}(A_2, A_3) \xrightarrow{\mathsf{cmp}_{A_1,A_2,A_3}} \mathsf{hom}_{\mathbb{A}}(A_1, A_3)\Big)_{A_1,A_2,A_3 \in |\mathbb{A}|} \tag{5}$$

of arrows in $\mathcal{C}$, where 1 is the terminal object of $\mathcal{C}$, satisfying the usual equations. Given $x: X \longrightarrow \mathsf{hom}_{\mathbb{A}}(A_1, A_2)$ and $y: X \longrightarrow \mathsf{hom}_{\mathbb{A}}(A_2, A_3)$ in $\mathcal{C}$, we shorten $\mathsf{cmp}_{A_1,A_2,A_3}\langle x, y\rangle$ as $x \cdot y$ and $\mathsf{inv}_{A_1,A_2} x$ as x^{i}.

A ***$\mathcal{C}$-enriched functor*** $F: \mathbb{A} \longrightarrow \mathbb{B}$ consists of a function $|F|: |\mathbb{A}| \longrightarrow |\mathbb{B}|$ and a family

$$\left(\mathsf{hom}_{\mathbb{A}}(A_1, A_2) \xrightarrow{F_{A_1,A_2}} \mathsf{hom}_{\mathbb{B}}(|F|(A_1), |F|(A_2)\,)\right)_{A_1,A_2 \in |\mathbb{A}|}$$

of arrows in $\mathcal{C}$ satisfying the usual functoriality conditions. We may drop subscripts when these are clear from the context. The standard reference for enriched category theory is Kelly (1982), but see also Borceux (1994, Chapter 6). One difference with the general theory, which makes the groupoid case more manageable, is that the symmetric monoidal structure in $\mathcal{C}$ used for the enrichment is cartesian.

Remark 3.1. As $\mathcal{C}$ has finite limits, it follows easily that $\mathcal{C}\text{-}\mathcal{Gpd}$ has finite limits as well. In particular note that, for any product diagram $\mathbb{A} \xleftarrow{P_1} \mathbb{P} \xrightarrow{P_2} \mathbb{B}$, it is $|\mathbb{P}| \cong |\mathbb{A}| \times |\mathbb{B}|$ and, for every $X, Y \in |\mathbb{P}|$, it is $\mathsf{hom}_{\mathbb{P}}(X, Y) \cong \mathsf{hom}_{\mathbb{A}}(|P_1|(X), |P_1|(Y)) \times \mathsf{hom}_{\mathbb{B}}(|P_2|(X), |P_2|(Y))$ and these isos commute with the structure maps 1, inv and cmp.

For a $\mathcal{C}$-enriched groupoid $\mathbb{A}$, we shall denote as $\Gamma(\mathbb{A})$ the groupoid whose set of objects is $|\mathbb{A}|$ and whose set of arrows $\Gamma(\mathbb{A})(A, A')$ consists of the global sections $1 \xrightarrow{y} \mathsf{hom}_{\mathbb{A}}(A, A')$. A $\mathcal{C}$-enriched functor $F: \mathbb{A} \longrightarrow \mathbb{B}$ induces a functor $\Gamma(F): \Gamma(\mathbb{A}) \longrightarrow \Gamma(\mathbb{B})$ and Γ is in fact a functor $\mathcal{C}\text{-}\mathcal{Gpd} \longrightarrow \mathcal{Gpd}$.

Recall that an ***algebraic weak factorisation system*** on a category $\mathcal{C}$ consists of a pair of functors $\mathsf{L}, \mathsf{R}: \mathcal{C}^{\mathbf{2}} \longrightarrow \mathcal{C}^{\mathbf{2}}$ that give rise to a functorial factorisation $f = (\mathsf{R}f)(\mathsf{L}f)$ of arrows of $\mathcal{C}$, together with suitable monad and comonad structures on R and L, respectively, with a distributivity law between them. We refer the reader to Grandis and Tholen (2006), Garner (2008), and Bourke and Garner (2016) for a precise definition and the basic properties of algebraic weak factorisation systems. We shall denote as M the functor $\mathsf{codL} = \mathsf{domR}\colon \mathcal{C}^{\mathbf{2}} \longrightarrow \mathcal{C}$. In particular, every $f\colon X \longrightarrow Y$ in $\mathcal{C}$ fits in a commutative triangle as shown below:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle \mathsf{R}f} \searrow \quad \nearrow {\scriptstyle \mathsf{L}f} & \\ & \mathsf{M}f & \end{array} \tag{6}$$

Proposition 3.2. *Let $\mathcal{C}$ be a category with finite limits. There is an algebraic weak factorisation system (L, R) on $\mathcal{C}$-$\mathcal{G}pd$.*

Proof. For a $\mathcal{C}$-enriched functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ between $\mathcal{C}$-enriched groupoids, we begin by constructing the factorisation in (6).

Define the $\mathcal{C}$-enriched groupoid $\mathsf{M}F$ as follows. The set $|\mathsf{M}F|$ consists of triples $(A, B, 1 \xrightarrow{x} \mathsf{hom}_{\mathbb{B}}(B, |F|(A)))$ with $A \in |\mathbb{A}|$ and $B \in |\mathbb{B}|$, and whose family of hom-objects is given, on objects (A, B, x) and (A', B', x'), by the equaliser below:

$$\mathsf{hom}_{\mathsf{M}F}(x, x') \xrightarrow{e_{x,x'}} \mathsf{hom}_{\mathbb{A}}(A, A') \times \mathsf{hom}_{\mathbb{B}}(B, B') \underset{\mathrm{pr}_2 \cdot (x'!)}{\overset{(x!) \cdot (F\mathrm{pr}_1)}{\rightrightarrows}} \mathsf{hom}_{\mathbb{B}}(B, |F|(A')) \tag{7}$$

The three families of arrows in (5) are easily induced from those of $\mathbb{A}$ and $\mathbb{B}$ using functoriality of F and the universal property of the above equalisers. The $\mathcal{C}$-enriched functor $\mathsf{R}F\colon \mathsf{M}F \longrightarrow \mathbb{B}$ maps an object (A, B, x) to B and its component on (A, B, x) nd (A', B', x') is given by:

$$\mathsf{hom}_{\mathsf{M}F}((A, B, x), (A', B', x')) \xrightarrow{(\mathsf{R}F)_{x,x'} := \mathrm{pr}_2 e_{x,x'}} \mathsf{hom}_{\mathbb{B}}(B, B')$$

The $\mathcal{C}$-enriched functor $\mathsf{L}F\colon \mathbb{A} \longrightarrow \mathsf{M}F$ is defined on objects by $|\mathsf{L}F|(A) = (A, |F|(A), 1_{|F|(A)})$. It follows that the composite:

$$\begin{array}{ccc} \mathsf{hom}_{\mathsf{M}F}(|\mathsf{L}F|(A), |\mathsf{L}F|(A')) & \xrightarrow{\qquad\sim\qquad} & \mathsf{hom}_{\mathbb{A}}(A, A') \\ & {\scriptstyle e_{1_A, 1_{A'}}} \searrow \qquad \nearrow {\scriptstyle \mathrm{pr}_1} & \\ & \mathsf{hom}_{\mathbb{A}}(A, A') \times \mathsf{hom}_{\mathbb{B}}(|F|(A), |F|(A')) & \end{array}$$

is an iso in $\mathcal{C}$ and we may take the component $(\mathsf{L}F)_{A,A'}$ to be its inverse.

The action of M extends to a functor $\mathsf{M}\colon \mathcal{C}\text{-}\mathcal{G}pd^{\mathbf{2}} \longrightarrow \mathcal{C}\text{-}\mathcal{G}pd$: given also $G\colon \mathbb{C} \longrightarrow \mathbb{D}$ and an arrow $(H, K)\colon F \longrightarrow G$ in $\mathcal{C}\text{-}\mathcal{G}pd^{\mathbf{2}}$, the $\mathcal{C}$-enriched functor $\mathsf{M}(H, K)\colon \mathsf{M}F \longrightarrow \mathsf{M}G$ is defined on objects by:

$$|\mathsf{M}(H, K)|(A, B, x) := \left(|H|(A),\ |K|(B),\ 1 \xrightarrow{K_{B,|F|(A)} x} \mathsf{hom}_{\mathbb{D}}(|K|(B), |GH|(A)) \right)$$

https://doi.org/10.1017/S096012952100030X Published online by Cambridge University Press

and the component of $\mathsf{M}(H, K)$ on $(A, B, x), (A', B', x')$ is the (unique) top arrow making the diagram below commute:

$$\begin{array}{ccc}
\mathsf{hom}_{\mathsf{M}F}(x, x') & \xrightarrow{\quad \mathsf{M}(H,K)_{x,x'} \quad} & \mathsf{hom}_{\mathsf{M}G}(Kx, Kx') \\
\downarrow{\scriptstyle e_{x,x'}} & & \downarrow{\scriptstyle e_{Kx,Kx'}} \\
\mathsf{hom}_{\mathbb{A}}(A, A') \times \mathsf{hom}_{\mathbb{B}}(B, B') & \xrightarrow{H_{A,A'} \times K_{B,B'}} & \mathsf{hom}_{\mathbb{C}}(|H|(A), |H|(A')) \times \mathsf{hom}_{\mathbb{D}}(|K|(B), |K|(B')) \\
{\scriptstyle \mathrm{pr}_2 \cdot (x'!)} \downdownarrows {\scriptstyle (x!) \cdot (F\mathrm{pr}_1)} & & {\scriptstyle \mathrm{pr}_2 \cdot (Kx'!)} \downdownarrows {\scriptstyle (Kx!) \cdot (G\mathrm{pr}_1)} \\
\mathsf{hom}_{\mathbb{B}}(B, |F|(A')) & \xrightarrow{\quad K_{A,|F|(A')} \quad} & \mathsf{hom}_{\mathbb{D}}(|K|(B), |GH|(A'))
\end{array}$$

The actions of L and R extend to functors $\mathcal{C}\text{-}\mathcal{G}pd^{\mathbf{2}} \longrightarrow \mathcal{C}\text{-}\mathcal{G}pd^{\mathbf{2}}$ similarly.

Clearly, $F = (\mathsf{R}F)(\mathsf{L}F)$. It follows that the functor R is pointed, with transformation $\mathrm{Id} \dashrightarrow \mathsf{R}$ given by L and, dually, that L is copointed.

The component of the multiplication $\mu\colon R^2 \longrightarrow R$ on F is defined as follows. Elements of $|\mathsf{MR}F|$ are those $((A, x), y)$ where $(A, 1 \xrightarrow{x} \mathrm{hom}_{\mathbb{B}}(B_1, |F|(A)) \in |\mathsf{M}F|$ and $1 \xrightarrow{y} \mathrm{hom}_{\mathbb{B}}(B_2, B_1))$ maps an element of $|\mathsf{MR}F|$. We define

$$\mu_F(A, x, y) := (A, y \cdot x).$$

The action on arrows is induced by:

$$\mathrm{hom}_{\mathbb{A}}(A, A') \times \mathrm{hom}_{\mathbb{B}}(B_1, B_1') \times \mathrm{hom}_{\mathbb{B}}(B_2, B_2') \xrightarrow{\mathrm{pr}_{1,3}} \mathrm{hom}_{\mathbb{A}}(A, A') \times \mathrm{hom}_{\mathbb{B}}(B_2, B_2').$$

The component of the comultiplication $\delta\colon \mathsf{L} \longrightarrow \mathsf{L}^2$ on F is defined as follows. Elements of $|\mathsf{ML}F|$ are those $(A_1, (A_2, x), a)$ where $A_1 \in |\mathbb{A}|$, $(A_2, 1 \xrightarrow{x} \mathrm{hom}_{\mathbb{B}}(B, A_2)) \in |\mathsf{M}F|$ and $1 \xrightarrow{a} \mathrm{hom}_{\mathbb{A}}(A_2, A_1)$. Note that a induces a unique global element of $\mathrm{hom}_{\mathsf{M}F}((A_2, x), |\mathsf{L}F|(A_1))$. We define

$$\delta_F(A, x) := (A, (A, x), 1_A).$$

The action on arrows is induced by:

$$\mathrm{hom}_{\mathbb{A}}(A, A') \times \mathrm{hom}_{\mathbb{B}}(B, B') \xrightarrow{\mathrm{pr}_{1,1,2}} \mathrm{hom}_{\mathbb{A}}(A, A') \times \mathrm{hom}_{\mathbb{A}}(A, A') \times \mathrm{hom}_{\mathbb{B}}(B, B').$$

It is now not difficult to see that μ_F and δ_F are natural in F and make the pointed functor R into a monad and the copointed functor L into a comonad and, in fact, make L and R the underlying functors of an algebraic weak factorisation system on $\mathcal{C}\text{-}\mathcal{G}pd$. □

When the category $\mathcal{C}$ is the category $\mathcal{S}et$ of sets and functions, it is well known that the algebras for the monad on R are split (cloven) isofibrations, see e.g. van Woerkom (2021) (Chapter 7). In the enriched case, a definition of a fibration enriched over a suitable fibration T between monoidal categories is given in Vasilakopoulou (2018). Specialising to the case where T is the identity functor on a cartesian category $\mathcal{C}$, this notion reduces to that of a fibration that is also a $\mathcal{C}$-enriched functor. By further specialising to the case of isofibrations, one reaches the notion of $\mathcal{C}$-enriched isofibration. As we show in Proposition 3.4, these are the algebras for the monad on R.

Let us first give an alternative characterisation of the $\mathcal{C}$-enriched groupoid $\mathsf{M}F$ for a $\mathcal{C}$-enriched functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$.

Remark 3.3. For a $\mathcal{C}$-enriched groupoid $\mathbb{B}$, let us denote the $\mathcal{C}$-enriched groupoid $\mathsf{MId}_{\mathbb{B}}$ as $\mathrm{Iso}(\mathbb{B})$. Its objects are triples $(B_2, B_1, 1 \xrightarrow{y} \mathsf{hom}_{\mathbb{B}}(B_1, B_2))$ and, unfolding (7), one sees that hom-objects consist of commuting squares in $\mathbb{B}$. There is a $\mathcal{C}$-enriched functor $\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle\colon \mathrm{Iso}(\mathbb{B}) \longrightarrow \mathbb{B} \times \mathbb{B}$ where

https://doi.org/10.1017/S096012952100030X Published online by Cambridge University Press

$\mathsf{d}_{\mathbb{B}} := \mathsf{R}\mathrm{Id}_{\mathbb{B}}$, the function $|\mathsf{c}_{\mathbb{B}}|$ maps (B_2, B_1, y) to B_2, and $(\mathsf{c}_{\mathbb{B}})_{y,y'} := \mathrm{pr}_1 e_{y,y'} : \hom_{\mathrm{Iso}(\mathbb{B})}(y, y') \longrightarrow \hom_{\mathbb{B}}(B_2, B_2')$.

Given a $\mathcal{C}$-enriched functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$, there is a pullback in $\mathcal{C}$-$\mathcal{Gpd}$:

$$\begin{array}{ccc} \mathsf{M}F & \xrightarrow{\quad F' \quad} & \mathrm{Iso}(\mathbb{B}) \\ {\scriptstyle \langle \mathsf{c}'_{\mathbb{B}}, \mathsf{R}F\rangle}\big\downarrow & & \big\downarrow{\scriptstyle \langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}}\rangle} \\ \mathbb{A} \times \mathbb{B} & \xrightarrow[F \times \mathrm{Id}_{\mathbb{B}}]{} & \mathbb{B} \times \mathbb{B} \end{array} \tag{8}$$

where $|\mathsf{c}'_{\mathbb{B}}|(A, B, x) = A$ and $(\mathsf{c}'_{\mathbb{B}})_{x,x'} = \mathrm{pr}_1 e_{x,x'}$, the $\mathcal{C}$-enriched functor F' is the identity on objects and $F'_{x,x'}$ is the unique arrow induced by $(F_{A,A'} \times \mathrm{id})e_{x,x'}$.

Note also that, since the global section functor $\Gamma\colon \mathcal{C} \longrightarrow \mathcal{Set}$ preserves limits, the $\mathcal{C}$-enriched groupoid $\mathsf{M}F$ is in fact the enrichment of the ($\mathcal{Set}$-enriched) groupoid which appears in (Emmenegger et al. 2021, Section 5) for the ($\mathcal{Set}$-enriched) functor F.

Proposition 3.4. *Let $\mathcal{C}$ be a category with finite limits. The algebras of the monad on* R *are the $\mathcal{C}$-enriched split isofibrations, namely $\mathcal{C}$-enriched functors $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ equipped with a function $c\colon |\mathsf{M}F| \longrightarrow |\mathrm{Iso}(\mathbb{A})|$ such that*

(i) $\mathsf{c}_{\mathbb{A}} \circ c(A, x) = A$
(ii) $F_{c_0(A,x),A} \circ c(A, x) = x$, *where* $c_0 := |\mathsf{d}_{\mathbb{A}}| \circ c\colon |\mathsf{M}F| \longrightarrow |\mathbb{A}|$,
(iii) $c(A, 1_{FA}) = 1_A$, *and*
(iv) $c(A, y \cdot x) = c(c_0(A, x), y)$.

Proof. Let $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ be an algebra with structure map $S\colon \mathsf{M}F \longrightarrow \mathbb{A}$ and define $c_0(A, x) := |S|(A, x)|$. Since $S \circ \mathsf{L}F = \mathrm{Id}_A$, in particular it is $|S|(A, 1_{|F|(A)}) = A$ and we define

$$\begin{array}{ccc} 1 & \xrightarrow{\qquad c(A,x) \qquad} & \hom_{\mathbb{A}}(c_0(A, x), A) \\ & {\scriptstyle \hat{x}}\searrow \qquad \nearrow{\scriptstyle S_{x,1_{|F|(A)}}} & \\ & \hom_{\mathsf{M}F}((A, x), (A, 1_{|F|(A)})) & \end{array}$$

where $\hat{x}$ is the global element induced on the equaliser (7) by the pair $(1_A, x)$. Condition (i) holds by construction. As $F_{c_0(A,x),A} \circ S_{x,1_{|F|(A)}} = \mathrm{pr}_2 \circ e_{x,1_{|F|(A)}}$, condition (ii) is satisfied. Condition (iii) follows immediately from the functoriality of S on identities since, for every $(A, 1 \xrightarrow{x} \hom_{\mathbb{B}}(B, A))$, the identity $1_{(A,x)}$ is the global element induced by the pair $(1_A, 1_B)$. Since $S\mu_F = S\mathsf{M}(\mathrm{Id}_{\mathbb{B}}, S)$, in particular it is $c_0(A, y \cdot x) = c_0(c_0(A, x), y)$. Condition (iv) then follows from commutativity of the diagram:

$$\begin{array}{ccc} \hom_{\mathsf{MR}F}((A, x, y), (A, 1_{|F|A}, 1_{|F|A})) & \xrightarrow{\mathsf{M}(\mathrm{Id}_{\mathbb{B}}, S)} & \hom_{\mathsf{M}F}((c_0(A, x), y), (A, 1_{|F|A})) \\ {\scriptstyle \mu_F}\big\downarrow & & \big\downarrow{\scriptstyle S} \\ \hom_{\mathsf{M}F}((A, y \cdot x), (A, 1_{|F|A})) & \xrightarrow[S]{} & \hom_{\mathbb{A}}(c_0(A, y \cdot x), A) \end{array}$$

by precomposing it with

$$\hom((A, x, y), (A, x, 1_B)) \times \hom((A, x, 1_B), (A, 1_{|F|A}, 1_{|F|A})) \xrightarrow{\mathsf{cmp}} \hom((A, x, y), (A, 1_{|F|A}, 1_{|F|A}))$$

https://doi.org/10.1017/S096012952100030X Published online by Cambridge University Press

and using functoriality of μ_F and S, as well as commutativity of

$$\begin{array}{ccc} & & \mathrm{hom}_{\mathsf{M}F}\,((A, 1_{|F|(A)}), (A', 1_{|F|(A')})) \\ & \overset{\mathsf{L}F_{A,A'}}{\nearrow} & \Big\downarrow S_{1_{|F|(A)}, 1_{|F|(A)}} \\ \mathrm{hom}_{\mathbb{A}}\,(A, A') & \xrightarrow{\quad \mathrm{id} \quad} & \mathrm{hom}_{\mathbb{A}}\,(A, A') \end{array}$$

which follows from $S \circ \mathsf{L}F = \mathrm{Id}_{\mathbb{A}}$. Thus F is an enriched split isofibration.

Conversely, let $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ be an enriched functor and let $c\colon |\mathsf{M}F| \longrightarrow |\mathrm{Iso}(\mathbb{A})|$ be as in the statement. We need to construct an enriched functor $S\colon \mathsf{M}F \longrightarrow \mathbb{A}$ that makes (F, S) an algebra for the monad on R. For $(A, x) \in |\mathsf{M}F|$, define $|S|(A, x) := c_0(A, x)$. For $(A, x), (A', x') \in |\mathsf{M}F|$, define $S_{x,x'}$ as the composite below:

$$\begin{array}{ccc} \mathrm{hom}\,((A, x), (A', x')) & \xrightarrow{\langle c(A,x)!,\, \mathrm{id},\, c(A',x')!\rangle} & \mathrm{hom}\,(c_0(A, x), A) \times \mathrm{hom}\,(A, A') \times \mathrm{hom}\,(c_0(A', x'), A') \\ & \searrow^{S_{x,x'}} & \Big\downarrow \mathrm{pr}_1 \cdot \mathrm{pr}_2 \cdot (\mathrm{pr}_3)^{\mathrm{i}} \\ & & \mathrm{hom}\,(c_0(A, x), c_0(A', x')) \end{array}$$

Functoriality of S then follows from the groupoid laws of $\mathbb{A}$. Condition (ii) ensures that S defines a morphism from F to $\mathsf{R}F$ over $\mathbb{B}$ and conditions (iii) and (iv) ensure that S is an algebra map. □

Notation 3.5. We adopt the notation in Gambino and Larrea (2021) and denote as R-**Alg** $\longrightarrow$ $\mathcal{C}$-$\mathcal{Gpd}$ the fibration of algebras for the monad on R.

Theorem 3.6. *The fibration* R-**Alg** $\longrightarrow \mathcal{C}$-$\mathcal{Gpd}$ *is elementary.*

The proof of Theorem 3.6 is given in the remainder of the section.

Thanks to the characterisation of elementary fibrations in Theorem 2.17, it is enough to check that R-**Alg** $\longrightarrow \mathcal{C}$-$\mathcal{Gpd}$ has strictly productive transporters (see Definitions 2.9 and 2.12), which we do in Lemmas 3.7 and 3.9, and that certain arrows are locally epic in Lemma 3.10. Not only are these conditions easier to verify than the existence of left adjoints to certain reindexing functors, but Lemmas 3.7 and 3.9 also expose part of the structure that makes $\mathcal{C}$-$\mathcal{Gpd}$ suitable to interpret Martin-Löf's identity types. We elaborate on this in the next section.

Lemma 3.7. *The fibration* R-**Alg** $\longrightarrow \mathcal{C}$-$\mathcal{Gpd}$ *has strictly productive loops.*

Proof. Let $\mathbb{B}$ be a $\mathcal{C}$-enriched groupoid. To provide a loop on $\mathbb{B}$, we first need an object $\mathrm{I}_{\mathbb{B}}$ of R-**Alg** in the fibre over $\mathbb{B} \times \mathbb{B}$. To this aim, we show that there is a $\mathcal{C}$-enriched functor $\mathsf{s}_{\mathbb{B}}\colon \mathsf{M}\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}}\rangle \longrightarrow \mathrm{Iso}(\mathbb{B})$ making the $\mathcal{C}$-enriched functor $\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}}\rangle$ defined in Remark 3.3 an algebra for the monad on R. The function $|\mathsf{s}_{\mathbb{B}}|$ maps $((B_2, B_1, y), b_2, b_1)$, with $b_i\colon 1 \longrightarrow \mathrm{hom}_{\mathbb{B}}(B'_i, B_i)$, $i = 1, 2$, to

$$(B'_2, B'_1, 1 \xrightarrow{b_1 \cdot y \cdot (b_2{}^{\mathrm{i}})} \mathrm{hom}_{\mathbb{B}}(B'_1, B'_2)).$$

and the family component on $(y, b_2, b_1), (z, c_2, c_1)$ is the unique arrow in

$$\begin{array}{ccc} \mathrm{hom}_{\mathsf{M}\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}}\rangle}((y, b_2, b_1), (z, c_2, c_1)) & \xrightarrow{\quad e \quad} & \mathrm{hom}_{\mathrm{Iso}(\mathbb{B})}(y, y') \times \mathrm{hom}_{\mathbb{B}}(B'_2, C'_2) \times \mathrm{hom}_{\mathbb{B}}(B'_1, C'_1) \\ \Big\downarrow (\mathsf{s}_{\mathbb{B}})_{(y,b_2,b_1),(z,c_2,c_1)} & & \Big\downarrow \mathrm{pr}_{2,3} \\ \mathrm{hom}_{\mathrm{Iso}(B)}(|\mathsf{s}_{\mathbb{B}}|(y, b_2, b_1), |\mathsf{s}_{\mathbb{B}}|(z, c_2, c_1)) & \xrightarrow{\quad e \quad} & \mathrm{hom}_{\mathbb{B}}(B'_2, C'_2) \times \mathrm{hom}_{\mathbb{B}}(B'_1, C'_1) \end{array}$$

https://doi.org/10.1017/S096012952100030X Published online by Cambridge University Press

given by the universal property of the equaliser in the bottom row. It follows that

$$\mathsf{I}_{\mathbb{B}} := (\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle, \mathsf{s}_{\mathbb{B}})$$

is an algebra for the monad on R.

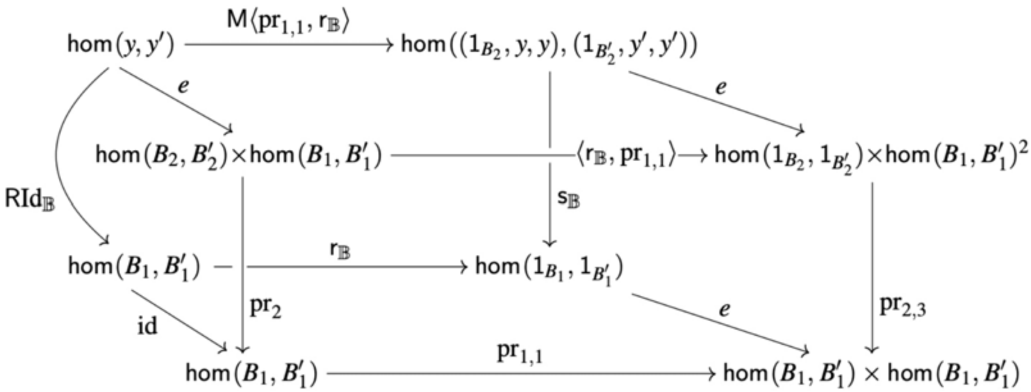
Let now $\mathsf{r}_{\mathbb{B}} := \mathsf{LId}_{\mathbb{B}} \colon \mathbb{B} \longrightarrow \mathrm{Iso}(\mathbb{B})$. To have a loop on $\mathbb{B}$, it is enough to show that $(\mathrm{pr}_{1,1}, \mathsf{r}_{\mathbb{B}}) \colon \mathrm{Id}_{\mathbb{B}} \longrightarrow \langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle$ is a morphism of algebras from $(\mathrm{Id}_{\mathbb{B}}, \mathsf{RId}_{\mathbb{B}})$ to $\mathsf{I}_{\mathbb{B}}$, that is, that the diagram of $\mathcal{C}$-enriched functor below commutes

$$\begin{array}{ccc} \mathsf{MId}_{\mathbb{B}} & \xrightarrow{\mathsf{M}(\mathrm{pr}_{1,1}, \mathsf{r}_{\mathbb{B}})} & \mathsf{M}\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle \\ {\scriptstyle \mathsf{RId}_{\mathbb{B}}}\downarrow & & \downarrow{\scriptstyle \mathsf{s}_{\mathbb{B}}} \\ \mathbb{B} & \xrightarrow{\mathsf{r}_{\mathbb{B}}} & \mathrm{Iso}(\mathbb{B}) \end{array}$$

On an object (B_2, B_1, y) it is

$$\mathsf{r}_{\mathbb{B}}(\mathsf{RId}_{\mathbb{B}})(B_2, B_1, y) = \mathsf{r}_{\mathbb{B}}(B_1) = (B_1, B_1, 1_{B_1}) = \mathsf{s}_{\mathbb{B}}(1_{B_2}, y, y) = \mathsf{s}_{\mathbb{B}}\mathsf{M}(\mathrm{pr}_{1,1}, \mathsf{r}_{\mathbb{B}})(B_2, B_1, y).$$

On arrows from (B_2, B_1, y) to (B'_2, B'_1, y'), it amounts to the commutativity of the back square below, which follows from the commutativity of the front square, where we dropped indices from hom-objects, families of arrows and equalisers of the form (7):



This choice of loops is strictly productive. Indeed, $|\mathrm{Iso}(\mathbb{A} \times \mathbb{B})| \cong |\mathrm{Iso}(\mathbb{A})| \times |\mathrm{Iso}(\mathbb{B})|$ by Remark 3.1 and, for $1 \xrightarrow{x} \mathsf{hom}_{\mathbb{A}}(A_1, A_2)$, $1 \xrightarrow{y} \mathsf{hom}_{\mathbb{B}}(B_1, B_2)$, $1 \xrightarrow{x'} \mathsf{hom}_{\mathbb{A}}(A'_1, A'_2)$ and $1 \xrightarrow{y'} \mathsf{hom}_{\mathbb{B}}(B'_1, B'_2)$,

$$\mathsf{hom}_{\mathrm{Iso}(\mathbb{A}\times\mathbb{B})}(\langle x, y\rangle, \langle x', y'\rangle) \cong \mathsf{hom}_{\mathrm{Iso}(\mathbb{A})}(x, x') \times \mathsf{hom}_{\mathrm{Iso}(\mathbb{B})}(y, y')$$

because Remark 3.1 ensures that the corresponding equalisers are isomorphic. The isomorphism $\mathrm{Iso}(\mathbb{A} \times \mathbb{B}) \cong \mathrm{Iso}(\mathbb{A}) \times \mathrm{Iso}(\mathbb{B})$ extends to an isomorphism of algebras:

$$(\langle \mathsf{c}_{\mathbb{A}\times\mathbb{B}}, \mathsf{d}_{\mathbb{A}\times\mathbb{B}} \rangle, \mathsf{s}_{\mathbb{A}\times\mathbb{B}}) \cong (\langle \mathsf{c}_{\mathbb{A}}, \mathsf{d}_{\mathbb{A}} \rangle, \mathsf{s}_{\mathbb{A}}) \times (\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle, \mathsf{s}_{\mathbb{B}}) \tag{9}$$

which clearly commutes with the loops. □

Remark 3.8. The isomorphism in (9) is the same as the one arising in Remark 2.15. That this must be the case will be clear after we have proved that $\mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathcal{C}\text{-}\mathcal{Gpd}$ is indeed elementary.

Lemma 3.9. *Given an algebra* $(F \colon \mathbb{A} \longrightarrow \mathbb{B}, S \colon \mathsf{M}F \longrightarrow \mathbb{A})$ *in* $\mathsf{R}\text{-}\mathbf{Alg}$*, there is exactly one carrier for the loop* $(\mathrm{pr}_{1,1}, \mathsf{r}_{\mathbb{B}}) \colon (\mathrm{Id}_{\mathbb{B}}, \mathsf{RId}_{\mathbb{B}}) \longrightarrow (\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle, \mathsf{s}_{\mathbb{B}})$ *and it is* $(\mathrm{pr}_2, S) \colon \mathrm{pr}_1{}^*(F, S) \wedge (\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle, \mathsf{s}_{\mathbb{B}}) \longrightarrow F$.

https://doi.org/10.1017/S096012952100030X Published online by Cambridge University Press

Proof. By Remark 3.3, we may assume, without loss of generality, that the underlying functor of the algebra $\mathrm{pr}_1{}^*(F,S) \wedge (\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle, \mathsf{s}_{\mathbb{B}})$ is the diagonal $D\colon \mathsf{M}F \longrightarrow \mathbb{B} \times \mathbb{B}$ in the pullback of Remark 3.3. The structure map $S_D\colon \mathsf{M}D \longrightarrow \mathsf{M}F$ is induced by those on F and $\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle$ and maps an object $\big((A,B,x), 1 \xrightarrow{\langle b_1, b_2 \rangle} \mathsf{hom}_{\mathbb{B}}(B_1, FA) \times \mathsf{hom}_{\mathbb{B}}(B_2, B)\big) \in |\mathsf{M}D|$ to the object $\big(S(A, B_1, b_1), B_2, 1 \xrightarrow{b_2 \cdot x \cdot b_1{}^{\mathrm{i}}} \mathsf{hom}_{\mathbb{B}}(B_2, B_1)\big) \in |\mathsf{M}F|$. A carrier, if it exists, is of the form (pr_2, T) where the $\mathcal{C}$-enriched functor $T\colon \mathsf{M}F \longrightarrow \mathbb{A}$ has to fit in the commutative diagram:

$$\begin{array}{ccc} \mathsf{M}F & \xrightarrow{\ T\ } & \mathbb{A} \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbb{B} \times \mathbb{B} & \xrightarrow{\ \mathrm{pr}_2\ } & \mathbb{B} \end{array}$$

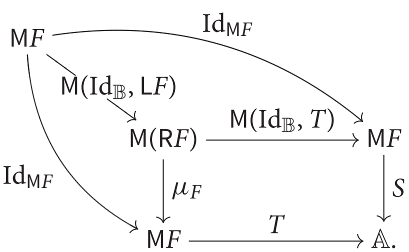
and, since it has to be a homomorphism of algebras, the following diagram must commute

$$\begin{array}{ccc} \mathsf{M}D & \xrightarrow{\ \mathsf{M}(\mathrm{pr}_2, T)\ } & \mathsf{M}F \\ {\scriptstyle S_D}\downarrow & & \downarrow{\scriptstyle S} \\ \mathsf{M}F & \xrightarrow{\ T\ } & \mathbb{A}. \end{array} \tag{10}$$

Moreover, the strictness condition imposes that the diagram:

$$\begin{array}{ccc} \mathbb{A} & & \\ {\scriptstyle \mathsf{L}F}\downarrow & \searrow^{\mathrm{Id}_{\mathbb{A}}} & \\ \mathsf{M}F & \xrightarrow{\ T\ } & \mathbb{A} \end{array} \tag{11}$$

commutes. Note also that there is a $\mathcal{C}$-enriched functor $H\colon \mathsf{M}(\mathsf{R}F) \longrightarrow \mathsf{M}D$ such that $S_D H = \mu_F$, the multiplication for the monad on R, and $\mathsf{M}(\mathrm{pr}_2, T)H = \mathsf{M}(\mathrm{Id}_{\mathbb{B}}, T)$. Precomposing diagram (10) with H and using (11) together with a triangular identity for the monad, the commutative diagram:



shows that the only possible choice for T is the structural functor $S\colon \mathsf{M}F \longrightarrow \mathbb{A}$, and it is straightforward to see that that choice makes diagrams (10) and (11) commute. □

It follows from Lemma 3.7 and Lemma 3.9 that the fibration R-**Alg** $\longrightarrow \mathcal{C}$-$\mathcal{Gpd}$ has strictly productive transporters.

Recall the definition of the class of arrows Λ_p as in (4) of Notation 2.11. When p is the fibration R-**Alg** $\longrightarrow \mathcal{C}$-$\mathcal{Gpd}$, it consists of those arrows of the form:

$$(F,S) \xrightarrow{\ \delta_F^{\mathbb{I}} = (\mathrm{pr}_{1,2,2}, \langle \mathrm{Id}_{\mathbb{A}}, \mathsf{r}_{\mathbb{B}} \mathrm{pr}_2 F \rangle)\ } (\mathrm{pr}_{1,2}{}^*(F,S)) \wedge (\mathrm{pr}_{2,3}{}^*(\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle, \mathsf{s}_{\mathbb{B}})) \tag{12}$$

for $(F\colon \mathbb{A} \longrightarrow \mathbb{I} \times \mathbb{B}, S\colon \mathsf{M}F \longrightarrow \mathbb{A})$ in the fibre of R-**Alg** over $\mathbb{I} \times \mathbb{B}$. In the following, we simply write Λ for this class.

Lemma 3.10. *The arrows in* Λ *are locally epic with respect to* $\mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathcal{C}\text{-}\mathcal{G}pd$.

Proof. Let $(F: \mathbb{A} \longrightarrow \mathbb{I} \times \mathbb{B}, S: \mathsf{M}F \longrightarrow \mathbb{A})$ be in $\mathsf{R}\text{-}\mathbf{Alg}$, write $D: \mathbb{A} \times_{\mathbb{B}} \mathrm{Iso}(\mathbb{B}) \longrightarrow \mathbb{I} \times \mathbb{B} \times \mathbb{B}$ for the underlying functor of $(\mathrm{pr}_{1,2}{}^{*}(F, S)) \wedge (\mathrm{pr}_{2,3}{}^{*}(\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle, \mathsf{s}_{\mathbb{B}}))$ and let $S_D: \mathsf{M}D \longrightarrow \mathbb{A} \times_{\mathbb{B}} \mathrm{Iso}(\mathbb{B})$ be its structure map. Note that there is a $\mathcal{C}$-enriched functor $K: \mathbb{A} \times_{\mathbb{B}} \mathrm{Iso}(\mathbb{B}) \longrightarrow \mathsf{M}(\mathrm{pr}_{1,2,2}F)$ mapping (A, B, x) to

$$\left(A, (|F|(A), B), 1 \xrightarrow{\langle 1_{|F|(A)}, x \rangle} \mathsf{hom}_{\mathbb{I} \times \mathbb{B}}(|F|(A), |F|(A)) \times \mathsf{hom}_{\mathbb{B}}(B, \mathrm{pr}_2 |F|(A)) \right)$$

and that the composite $\mathsf{M}(\mathrm{Id}_{\mathbb{I} \times \mathbb{B} \times \mathbb{B}}, \langle \mathrm{Id}_{\mathbb{A}}, \mathsf{r}_{\mathbb{B}} \mathrm{pr}_2 F \rangle) K: \mathbb{A} \times_{\mathbb{B}} \mathrm{Iso}(\mathbb{B}) \longrightarrow \mathsf{M}D$ is a section of S_D. Then for every vertical morphism $H: (D, S_D) \longrightarrow (G, T)$, it is

$$H = H S_D \mathsf{M}(\mathrm{Id}_{\mathbb{I} \times \mathbb{B} \times \mathbb{B}}, \langle \mathrm{Id}_{\mathbb{A}}, \mathsf{r}_{\mathbb{B}} \mathrm{pr}_2 F \rangle) K = T \mathsf{M}(\mathrm{Id}_{\mathbb{I} \times \mathbb{B} \times \mathbb{B}}, H \langle \mathrm{Id}_{\mathbb{A}}, \mathsf{r}_{\mathbb{B}} \mathrm{pr}_2 F \rangle) K.$$

As $\delta^{\mathbb{I}}_F = (\mathrm{pr}_{1,2,2}, \langle \mathrm{Id}_{\mathbb{A}}, \mathsf{r}_{\mathbb{B}} \mathrm{pr}_2 F \rangle)$, algebra morphisms out of $(\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle, \mathsf{s}_{\mathbb{B}})$ are determined by their precomposition with $\delta^{\mathbb{I}}_F$. □

This concludes the proof of Theorem 3.6, that the fibration $\mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathcal{C}\text{-}\mathcal{G}pd$ is elementary.

Remark 3.11. The value of the left adjoint to reindexing along a parametrised diagonal $\mathrm{pr}_{1,2,2}: \mathbb{I} \times \mathbb{B} \longrightarrow \mathbb{I} \times \mathbb{B} \times \mathbb{B}$ at an algebra (F, S) over $\mathbb{I} \times \mathbb{B}$ is given by a cocartesian lift of $\mathrm{pr}_{1,2,2}$ at (F, S), see Remark 2.8. By Theorem 3.6, the class Λ provides a choice of cocartesian lifts. It follows that the value of the left adjoint at (F, S) is the codomain of the arrow $\delta^{\mathbb{I}}_F$ in (12).

Remark 3.12. We have seen in Remark 2.19 that a sufficient condition for the right class of a weak factorisation system to give rise to an elementary fibration is the factorisation system being orthogonal. The underlying weak factorisation system of the a.w.f.s. on $\mathcal{C}\text{-}\mathcal{G}pd$ from Proposition 3.2 is not orthogonal and the above proof rather makes use of the structure given by the a.w.f.s. itself and, crucially, of the possibility to factor a suitable section of S_D through $\mathsf{M}(\mathrm{Id}_{\mathbb{I} \times \mathbb{B} \times \mathbb{B}}, \langle \mathrm{Id}_{\mathbb{A}}, \mathsf{r}_{\mathbb{B}} \mathrm{pr}_2 F \rangle)$.

4. Enriched Groupoid Models of Identity Types

We have seen in the previous section that there is an algebraic weak factorisation system (L, R) on $\mathcal{C}\text{-}\mathcal{G}pd$ such that the fibration $\mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathcal{C}\text{-}\mathcal{G}pd$ is elementary. That fibration is also part of a comprehension category:

$$\begin{array}{ccc} \mathsf{R}\text{-}\mathbf{Alg} & \xrightarrow{\quad U \quad} & \mathcal{C}\text{-}\mathcal{G}pd^{\mathbf{2}} \\ & \searrow \quad \swarrow & \\ & \mathcal{C}\text{-}\mathcal{G}pd. & \end{array} \tag{13}$$

where the functor U forgets the algebra structure.

When $\mathcal{C}$ is the category $\mathcal{S}et$ of sets and functions, the comprehension category (13) is equivalent to the Hofmann–Streicher groupoid model in Hofmann and Streicher (1998). The choice of a loop on a groupoid $\mathbb{B}$ given in the proof of Lemma 3.7 coincides with the interpretation, in the groupoid model, of the identity type and its reflexivity term on the type interpreted by $\mathbb{B}$. In fact, strict productive transporters in $\mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathcal{G}pd$ ensure that the upper component $\mathsf{r}_{\mathbb{B}}: \mathbb{B} \longrightarrow \mathrm{Iso}(\mathbb{B})$ of a loop $(\mathrm{pr}_{1,1}, \mathsf{r}_{\mathbb{B}}): \mathrm{Id}_{\mathbb{B}} \longrightarrow \mathrm{I}_{\mathbb{B}}$ has the left lifting property against algebras for the monad and, also, algebras for the pointed endofunctor (Emmenegger et al. 2021, Section 5). These lifts provide an interpretation for the eliminator of the identity type on $\mathbb{B}$.

It is then natural to ask under which hypotheses on $\mathcal{C}$ the comprehension category of enriched groupoids in (13) interprets identity types. As described in Gambino and Larrea (2021)

https://doi.org/10.1017/S096012952100030X Published online by Cambridge University Press

(Section 2), in the context of an algebraic weak factorisation system (L, R) on a category $\mathcal{C}$ with finite limits, an interpretation of the identity type can be obtained from a suitable functorial factorisation of diagonals $A \longrightarrow A \times_X A$, for $f\colon A \longrightarrow X$ an object in $\mathcal{C}^2$. We hasten to add that definitions and results in Gambino and Larrea (2021) (Section 2) are cast in terms of algebras for R as a pointed endofunctor. Nevertheless, these definitions and results can be phrased and proved in terms of algebras for R as a monad as well, see van Woerkom (2021).

A functorial factorisation of diagonals in a category $\mathcal{C}$ with finite limits is a functor $\mathsf{P}\colon \mathcal{C}^2 \longrightarrow \mathcal{C}^2 \times_{\mathcal{C}} \mathcal{C}^2$ that maps $f\colon A \longrightarrow X$ to a factorisation of the diagonal $A \longrightarrow A \times_X A$. Recall from Gambino and Larrea (2021) and van Woerkom (2021) that a ***stable functorial choice of path objects*** for an algebraic weak factorisation system (L, R) consists of a lift:

$$\begin{array}{ccc} \mathsf{R}\text{-}\mathbf{Alg} & \xrightarrow{\mathsf{P}} & \mathsf{L}\text{-}\mathbf{Coalg} \times_{\mathcal{C}} \mathsf{R}\text{-}\mathbf{Alg} \\ \downarrow & & \downarrow \\ \mathcal{C}^2 & \xrightarrow{\mathsf{P}} & \mathcal{C}^2 \times_{\mathcal{C}} \mathcal{C}^2 \end{array}$$

of a functorial factorisation of diagonals P that is stable, in the sense that the right-hand component:

$$\mathcal{C}^2 \xrightarrow{\mathsf{P}} \mathcal{C}^2 \times_{\mathcal{C}} \mathcal{C}^2 \xrightarrow{\mathrm{pr}_2} \mathcal{C}^2$$

maps cartesian squares to cartesian squares, that is, taking the right-hand component commutes with pullback.

Note that the functorial factorisation of diagonals provided by the algebraic weak factorisation system itself

$$A \xrightarrow{f} X \qquad \longmapsto \qquad A \xrightarrow{\mathsf{L}\mathrm{pr}_{1,1}} \mathsf{M}\mathrm{pr}_{1,1} \xrightarrow{\mathsf{R}\mathrm{pr}_{1,1}} A \times_X A$$

does lift to a functor between algebras as above, simply by equipping its values with the cofree and free structure, respectively. However, it is an observation that goes back to van den Berg and Garner (van den Berg and Garner 2012, Remark 3.3.4) that this factorisation of diagonals is seldom stable as free structures need not be stable under pullback. In what follows, we investigate how to obtain a stable functorial choice of path objects for the algebraic weak factorisation system on $\mathcal{C}$-enriched groupoids described in the proof of Proposition 3.2 from the elementary structure on $\mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathcal{C}\text{-}\mathcal{G}pd$.

Proposition 4.1. *Let $\mathcal{C}$ be a category with finite limits. The algebraic weak factorisation system* (L, R) *on $\mathcal{C}\text{-}\mathcal{G}pd$ from Proposition 3.2 has a stable functorial choice of path objects.*

Proof. The choice of loops given in the proof of Lemma 3.7 provides a factorisation of diagonals over terminal arrows:

$$\mathbb{B} \qquad \longmapsto \qquad \mathbb{B} \xrightarrow{\mathsf{r}_{\mathbb{B}}} \mathrm{Iso}(\mathbb{B}) \xrightarrow{\mathrm{I}_{\mathbb{B}}} \mathbb{B} \times \mathbb{B}$$

and, as loops are cocartesian arrows, this assignment extends to a functor:

$$\mathcal{C}\text{-}\mathcal{G}pd \xrightarrow{(\mathsf{r}_{-}, \mathrm{Iso}(-), \mathrm{I}_{-})} \mathcal{C}\text{-}\mathcal{G}pd^2 \times_{\mathcal{C}\text{-}\mathcal{G}pd} \mathcal{C}\text{-}\mathcal{G}pd^2.$$

https://doi.org/10.1017/S096012952100030X Published online by Cambridge University Press

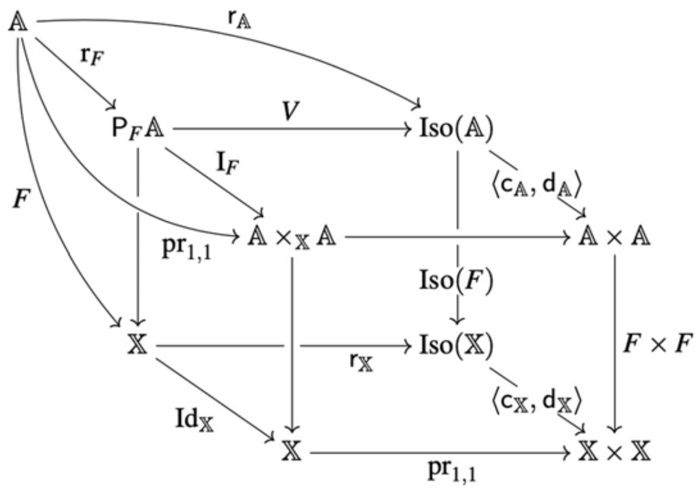
In order to extend this functor further to a functorial factorisation of diagonals $\mathcal{C}\text{-}\mathcal{G}pd^{\mathbf{2}} \longrightarrow \mathcal{C}\text{-}\mathcal{G}pd^{\mathbf{2}} \times_{\mathcal{C}\text{-}\mathcal{G}pd} \mathcal{C}\text{-}\mathcal{G}pd^{\mathbf{2}}$, consider the set:

$$|\mathsf{P}_F\mathbb{A}| := \sum_{\substack{(X,A_1,A_2)\in|\mathbb{X}|\times|\mathbb{A}|\times|\mathbb{A}| \\ |F|(A_1)=X=|F|(A_2)}} \left\{ 1 \xrightarrow{a} \mathsf{hom}_{\mathbb{A}}(A_1, A_2) \;\middle|\; F_{A_1,A_2}a = 1_{|F|(X)} \right\}$$

for $F\colon \mathbb{A} \longrightarrow \mathbb{X}$ an object in $\mathcal{C}\text{-}\mathcal{G}pd^{\mathbf{2}}$. There is an obvious inclusion $|V|\colon |\mathsf{P}_F\mathbb{A}| \longrightarrow |\mathrm{Iso}(\mathbb{A})|$, and functions $|\mathrm{r}_F|\colon |\mathbb{A}| \longrightarrow |\mathsf{P}_F\mathbb{A}|$ and $|\mathrm{I}_F|\colon |\mathsf{P}_F\mathbb{A}| \longrightarrow |\mathbb{A} \times_{\mathbb{X}} \mathbb{A}|$ mapping A to $(|F|(A), A, A, 1_A)$ and (X, A_1, A_2, a) to (X, A_1, A_2), respectively. Note that

$$\mathsf{hom}_{\mathrm{Iso}(\mathbb{X})}(F_{A_1,A_2}a, F_{A'_1,A'_2}a') = \mathsf{hom}_{\mathrm{Iso}(\mathbb{X})}(1_{|F|(X)}, 1_{|F|(X')}),$$

for $(X, A_1, A_2, a), (X', A'_1, A'_2, a') \in |\mathsf{P}_F\mathbb{A}|$. Defining $\mathsf{hom}_{\mathsf{P}_F\mathbb{A}}(a, a') := \mathsf{hom}_{\mathrm{Iso}(\mathbb{A})}(a, a')$, we obtain a $\mathcal{C}$-enriched category $\mathsf{P}_F\mathbb{A}$ and a $\mathcal{C}$-enriched functor $V\colon \mathsf{P}_F\mathbb{A} \longrightarrow \mathrm{Iso}(\mathbb{A})$ which is full, faithful and injective on objects, that is, $\mathsf{P}_F(\mathbb{A})$ is a $\mathcal{C}$-enriched full subcategory of $\mathrm{Iso}(\mathbb{A})$. These fit in a pullback square in $\mathcal{C}\text{-}\mathcal{G}pd$ which is the one in the back of diagram (**??**). It is then easy to see that the functions $|\mathrm{r}_F|$ and $|\mathrm{I}_F|$ extend to $\mathcal{C}$-enriched functors making the diagram (**??**) commute:



(14)

The assignment $F \mapsto (\mathrm{r}_F, \mathsf{P}_F\mathbb{A}, \mathrm{I}_F)$ is easily seen to be functorial. To see that it is also stable, it is enough to observe that, whenever the left-hand square in $\mathcal{C}\text{-}\mathcal{G}pd$ below is a pullback, so is the right-hand square:

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{H} & \mathbb{B} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbb{X} & \xrightarrow{K} & \mathbb{Y} \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{Iso}(\mathbb{A}) & \xrightarrow{\mathrm{Iso}(H)} & \mathrm{Iso}(\mathbb{B}) \\ {\scriptstyle \mathrm{Iso}(F)}\downarrow & & \downarrow{\scriptstyle \mathrm{Iso}(G)} \\ \mathrm{Iso}(\mathbb{X}) & \xrightarrow{\mathrm{Iso}(K)} & \mathrm{Iso}(\mathbb{Y}) \end{array}$$

Therefore we have constructed a stable functorial factorisation of diagonals $\mathsf{P}\colon \mathcal{C}\text{-}\mathcal{G}pd^{\mathbf{2}} \longrightarrow \mathcal{C}\text{-}\mathcal{G}pd^{\mathbf{2}} \times_{\mathcal{C}\text{-}\mathcal{G}pd} \mathcal{C}\text{-}\mathcal{G}pd^{\mathbf{2}}$. To obtain a stable functorial choice of path objects it is enough to equip I_F and r_F with algebra and coalgebra structures, respectively, for F a $\mathcal{C}$-enriched functor.

https://doi.org/10.1017/S096012952100030X Published online by Cambridge University Press

For the former, by functoriality of F the function $|\mathsf{s}_{\mathbb{A}}|\colon |\mathsf{M}\langle \mathsf{c}_{\mathbb{A}}, \mathsf{d}_{\mathbb{A}}\rangle| \longrightarrow |\mathrm{Iso}(\mathbb{A})|$ restricts along the inclusion $|V|\colon |\mathsf{P}_F\mathbb{A}| \longrightarrow |\mathrm{Iso}(\mathbb{A})|$ as shown below:

$$\begin{array}{ccc} |\mathsf{MI}_F| & \xrightarrow{\;|\mathsf{M}(F\times F,\,V)|\;} & |\mathsf{M}\langle \mathsf{c}_{\mathbb{A}}, \mathsf{d}_{\mathbb{A}}\rangle| \\ {\scriptstyle |\mathsf{s}_F|}\downarrow & & \downarrow{\scriptstyle |\mathsf{s}_{\mathbb{A}}|} \\ |\mathsf{P}_F\mathbb{A}| & \xrightarrow{\;|V|\;} & |\mathrm{Iso}(\mathbb{A})| \end{array}$$

where an element of $|\mathsf{MI}_F|$ consists of an element $(X, A_1, A_2, a) \in \mathsf{P}_F\mathbb{A}$, an element $(X', A_1', A_2') \in |\mathbb{A} \times_{\mathbb{X}} \mathbb{A}|$ and a triple $1 \xrightarrow{\langle x, a_1, a_2\rangle} \mathsf{hom}_{\mathbb{X}}(X', X) \times \mathsf{hom}_{\mathbb{A}}(A_1', A_1) \times \mathsf{hom}_{\mathbb{A}}(A_2', A_2)$ such that $F_{A_1', A_1} a_1 = x = F_{A_2', A_2} a_2$, and it is sent by $|\mathsf{s}_F|$ to $(X', A_1', A_2', a_1 \cdot a \cdot a_2{}^{\mathrm{i}})$. As $\mathsf{P}_F\mathbb{A}$ is a $\mathcal{C}$-enriched full subcategory of $\mathrm{Iso}(\mathbb{A})$, the component of s_F on a pair of elements of $|\mathsf{MI}_F|$ is simply given composing the corresponding components of $\mathsf{M}(F \times F, V)$ and $\mathsf{s}_{\mathbb{A}}$. It is straightforward to check that the pair $(\mathrm{I}_F, \mathsf{s}_F)$ is an algebra for the monad on R.

To construct a coalgebra structure on r_F, note that an element of $|\mathsf{Mr}_F|$ consists of $A \in |\mathbb{A}|$, $(X, A_1, A_2, a) \in |\mathsf{P}_F\mathbb{A}|$ and a pair $1 \xrightarrow{\langle a_1, a_2\rangle} \mathsf{hom}_{\mathbb{A}}(A_1, A) \times \mathsf{hom}_{\mathbb{A}}(A_2, A)$ such that $a_2 \cdot a = a_1$. Consider the function $|\mathsf{t}_F|\colon |\mathsf{P}_F\mathbb{A}| \longrightarrow |\mathsf{Mr}_F|$ that maps (X, A_1, A_2, a) to $(A_2, (X, A_1, A_2, a), 1 \xrightarrow{\langle a, 1_{A_2}\rangle} \mathsf{hom}_{\mathbb{A}}(A_1, A_2) \times \mathsf{hom}_{\mathbb{A}}(A_2, A_2))$. The component of t_F on a pair $(X, A_1, A_2, a), (X'A_1', A_2', a')$ is the arrow induced by the equaliser defining $\mathsf{hom}_{\mathsf{Mr}_F}$ as depicted below:

$$\begin{array}{ccc} \mathsf{hom}_{\mathsf{P}_F\mathbb{A}}(a, a') & \xrightarrow{\;(\mathsf{t}_F)_{a,a'}\;} & \mathsf{hom}_{\mathsf{Mr}_F}((A_2, a, \langle a, 1_{A_2}\rangle), (A_2', a', \langle a', 1_{A_1'}\rangle)) \\ & \searrow^{\langle (\mathsf{c}_{\mathbb{A}})_{a,a'}, \mathrm{id}\rangle} & \downarrow{\scriptstyle e} \\ & & \mathsf{hom}_{\mathbb{A}}(A_2, A_2') \times \mathsf{hom}_{\mathsf{P}_F\mathbb{A}}(a, a') \\ & & {\scriptstyle (\langle a, 1_{A_2}\rangle !)\cdot\langle \mathrm{pr}_1, \mathrm{pr}_1\rangle}\downarrow\;\downarrow{\scriptstyle \mathrm{pr}_2\cdot(\langle a', 1_{A_2'}\rangle !)} \\ & \xrightarrow{\;\langle (\mathsf{d}_{\mathbb{A}})_{a,a'}\cdot a', (\mathsf{c}_{\mathbb{A}})_{a,a'}\rangle\;} & \mathsf{hom}_{\mathsf{P}_F\mathbb{A}}(a, 1_{A_2'}) \end{array}$$

□

A comprehension category is suitable to interpret identity types if it has a pseudo-stable choice of Id-types, see Gambino and Larrea (2021) (Definition 1.4). Indeed in this case, its right adjoint splitting can be equipped with a strictly stable choice of Id-types that allows for a sound interpretation of identity types, see Gambino and Larrea (2021) (Theorem 1.6) and Warren (2008) (Theorem 2.48).

Theorem 4.2. *The comprehension category:*

$$\begin{array}{ccc} \mathsf{R}\text{-}\mathbf{Alg} & \xrightarrow{\;U\;} & \mathcal{C}\text{-}\mathcal{Gpd}^{\mathbf{2}} \\ & \searrow\quad\swarrow & \\ & \mathcal{C}\text{-}\mathcal{Gpd} & \end{array}$$

has a pseudo-stable choice of Id-types. Hence, its right adjoint splitting models identity types.

Proof. Proposition 59 in van Woerkom (2021) ensures that a stable functorial choice of path objects in $\mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathcal{C}\text{-}\mathcal{Gpd}$ yields a pseudo-stable choice of Id-types in the associated comprehension category (13). It thus follows from Proposition 4.1 that the comprehension category

https://doi.org/10.1017/S096012952100030X Published online by Cambridge University Press

associated with the fibration R-**Alg** $\longrightarrow \mathcal{C}\text{-}\mathcal{G}pd$ has a pseudo-stable choice of Id-types. By Gambino and Larrea (2021) (Proposition 1.9 and Theorem 1.6), its right adjoint splitting models identity types. □

5. Iterations of Enrichment

In this section, we intend to analyse the enrichment over the category $\mathcal{G}pd$ of groupoids themselves. The first direct consequence of Theorem 3.6 is the following.

Theorem 5.1. *The fibration* R-**Alg** $\longrightarrow \mathcal{G}pd\text{-}\mathcal{G}pd$ *is elementary.*

Proof. As remarked in Remark 3.1, the category $\mathcal{G}pd$ has limits. □

Remark 5.2. One may wonder if there is any gain in enriching over the category $\mathcal{C}at$ of all categories. But $\mathcal{C}at\text{-}\mathcal{G}pd = \mathcal{G}pd\text{-}\mathcal{G}pd$.

Indeed, $\mathcal{G}pd\text{-}\mathcal{G}pd$ is clearly a full subcategory of $\mathcal{C}at\text{-}\mathcal{G}pd$. But a $\mathcal{C}at$-enriched groupoid is a 2-category with an inverse:

$$B \underset{g^{-1}}{\overset{f^{-1}}{\underset{\alpha^*\Downarrow}{\rightrightarrows}}} A$$

with respect to horizontal composition for each 2-arrow $A \underset{g}{\overset{f}{\underset{\alpha\Downarrow}{\rightrightarrows}}} B$. Taking the following whiskering of α^*

$$A \xrightarrow{\;g\;} B \underset{g^{-1}}{\overset{f^{-1}}{\underset{\alpha^*\Downarrow}{\rightrightarrows}}} A \xrightarrow{\;f\;} B$$

gives the required 2-arrow, inverse for $A \underset{g}{\overset{f}{\underset{\alpha\Downarrow}{\rightrightarrows}}} B$ with respect to vertical composition.

The next result is the application of Theorem 4.2 to the case at hand.

Theorem 5.3. *The full comprehension category associated with the fibration* R-**Alg** $\longrightarrow \mathcal{G}pd\text{-}\mathcal{G}pd$ *has a pseudo-stable choice of Id-types.*

Since the category $\mathcal{G}pd\text{-}\mathcal{G}pd$ has finite limits by Remark 3.1, one sees immediately that there is the possibility to iterate the last two results. Write $n\mathcal{G}pd$ for the category of n-groupoids, that is, let $0\mathcal{G}pd$ be the category $\mathcal{S}et$ and let $(n+1)\mathcal{G}pd$ be the category of $n\mathcal{G}pd$-enriched groupoids. Write $(\mathsf{L}_{n+1}, \mathsf{R}_{n+1})$ for the algebraic weak factorisation system on $(n+1)\mathcal{G}pd$ from Proposition 3.2.

Theorem 5.4. *The fibration* R_{n+1}-**Alg** $\longrightarrow (n+1)\mathcal{G}pd$ *is elementary and the comprehension category associated with the fibration* R_{n+1}-**Alg** $\longrightarrow (n+1)\mathcal{G}pd$ *has a pseudo-stable choice of Id-types.*

https://doi.org/10.1017/S096012952100030X Published online by Cambridge University Press

These models can be seen as the Hofmann and Streicher's groupoid model lifted to $(n+1)$-groupoids. In particular, in these models, Uniqueness of Identity Proofs holds for all the identity types, which are constructed in the proof of Proposition 4.1. In fact, the fibre of $\mathrm{I}_{\mathbb{B}}$ over a pair of objects (B_2, B_1) is the discrete $(n+1)$-groupoid on the set $|\hom_{\mathbb{B}}(B_1, B_2)|$. Indeed, an arrow $(b_2, b_1)\colon (B_2, B_1, 1 \xrightarrow{x} \hom_{\mathbb{B}}(B_1, B_2)) \longrightarrow (B_2', B_1', 1 \xrightarrow{x'} \hom_{\mathbb{B}}(B_1', B_2'))$ in $\mathrm{Iso}(\mathbb{B})$ is vertical with respect to $\mathrm{I}_{\mathbb{B}}$ if and only if $b_1 = \mathrm{id}_{B_1}$ and $b_2 = \mathrm{id}_{B_2}$ and thus, necessarily, $x = x'$.

The elementary fibrations underlying these models arrange into a chain of forgetful functors:

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathsf{R}_{n+1}\text{-}\mathbf{Alg} & \longrightarrow & \mathsf{R}_{n}\text{-}\mathbf{Alg} & \longrightarrow & \cdots & \longrightarrow & \mathsf{R}_{1}\text{-}\mathbf{Alg} \\
 & & \downarrow & & \downarrow & & \cdots & & \downarrow \\
\cdots & \longrightarrow & (n+1)\mathcal{G}pd & \longrightarrow & n\mathcal{G}pd & \longrightarrow & \cdots & \longrightarrow & \mathcal{G}pd
\end{array}
$$

each of which preserves limits and the elementary structure. The limit of the bottom chain of categories is the category $\omega\mathcal{G}pd$ of strict ω-groupoids. It is reasonable to expect that the elementary fibration over it interprets identity types. However, types in this model would most likely be 1-types, as this is the case for each fibration in the above limit diagram. Thus, we expect this model to be different from the model in strict ω-groupoids of Warren (2011).

Note

1 Jacopo Emmenegger's work was partially funded by EPSRC grant EP/T000252/1. Giuseppe Rosolini's work was partially supported by INdAM-GNSAGA.

References

Borceux, F. (1994). *Handbook of Categorical Algebra*, vol. II. Cambridge University Press.

Bourke, J. and Garner, R. (2016). Algebraic weak factorisation systems I: Accessible AWFS. *Journal of Pure and Applied Algebra* **220** (1) 108–147.

Emmenegger, J., Pasquali, F. and Rosolini, G. (2020). Elementary doctrines as coalgebras. *Journal of Pure and Applied Algebra* **224** (12) 106445, 16.

Emmenegger, J., Pasquali, F. and Rosolini, G. (2021). A characterization of elementary fibrations. Available as arXiv:2007.16180.

Gambino, N. and Larrea, M. F. (2021). Models of Martin-Löf type theory from algebraic weak factorisation systems. *Journal of Symbolic Logic* (In Press). Preprint version available as arXiv:1906.01491.

Garner, R. (2008). Understanding the small object argument. *Applied Categorical Structures* **17** (3) 247–285.

Grandis, M. and Tholen, W. (2006). Natural weak factorization systems. *Archiv der Mathematik (Brno)* **42**(4) 397–408.

Hermida, C. (1994). On fibred adjunctions and completeness for fibred categories. In: *Recent Trends in Data Type Specification (Caldes de Malavella, 1992)*, vol. 785. Lecture Notes in Computer Science. Springer, 235–251.

Hofmann, M. (1997). *Extensional Constructs in Intensional Type Theory*. CPHC/BCS Distinguished Dissertations. Springer-Verlag London Ltd.

Hofmann, M. and Streicher, T. (1994). The groupoid model refutes uniqueness of identity proofs. In: *Proceedings Ninth Annual IEEE Symposium on Logic in Computer Science*, IEEE Computer Society, 208–212.

Hofmann, M. and Streicher, T. (1998). The groupoid interpretation of type theory. In: *Twenty-Five Years of Constructive Type Theory* (*Venice, 1995*), vol. 36. Oxford Logic Guides. Oxford Univ. Press, 83–111.

Jacobs, B. (1999). *Categorical Logic and Type Theory*, vol. 141. Studies in Logic and the Foundations of Mathematics. North Holland Publishing Company.

Kelly, G. (1982). *Basic Concepts of Enriched Category Theory*, vol. 64. London Mathematical Society Lecture Note Series. Cambridge University Press.

Lawvere, F. (1970). Equality in hyperdoctrines and comprehension schema as an adjoint functor. In: Heller, A. (ed.) *Proc. New York Symposium on Application of Categorical Algebra*. American Mathematical Society, 1–14.

Maietti, M. and Rosolini, G. (2013a). Elementary quotient completion. *Theory and Applications of Categories* **27** 445–463.

Maietti, M. and Rosolini, G. (2013b). Quotient completion for the foundation of constructive mathematics. *Logica Universalis* **7**(3) 371–402.

Maietti, M. and Rosolini, G. (2015). Unifying exact completions. *Applied Categorical Structures* **23** 43–52.

Streicher, T. 2020. Fibred Categories à la Jean Bénabou. Available at arXiv:1801.02927v7.

The Univalent Foundations Program (2013). *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study.

van den Berg, B. and Garner, R. (2012). Topological and simplicial models of identity types. *ACM Transactions on Computational Logic* **13**(1) 1–44.

van Woerkom, W. (2021). Algebraic models of type theory. Master's thesis, Universiteit van Amsterdam.

Vasilakopoulou, C. (2018). On enriched fibrations. *Cahiers de Topologie et Géométrie Différentielle Catégoriques* **LIX**(4) 354–387.

Warren, M. (2008). *Homotopy Theoretic Aspects of Constructive Type Theory*. PhD thesis, Carnegie Mellon University.

Warren, M. (2011). The strict ω-groupoid interpretation of type theory. In: B. Hart, et al. (eds.) *Models, Logics and Higher-Dimensional Categories*, CRM Proceedings and Lecture Notes, vol. 53, American Mathematical Society, 291–340

Cite this article: Emmenegger J, Pasquali F and Rosolini G (2021). Elementary fibrations of enriched groupoids. *Mathematical Structures in Computer Science*. https://doi.org/10.1017/S096012952100030X