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GRÖBNER BASES FOR QUADRATIC ALGEBRAS OF SKEW TYPE

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Abstract Non-degenerate monoids of skew type are considered. This is a class of monoids S defined by n generators and $\binom{n}{2}$ quadratic relations of certain type, which includes the class of monoids yielding set-theoretic solutions of the quantum Yang–Baxter equation, also called binomial monoids (or monoids of I-type with square-free defining relations). It is shown that under any degree-lexicographic order on the associated free monoid FM_n of rank n the set of normal forms of elements of S is a regular language in FM_n. As one of the key ingredients of the proof, it is shown that an identity of the form $x^N y^N = y^N x^N$ holds in S. The latter is derived via an investigation of the structure of S viewed as a semigroup of matrices over a field. It also follows that the semigroup algebra $K[S]$ is a finite module over a finitely generated commutative subalgebra of the form $K[A]$ for a submonoid A of S.

Keywords: finitely presented semigroup; semigroup ring; semigroup; normal form; regular language

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1. Introduction

Automaton algebras (and automaton semigroups) were defined by Ufnarovskii [**16**] with the condition that the set of normal forms of elements of the algebra (a semigroup) is a regular language. Namely, let A be a finitely generated algebra over a field K with a set of generators $a_1, \dots, a_n$. Let $K\langle x_1, \dots, x_n\rangle$ denote the free K-algebra of rank n and let $\pi\colon K\langle x_1, \dots, x_n\rangle \to A$ be the homomorphism such that $\pi(x_i) = a_i$ for all i. Assume that a well order is fixed on the free monoid $\mathrm{FM}_n = \langle x_1, \dots, x_n\rangle$ which is compatible with the multiplication in FM_n. Let I be the ideal of FM_n consisting of all leading monomials of elements of $\ker(\pi)$. Then the set $N(A) = \mathrm{FM}_n \setminus I$ is called the set of normal words corresponding to the chosen presentation and the chosen order on FM_n, and the minimal set of generators of I is called the set of obstructions. One says that A is an automaton algebra if $N(A)$ is a regular language. Recall that the latter means that this set is obtained from a finite subset of FM_n by applying a finite sequence of operations of union, multiplication and operation $*$ defined by $T^* = \bigcup_{i\geqslant 1} T^i$ for $T \subseteq \mathrm{FM}_n$. If $T = \{w\}$ for some $w \in \mathrm{FM}_n$, then for the sake of simplicity we sometimes write $T^* = w^*$. For basic facts on regular languages and automata theory we refer the reader to [**8**]. If S is a semigroup, then S is an automaton semigroup if the semigroup algebra $K[S]$ is an automaton algebra. In this case we also write $N(S) = N(K[S])$. The class of automaton algebras contains the class of algebras with a finite Gröbner basis (or, equivalently, algebras with a finitely generated ideal of obstructions). There are several results indicating that not only has this class better computational properties, but also several algebraic and structural properties behave better than in the class of arbitrary finitely generated (or even finitely presented) algebras. For example, in this context one can quote results on the growth and Gelfand–Kirillov dimension [**16**, § 5.10], results on the radical in the case of monomial automaton algebras [**16**, § 7.6], results on prime algebras of this type [**2**] and also results concerning the special case of finitely presented monomial algebras in [**12**, Chapter 24] and [**13**].

In general, Ufnarovskii's notion depends not only on the given presentation but also on the chosen order on the corresponding free monoid FM_n. His approach was later continued in [**9**, **10**].

The object of our study in this paper is the class of so-called monoids (and algebras) of skew type. Let $X = \{x_1, x_2, \dots, x_n\}$. Assume that a function $r\colon X^2 \to X^2$ is given. Then $r(x_i, x_j) = (x_{\sigma_i(j)}, x_{\gamma_j(i)})$ for some maps $\sigma_1, \dots, \sigma_n, \gamma_1, \dots, \gamma_n\colon \{1, \dots, n\} \to \{1, \dots, n\}$. Assume also that $r^2 = \mathrm{id}_{X^2}$ and $r(x, x) = (x, x)$ for all $x \in X$. By the monoid of skew type $S = \langle x_1, \dots, x_n \rangle$ associated to r, we mean the monoid presented with generators $x_1, \dots, x_n$ and with the defining relations $x_i x_j = x_{\sigma_i(j)} x_{\gamma_j(i)}$ for all $1 \leqslant i, j \leqslant n$. We shall also write $S = \langle X; R \rangle$, where R denotes the set of defining relations. We refer the reader to [**7**], where it is shown in particular that these monoids provide us with intriguing classes of Noetherian PI algebras with additional nice properties. Algebras of this type are also called algebras with quantum binomial relations [**4**]. In particular, they include the class of so-called square-free algebras of I-type, which are semigroup algebras of monoids yielding set-theoretic solutions to the quantum Yang–Baxter equation [**3**, **5**, **7**, **15**].

Let Sym_n be the symmetric group of degree n. If $\sigma_1, \dots, \sigma_n \in \mathrm{Sym}_n$, then we say that S is right non-degenerate, and if $\gamma_1, \dots, \gamma_n \in \mathrm{Sym}_n$, then S is left non-degenerate. If both conditions are satisfied, then we say that S is a non-degenerate monoid of skew type. Our main result shows that under any degree-lexicographic order on the associated free monoid FM_n of rank n the set $N(S)$ of normal forms of elements of S is a regular language in FM_n. One of our main motivations is a fundamental result saying that if A is a square-free algebra of I-type, then there exists a degree-lexicographic order on FM_n such that the corresponding ideal I of obstructions is generated by the set $\{x_i x_j \mid x_i x_j = x_k x_l \text{ and } x_i x_j > x_k x_l\}$, or, equivalently, that $N(A) = \{x_{i_1}^{\alpha_1} \cdots x_{i_n}^{\alpha_n} \mid x_{i_1} < \cdots < x_{i_n},\ \alpha_i \geqslant 0\}$. (Actually, so-called binomial monoids were first defined in terms of the Gröbner basis in [**3**]; then, in [**5**], it was shown that binomial monoids are of I-type, while in [**15**] it was shown that every square-free monoid of I-type is a binomial monoid.) As one of the key ingredients of the proof, it is shown that an identity of the form $x^N y^N = y^N x^N$ holds in S. The latter is derived via an investigation of the structure of S viewed as a semigroup of matrices over a field. This is then used to show that $C = \langle s^N \mid s \in S\rangle$ is a finitely generated commutative submonoid of S such that

$$S = \bigcup_{f \in F} fC = \bigcup_{f \in F} Cf$$

for a finite subset $F \subseteq S$, which seems to be a result of independent interest.

Notice that every monoid of skew type S has a degree function induced by the length of the words in FM_n, because the defining relations of S are homogeneous. We will denote by $\deg(a)$ the degree of $a \in S$.

2. Monoids of skew type as linear semigroups

From [**7**, Theorem 9.4.2], we know that, for every field K, the algebra $K[S]$ of a non-degenerate monoid S of skew type is a Noetherian PI algebra. Therefore, by a result of Anan′in [**1**], the algebra $K[S]$ is representable, which means that it embeds into the algebra $M_m(L)$ of matrices over a field extension L of K for some $m \geqslant 1$. Our aim in this section is to prove certain structural properties of S viewed as a semigroup of matrices over a field and to derive an important combinatorial property that will be crucial in the next section.

First we prove the following technical lemma.

Lemma 2.1. *Let $S = \langle X; R\rangle$ be a right non-degenerate monoid of skew type. Then, for every $a, b \in S$, $a^{(n!)^m} b \in bS$, where $n = |X|$ and $m = \deg(b)$.*

Proof. We shall prove the result by induction on $\deg(b)$. If $\deg(b) = 0$, then $b = 1$ and the result is clear in this case.

Suppose that $\deg(b) = m \geqslant 1$ and the result is true for all $a, b' \in S$ with $\deg(b') < m$. Since $\deg(b) \geqslant 1$, there exist $b' \in S$ and $x_i \in X$ such that $b = b'x_i$. Since $\deg(b') = m-1$, by the induction hypothesis, there exists $c \in S$ such that $a^{(n!)^{m-1}} b' = b'c$. Therefore,

$$a^{(n!)^m} b = a^{(n!)^m} b' x_i = b' c^{n!} x_i.$$

Let $c = x_{i_1} \cdots x_{i_k}$. Then, by using the defining relations $x_j x_k = x_{\sigma_j(k)} x_{\gamma_k(j)}$, we have

$$\begin{aligned} c x_i &= x_{i_1} \cdots x_{i_k} x_i = x_{i_1} \cdots x_{i_{k-1}} x_{\sigma_{i_k}(i)} x_{\gamma_i(i_k)} \\ &= x_{i_1} \cdots x_{i_{k-2}} x_{\sigma_{i_{k-1}} \sigma_{i_k}(i)} x_{\gamma_{\sigma_{i_k}(i)}(i_{k-1})} x_{\gamma_i(i_1)} \\ &\;\;\vdots \\ &= x_{\sigma_{i_1} \cdots \sigma_{i_k}(i)} c' \end{aligned}$$

for some $c' \in S$. Since S is right non-degenerate, $\sigma_1, \ldots, \sigma_n \in \mathrm{Sym}_n$. Hence, there exists $c'' \in S$ such that $c^{n!} x_i = x_{(\sigma_{i_1} \cdots \sigma_{i_k})^{n!}(i)} c'' = x_i c''$. Therefore,

$$a^{(n!)^m} b = b' c^{n!} x_i = b' x_i c'' = bc'' \in bS,$$

and the result follows by induction. □

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Now we are ready to prove the main result of this section. For basic background on the structure of semigroups of matrices, including the description of Green's relations on the full multiplicative monoid $M_m(L)$ over a field L, we refer the reader to [**12**].

Theorem 2.2. *Let S be a non-degenerate monoid of skew type and let K be any field. Then S embeds into the multiplicative monoid $M_m(L)$ of matrices over a field extension L of K for some $m \geqslant 1$ and, if S is viewed as a subsemigroup of $M_m(L)$, then the following conditions hold.*

(i) *S intersects finitely many $\mathcal{H}$-classes of the multiplicative monoid $M_m(L)$.*

(ii) *If H is a maximal subgroup of $M_m(L)$, then $S \cap H$ generates a finitely generated abelian-by-finite subgroup of H that is the group of quotients of $S \cap H$.*

(iii) *If e, f are idempotents such that $e \in H_1$, $f \in H_2$ for some maximal subgroups H_1, H_2 of $M_m(L)$ intersecting S, then $ef = fe$.*

(iv) *There exists a positive integer N such that $a^N b^N = b^N a^N$ for every $a, b \in S$.*

Proof. As noted at the beginning of this section, $K[S]$ embeds into the algebra $M_m(L)$ of matrices over a field extension L of K for some $m \geqslant 1$. Thus, in order to prove assertions (i)–(iv), we view S as a subsemigroup of the multiplicative monoid $M_n(L)$. The first assertion is an easy consequence of the fact that $K[S]$ is right and left Noetherian [**7**, Proposition 5.1.1]. Then, from [**12**, Proposition 3.16], it follows that for every maximal subgroup H of $M_m(L)$ intersecting S the subgroup of H generated by $S \cap H$ is finitely generated. Since $K[S]$ is a PI algebra, this group must be abelian-by-finite [**14**, Theorems 5.3.7 and 5.3.9], and it is the group of quotients of $S \cap H$. Thus, the second assertion follows.

Let $a \in S \cap H_1$, $b \in S \cap H_2$ for maximal subgroups H_1, H_2 of $M_m(L)$ intersecting S. From Lemma 2.1 and its dual we know that

$$a^{\alpha} b = bc, \qquad ab^{\beta} = c'a, \qquad ba^{\gamma} = db, \qquad b^{\delta} a = ad' \tag{2.1}$$

for some positive integers α, β, γ, δ and some elements $c, c', d, d' \in S$. Let $b' \in H_2$ be the inverse of b in H_2. Then

$$a^{\alpha} f = a^{\alpha} bb' = bcb' = fbcb'.$$

Hence,

$$a^{\alpha} f = fa^{\alpha} f.$$

Since $a \in H_1$ and e is the identity of H_1, by the Cayley–Hamilton Theorem we know that $e = \sum_{i=1}^{j} \lambda_i (a^{\alpha})^i$ for some $\lambda_i \in K$, where j is the rank of all matrices in H_1. Then $f(a^{\alpha})^i f = (a^{\alpha})^i f$ for every $i \geqslant 1$. Hence,

$$ef = \sum_{i=1}^{j} \lambda_i (a^{\alpha})^i f = \sum_{i=1}^{j} \lambda_i f (a^{\alpha})^i f = fef.$$

In a similar way, the remaining three equalities in (2.1) imply that

$$ef = efe, \qquad fe = fef, \qquad fe = efe.$$

Then $ef = efe = fe$, which proves assertion (iii).

Let H_3 be the maximal subgroup of $M_m(L)$ containing the idempotent ef. Then every element of the set H_1H_2 is $\mathcal{J}$-related in the monoid $M_m(L)$ to ef.

In view of (i), the set C of $\mathcal{R}$-classes of $M_m(L)$ intersecting the set $(S\cap H_1)(S\cap H_2)$ is finite. Let $\hat{H}_i = (S\cap H_i)(S\cap H_i)^{-1}$ for $i = 1, 2, 3$. Let $a_1, a_2 \in S\cap H_1$ and $b_1, b_2 \in S\cap H_2$. We have seen above that there exists a positive integer α such that $a_2^{\alpha} b_1 \in b_1 S$ and this leads to $fa_2^{\alpha} f = a_2^{\alpha} f$. Since $a_2^{\alpha} f \;\mathcal{L}\; ef$ in $M_m(L)$ and $fa_2^{\alpha} f \in feM_m(L) = efM_m(L)$, this implies that $fa_2^{\alpha} f = a_2^{\alpha} f \in H_3$. If a_2^{-1} denotes the inverse of a_2 in H_1, then we also get $a_2^{-\alpha} fM_m(L) = a_2^{-\alpha} fa_2^{\alpha} fM_m(L) = efM_m(L)$. Thus,

$$\begin{aligned} a_1a_2^{-1}b_1b_2^{-1}M_m(L) &= a_1a_2^{\alpha-1}a_2^{-\alpha} fM_m(L) \\ &= a_1a_2^{\alpha-1}efM_m(L) \\ &= a_1a_2^{\alpha-1}b_1M_m(L). \end{aligned}$$

Therefore, the set of $\mathcal{R}$-classes of $M_m(L)$ intersecting $\hat{H}_1\hat{H}_2$ coincides with the set C, so it is finite. The elements of $S\cap H_1$ act by left multiplication on C. So, this gives a homomorphism $\phi\colon S\cap H_1 \to \mathrm{Sym}_t$, where t is the cardinality of C. This homomorphism ϕ can be extended to a homomorphism $\phi'\colon \hat{H}_1 \to \mathrm{Sym}_t$. It follows that the kernel of ϕ' is a normal subgroup of finite index dividing $t!$ in $\hat{H}_1$. So, the action of every $a^{t!}$, for $a \in S\cap H_1$, is trivial. Let $k_1 = t!$. It follows that $a^{k_1}ef \;\mathcal{R}\; ef$ in $M_m(L)$. Since $a^{k_1}ef \;\mathcal{L}\; ef$, it follows that $a^{k_1}ef \in H_3$. In particular, $a^{k_1}ef = efa^{k_1}ef$. A symmetric argument shows that there exists a positive integer k_2 (which depends on the cardinality of the finite set of $\mathcal{L}$-classes of $M_m(L)$ intersecting $(S\cap H_2)(S\cap H_1)$) such that $fea^{k_2} = fea^{k_2}fe$. Let $k = k_1k_2$. Since $ef = fe$, we get that $a^k ef = efa^k$. Similarly, it follows that $b^l ef = efb^l$ for some $l \geqslant 1$ (which depends on the cardinality of the finite set of $\mathcal{R}$-classes of $M_m(L)$ intersecting $(S\cap H_2)(S\cap H_1)$ and on the cardinality of the finite set of $\mathcal{L}$-classes of $M_m(L)$ intersecting $(S\cap H_1)(S\cap H_2)$). From (ii) we know that there exists a normal abelian subgroup A of finite index in the subgroup $\hat{H}_3$ of H_3 generated by $S\cap H_3$. If $r = [\hat{H}_3 : A]$, then $(a^k ef)^r (b^l ef)^r = (b^l ef)^r (a^k ef)^r$. Therefore,

$$a^{kr}b^{lr} = (a^{kr}e)(fb^{lr}) = (a^{kr}ef)(efb^{lr}) = (b^{lr}ef)(efa^{kr}) = b^{lr}a^{kr}.$$

Notice that for every $a \in S$ there exists a maximal subgroup H of $M_m(L)$ such that $a^m \in H$. Since the set Z of maximal subgroups of $M_m(L)$ intersecting S is finite, assertion (iv) follows with $N = mr'$, where r' is the least common multiple of indices of abelian normal subgroups (of finite index) for all groups of the form $(S\cap H)(S\cap H)^{-1}$, where H runs through the set Z and of the finite set of all possible integers k, l defined as above (for all pairs of maximal subgroups from the set Z). □

As mentioned above, a right non-degenerate monoid of skew type satisfies the ascending chain condition on right ideals. For arbitrary submonoids of nilpotent-by-finite groups with the latter property, the assertion of Lemma 2.1 can also be proved.

Lemma 2.3. *Let S be a submonoid of a nilpotent-by-finite group G. Assume that S satisfies the ascending chain condition on right ideals. Then there exists a positive integer q such that for every $s, t \in S$ we have $s^q t \in tS$ and $ts^q \in St$.*

Proof. By [**7**, Lemma 4.1.5], for every $u \in S$ there exists $n_u \geqslant 1$ such that $s^{n_u} u \in uS$. Moreover, S has a group of quotients. Hence, we may assume that $G = SS^{-1} = S^{-1}S$. Theorem 4.4.6 of [**7**] also implies that G has a normal subgroup A of finite index such that the commutator subgroup $[A, A]$ is contained in S.

First, consider the case where A is abelian. Let F be a finite set of coset representatives of A in G. Since A has finite index in G, such an F can be chosen so $F \subseteq S$. Let $m = |F|$. Then, for every $t \in S$, we get

$$t^{-1}s^m t = (t^{-1}st)^m = (f^{-1}sf)^m = f^{-1}s^m f,$$

where $f \in F$ is chosen so that $At = Af$. Therefore,

$$t^{-1}s^{mn_f}t = (f^{-1}s^{mn_f}f) = (f^{-1}s^{n_f}f)^m \in S.$$

Hence, $s^q t \in tS$ follows, with $q = m\prod_{f\in F} n_f$.

Now, consider the general case. Let $\bar{S} = S/[A, A] \subseteq \bar{G} = G/[A, A]$. Applying the previous case to any elements $s, t \in S$ we get $\bar{s}^q\bar{t} \in \bar{t}\bar{S}$, where $\bar{u}$ denotes the image of $u \in S$ in $\bar{S}$. This means that $s^q t \in tS[A, A] \subseteq tS$ because $[A, A] \subseteq S$. The symmetric assertion follows from the fact that S also satisfies the ascending chain condition on left ideals [**7**, Theorem 4.4.7]. So, there exists $q' \geqslant 1$ such that $ts^{q'} \in St$ for every $s, t \in S$. The result follows. □

Assume that $S = \langle a_1, \dots, a_n\rangle$ is a cancellative monoid. Let $\pi\colon \mathrm{FM}_n \to S$ be the natural homomorphism and assume that an order on FM_n is given such that $S = \pi(N)$ for a regular subset $N \subseteq \mathrm{FM}_n$ that is a union of finitely many subsets, each of the form $w_1y_1^*w_2y_2^*\cdots w_ky_k^*$ for some k and some $w_i, y_i \in \mathrm{FM}_n$. The growth of S is then polynomial, whence by the theorem of Grigorchuk [**11**, Theorem 8.3], S has a nilpotent-by-finite group of quotients. So, if S satisfies the ascending chain condition on right ideals, then the previous lemma can be applied. Observe that if S does not satisfy the ascending chain condition on right ideals, then this is no longer true, as the following example shows [**7**, Example 4.3.4]. Let $G = A \rtimes C$, where A is a free abelian group of rank 2 with basis a, b and $C = \langle c\rangle$ is the cyclic group of order 2 with the action $ca = bc$, $cb = ac$. Then in the submonoid $S = \langle a, ac\rangle$ of G we have $(ac)^{-1}a^i(ac) = c^{-1}a^ic = b^i \notin S$ for every i.

3. Normal forms of elements as regular languages

It is known that, in general, changing the order on the free monoid FM_n with basis X may result in a dramatic change of the properties of the subset $N(S)$ of normal words of a monoid S defined by a presentation $S = \langle X; R\rangle$. The following example comes from [**9**].

Example 3.1. Let $S = \langle x, y \mid xyx = yx^2 \rangle$. If $y > x$, then $xyx - yx^2$ forms a Gröbner basis for S, so the ideal of obstructions is of the form $I = (yx^2)$. If $x > y$, then $xy^i x^i - y^i x^{i+1}$, $i \geqslant 1$, forms a Gröbner basis and $I = (xy^i x^i \mid i \geqslant 0)$. Hence, in the latter case the set of normal forms in FM_2 of elements of S is not a regular language.

Our main aim in this section is to prove that, for every non-degenerate monoid of skew type $S = \langle X; R \rangle$ and any degree-lexicographic order on the free monoid FM_n with basis $X = \{x_1, \dots, x_n\}$, the subset of all normal forms in FM_n of the elements of S is a regular language.

However, it is not true that if $S = \langle X; R \rangle$ is a non-degenerate monoid of skew type and S has a finite Gröbner basis for some degree-lexicographic order on the free monoid FM_n with basis X, then it has a finite Gröbner basis for every degree-lexicographic order on FM_n. Even in the class of binomial monoids this is not true, as the following example shows.

Example 3.2. Let $M = \langle x_1, x_2, x_3, x_4 \rangle$ be the monoid of skew type defined by the relations

$$\begin{aligned} x_1x_2 &= x_3x_4, & x_1x_3 &= x_2x_4, & x_2x_1 &= x_4x_3, \\ x_3x_1 &= x_4x_2, & x_1x_4 &= x_4x_1, & x_2x_3 &= x_3x_2. \end{aligned}$$

Note that M is a binomial monoid (isomorphic to the monoid $B^{4,5}$ of [**7**, Proposition 10.2.1]). If FM_4 is ordered by $x_1 < x_4 < x_2 < x_3$, then $N(M) = \{x_1^i x_4^j x_2^k x_3^l \mid i, j, k, l \geqslant 0\}$ and the defining relations yield a Gröbner basis for M. So, the ideal of obstructions $I = (x_2x_1, x_3x_1, x_4x_1, x_2x_4, x_3x_4, x_3x_2)$. On the other hand, it is easy to see that, when the degree-lexicographic order on FM_4 is defined by $x_1 < x_2 < x_3 < x_4$, M does not admit a finite Gröbner basis. In other words, the corresponding ideal of obstructions I of FM_4 is not finitely generated. In fact,

$$x_3x_1x_3 = x_4x_2x_3 = x_4x_3x_2 = x_2x_1x_2 = x_2x_3x_4 = x_3x_2x_4$$

shows that $x_3x_1x_3 \in I$. Then, by an easy induction we get $x_3x_1^{2k+1}x_3 \in I$ for every $k \geqslant 1$. On the other hand, one verifies that $x_3x_1^{2k}x_3 \in (x_3M \cap x_4M) \setminus (x_1M \cup x_2M)$ for every $k \geqslant 1$. Since M is cancellative, this easily implies that $x_3x_1^{2k}x_3 \notin I$. Therefore, I is not a finitely generated ideal of FM_4.

In order to distinguish the generators of S from the generators of FM_n, we denote the generators $x_1, \dots, x_n$ of S by $a_1, \dots, a_n$, respectively. Thus, $S = \langle a_1, \dots, a_n \rangle$ and $\mathrm{FM}_n = \langle x_1, \dots, x_n \rangle$. We will denote by π the unique homomorphism $\pi\colon \mathrm{FM}_n \to S$ such that $\pi(x_i) = a_i$ for all $i = 1, \dots, n$. Assume that we order FM_n by the degree-lexicographic order with $x_1 < x_2 < \dots < x_n$. For $a \in S$ define its normal form by $\min(\pi^{-1}(a))$, i.e. the minimum of all words in FM_n that represent a.

For a subset Y of $\{1, 2, \dots, n\}$ define

$$S_Y = \bigcap_{i \in Y} a_i S, \qquad S'_Y = \bigcap_{i \in Y} S a_i$$

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and

$$D_Y = \{a \in S_Y \mid \text{if } a = a_i b \text{ for some } b \in S, \text{ then } i \in Y\},$$
$$D'_Y = \{a \in S'_Y \mid \text{if } a = b a_i \text{ for some } b \in S, \text{ then } i \in Y\}.$$

For $i \in \{1, 2, \dots, n\}$ define

$$S_i = \bigcup_{Y, |Y| = i} S_Y, \qquad S'_i = \bigcup_{Y, |Y| = i} S'_Y.$$

By [**7**, Theorem 9.3.7], we know that if S is right non-degenerate, then S_i is an ideal of S and

$$S_n \subseteq S_{n-1} \subseteq \cdots \subseteq S_1 \subseteq S,$$

and $S_1 \setminus S_2 = \bigcup_{i=1}^n a_i \langle a_i \rangle$. Similarly, if S is left non-degenerate, then S'_i is an ideal of S and

$$S'_n \subseteq S'_{n-1} \subseteq \cdots \subseteq S'_1 \subseteq S,$$

and $S'_1 \setminus S'_2 = \bigcup_{i=1}^n a_i \langle a_i \rangle$.

Lemma 3.3. *Assume that S is right non-degenerate. Let $a \in S \setminus \{1\}$. Then there exists a non-empty subset Y of $\{1, \dots, n\}$ such that $a \in D_Y$ and the normal form $w \in \mathrm{FM}_n$ of a is of the following form*

$$w = w_1 y_1^{q_1} w_2 y_2^{q_2} \cdots w_m y_m^{q_m}, \tag{3.1}$$

where $m, q_1, \dots, q_m$ are positive integers, $m \leqslant |Y| \leqslant n$, $\deg(w_i) \leqslant 2^n$, and $y_i \in \{x_1, \dots, x_n\}$ for all $i = 1, \dots, m$.

Proof. We shall prove the result by induction on $|Y| = k$.

If $k = 1$, then $Y = \{i\}$ for some $i \in \{1, \dots, n\}$, and $a = a_i^q$ for some positive integer q. Furthermore, $\pi^{-1}(a) = \{x_i^q\}$. Therefore, x_i^q is the normal form of a, and we get the result in this case.

Suppose that $k > 1$ and the result is true for all $b \in D_Z$, where Z is any subset of $\{1, \dots, n\}$ of cardinality less than k.

Let $w = y'_1 y'_2 \cdots y'_r$ be the normal form of a, where $y'_1, \dots, y'_r \in \{x_1, \dots, x_n\}$. Let $w'_j = y'_j y'_{j+1} \cdots y'_r$ and $b_j = \pi(w'_j)$ for $j = 1, 2, \dots, r$. It is clear that w'_j is the normal form of b_j. Let $Z_1, \dots, Z_r \subseteq \{1, \dots, n\}$ be the subsets such that $b_j \in D_{Z_j}$. Since $a = b_1$, we have $Z_1 = Y$. Since $S_n \subseteq S_{n-1} \subseteq \cdots \subseteq S_1 \subseteq S$ is a chain of ideals of S, it is easy to see that

$$k = |Y| = |Z_1| \geqslant |Z_2| \geqslant \cdots \geqslant |Z_r| = 1.$$

Let s be the greatest integer such that $s < r$ and $|Z_s| = k$. Then $b_{s+1} \in D_{Z_{s+1}}$ and $|Z_{s+1}| < k$. Therefore, by the induction hypothesis w'_{s+1} is of the form (3.1), with $m \leqslant |Z_{s+1}| < k$. If $s \leqslant 2^n$, then clearly the normal form of a is of the form (3.1). Thus, we may assume that $s > 2^n$. Since the sets Z_j are subsets of $\{1, \dots n\}$, there exist positive integers $s_1 < s_2$ such that $s_2 \leqslant s$ and $Z_{s_1} = Z_{s_2}$. Suppose that s_1 is the smallest positive integer such that there exists a positive integer s_2 satisfying the above properties. Then clearly $s_1 \leqslant 2^n$. In order to prove that a has a normal form of the form (3.1), it is sufficient to prove that

$$y'_{s_1} = y'_{s_1+1} = \cdots = y'_s.$$

Let $i_j = \min(Z_j)$. We have that $y'_1 y'_2 \cdots y'_s = x_{i_1} x_{i_2} \cdots x_{i_s}$. Recall that $a_i a_j = a_{\sigma_i(j)} a_{\gamma_j(i)}$ for all $i, j \in \{1, 2, \dots, n\}$. Since S is right non-degenerate, the maps $\sigma_i \in \mathrm{Sym}_n$. Since $|Z_1| = |Z_2| = \cdots = |Z_s|$ and $a_{i_j} b_{j+1} = \pi(y'_j) b_{j+1}$ with $b_j \in D_{Z_j}$, the restriction of σ_{i_j} to Z_{j+1} is a bijection from Z_{j+1} to Z_j for all $j = 1, 2, \dots, s-1$. Furthermore, since $i_j \in Z_j$ and $\sigma_{i_j}(i_j) = i_j$, we have that $i_j \in Z_{j+1}$. Therefore,

$$i_1 \geqslant i_2 \geqslant \cdots \geqslant i_s.$$

Since $Z_{s_1} = Z_{s_2}$, we have that $i_{s_1} = i_{s_1+1} = \cdots = i_{s_2}$. We have that the maps

$$\varphi_1 \colon Z_{i_{s_1}+1} \to Z_{i_{s_1}}, \dots, \varphi_{s_2 - s_1} \colon Z_{s_2} \to Z_{s_2-1},$$

defined by $\varphi_j(i) = \sigma_{i_{s_1}}(i)$ for all $i \in Z_{s_1+j}$ and all $1 \leqslant j \leqslant s_2 - s_1$, are bijections. Since $Z_{s_1} = Z_{s_2}$, the bijection from Z_{s_2+1} to Z_{s_2} given by the restriction of $\sigma_{i_{s_2}}$ to Z_{s_2+1} is the bijection $\varphi_1 \colon Z_{i_{s_1}+1} \to Z_{i_{s_1}}$. Hence, $Z_{s_2+1} = Z_{s_1+1}$ and thus $i_{s_2+1} = i_{s_1+1} = i_{s_1}$. An easy inductive argument shows that $Z_j = Z_{j-s_2+s_1}$ for all $s_2 < j \leqslant s$. Therefore, $i_{s_1} = i_{s_1+1} = \cdots = i_s$. Hence,

$$y'_{s_1} = y'_{s_1+1} = \cdots = y'_s,$$

and thus a has a normal form of the form (3.1). Therefore, the result follows by induction. □

Lemma 3.4. *Assume that S is right non-degenerate. Let $w_1, w_2, \dots, w_m \in \mathrm{FM}_n$, $y_1, y_2, \dots, y_m \in X$ and let N be a positive integer. Then there exist positive integers $N_1, N_2, \dots, N_m$, all divisible by N, such that for every positive integer $i < m$ and for all $0 \leqslant r_j < N_j$, with $i < j \leqslant m$, we have*

$$\pi(y_i^{N_i} w_{i+1} y_{i+1}^{r_{i+1}} w_{i+2} y_{i+2}^{r_{i+2}} \cdots w_m y_m^{r_m}) = \pi(w_{i+1} y_{i+1}^{r_{i+1}} w_{i+2} y_{i+2}^{r_{i+2}} \cdots w_m y_m^{r_m}) a^N \tag{3.2}$$

for some $a \in S$.

Proof. Let l be a positive integer such that $\deg(w_1), \dots, \deg(w_m) < l$. We choose $N_m = N$. We shall prove by induction on $m - i$ that if $N_{i+1}, \dots, N_m$ are chosen, then there exists a positive integer N_i satisfying the statement of the result.

Let $i < m$ and suppose that $N_{i+1}, \dots, N_m$ are chosen. Let $0 \leqslant r_j < N_j$, with $i < j \leqslant m$. We have that

$$\deg(w_{i+1} y_{i+1}^{r_{i+1}} w_{i+2} y_{i+2}^{r_{i+2}} \cdots w_m y_m^{r_m}) \leqslant (m-i)l + N_{i+1} + \cdots + N_m.$$

Let $N_i = N \cdot (n!)^{(m-i)l + N_{i+1} + \cdots + N_m}$. By Lemma 2.1,

$$\pi(y_i^{N_i} w_{i+1} y_{i+1}^{r_{i+1}} w_{i+2} y_{i+2}^{r_{i+2}} \cdots w_m y_m^{r_m}) = \pi(w_{i+1} y_{i+1}^{r_{i+1}} w_{i+2} y_{i+2}^{r_{i+2}} \cdots w_m y_m^{r_m}) a^N$$

for some $a \in S$. Thus, the result follows by induction. □

We denote by $N(S)$ the subset of FM_n of all the normal forms of elements of the monoid of skew type S.

Theorem 3.5. *If S is a non-degenerate monoid of skew type, then $N(S)$ is a regular language.*

Proof. Note that the subset

$$T = \{w_1 y_1^{q_1} \cdots w_m y_m^{q_m} \in \mathrm{FM}_n \mid m \leqslant n,\ y_j \in X,\ q_j \geqslant 0,\ \deg(w_j) \leqslant 2^n,\ \forall 1 \leqslant j \leqslant m\}$$

is a regular language. By Lemma 3.3, we have that $N(S) \subseteq T$.

Let m be a positive integer such that $m \leqslant n$. Fix $y_1, \ldots, y_m \in X$ and $w_1, \ldots, w_m \in \mathrm{FM}_n$ such that $\deg(w_1), \ldots, \deg(w_m) \leqslant 2^n$. Consider the following subset of T:

$$T(y_1, \ldots y_m, w_1, \ldots, w_m) = w_1 y_1^* \cdots w_m y_m^*.$$

Since T is a finite union of subsets of the above form, in order to prove the result it is sufficient to show that every $N(S) \cap T(y_1, \ldots, y_m, w_1, \ldots, w_m)$ is a regular language.

By Theorem 2.2, there exists a positive integer N such that

$$a^N b^N = b^N a^N \quad \text{for every } a, b \in S. \tag{3.3}$$

By Lemma 3.4, there exist positive integers $N_1, N_2, \ldots, N_m$, multiples of N, such that, for every positive integer $i < m$ and for all $0 \leqslant r_j < N_j$ with $i < j \leqslant m$, equality (3.2) holds for some $a \in S$. Since S also is left non-degenerate, by the dual of Lemma 3.4, there exist positive integers $N'_1, N'_2, \ldots, N'_m$, multiples of N, such that, for every positive integer $i \leqslant m$ and for all $0 \leqslant r_j < N'_j$ with $1 \leqslant j \leqslant i$, we have

$$\pi(w_1 y_1^{r_1} w_2 y_2^{r_2} \cdots w_i y_i^{r_i} y_i^{N'_i}) = a^N \pi(w_1 y_1^{r_1} w_2 y_2^{r_2} \cdots w_i y_i^{r_i}) \tag{3.4}$$

for some $a \in S$.

Let $M_i = N_i N'_i$ for $i = 1, \ldots, m$. Define for $(r_1, \ldots, r_m)$, such that $0 \leqslant r_j < M_j$, the subset of $T(y_1, \ldots, y_m, w_1, \ldots, w_m)$,

$$T_{(r_1,\ldots,r_m)} = w_1 y_1^{r_1} (y_1^{M_1})^* \cdots w_m y_m^{r_i} (y_m^{M_m})^*.$$

Since $T(y_1, \ldots, y_m, w_1, \ldots, w_m)$ is a finite union of subsets of the above form, in order to prove the result it is sufficient to show that $N(S) \cap T_{(r_1,\ldots,r_m)}$ is a regular language.

Claim 3.6. *The set*

$$I = \{(t_1, \ldots, t_m) \in \mathbb{N}^m \mid w_1 y_1^{r_1} y_1^{M_1 t_1} \cdots w_m y_m^{r_m} y_m^{M_m t_m} \notin N(S)\}$$

is an ideal of the additive monoid $\mathbb{N}^m$.

Proof of Claim 3.6. Let $(t_1, \ldots, t_m) \in I$. In order to prove the claim, we shall show that $(t_1, \ldots, t_{j-1}, t_j + 1, t_{j+1}, \ldots, t_m) \in I$ for every $1 \leqslant j \leqslant m$. Let

$$z = w_1 y_1^{r_1} y_1^{M_1 t_1} \cdots w_m y_m^{r_m} y_m^{M_m t_m}. \tag{3.5}$$

Also define

$$z_1 = w_1 y_1^{r_1} y_1^{M_1 t_1} \cdots w_{j-1} y_{j-1}^{r_{j-1}} y_{j-1}^{M_{j-1} t_{j-1}} w_j y_j^{r_j} \tag{3.6}$$

and

$$z_2 = w_{j+1} y_{j+1}^{r_{j+1}} y_{j+1}^{M_{j+1} t_{j+1}} \cdots w_m y_m^{r_m} y_m^{M_m t_m}, \tag{3.7}$$

so that $z = z_1 y_j^{M_j t_j} z_2$. Let

$$z' = z_1 y_j^{M_j (t_j+1)} z_2. \tag{3.8}$$

Let $w = uv \in \mathrm{FM}_n$ be the normal form of $\pi(z)$, with $\deg(u) = \deg(z_1)$ and $\deg(v) = M_j t_j + \deg(z_2)$. Note that $\deg(z) = \deg(w)$ and, since $(t_1, \dots, t_m) \in I$, it follows that $w = uv < z$.

Case 1 ($u < z_1$). Let r_i', t_i' be non-negative integers such that $r_i = r_i' + N_i t_i'$ and $0 \leqslant r_i' < N_i$, for $i = 1, \dots, m$. By Lemma 3.4 we get that there exist $b_j, b_{j+1}, \dots, b_{m-1} \in S$ such that

$$\pi(y_i^{N_i} w_{i+1} y_{i+1}^{r_{i+1}'} \cdots w_m y_m^{r_m'}) = \pi(w_{i+1} y_{i+1}^{r_{i+1}'} \cdots w_m y_m^{r_m'}) b_i^N \tag{3.9}$$

for all $i = j, j+1, \dots, m-1$. Let $b_m = \pi(y_m)$. Hence, first using (3.7) and the fact that $M_i = N_i N_i'$ and then applying (3.9) several times, we get

$$\begin{aligned}
\pi(y_j^{M_j} z_2) &= \pi(y_j^{M_j} w_{j+1} y_{j+1}^{r_{j+1}} y_{j+1}^{M_{j+1} t_{j+1}} \cdots w_m y_m^{r_m} y_m^{M_m t_m}) \\
&= \pi(y_j^{N_j N_j'} w_{j+1} y_{j+1}^{r_{j+1}'} y_{j+1}^{N_{j+1}(t_{j+1}' + N_{j+1}' t_{j+1})} \cdots w_m y_m^{r_m'} y_m^{N_m (t_m' + N_m' t_m)}) \\
&= \pi(y_j^{N_j N_j'} w_{j+1} y_{j+1}^{r_{j+1}'} y_{j+1}^{N_{j+1}(t_{j+1}' + N_{j+1}' t_{j+1})} \cdots w_m y_m^{r_m'}) b_m^{N_m (t_m' + N_m' t_m)} \\
&= \pi(y_j^{N_j N_j'} w_{j+1} y_{j+1}^{r_{j+1}'} y_{j+1}^{N_{j+1}(t_{j+1}' + N_{j+1}' t_{j+1})} \cdots w_{m-2} y_{m-2}^{r_{m-2}'} y_{m-2}^{N_{m-2}(t_{m-2}' + N_{m-2}' t_{m-2})} \\
&\qquad \times w_{m-1} y_{m-1}^{r_{m-1}'} w_m y_m^{r_m'}) b_{m-1}^{N_{m-1}(t_{m-1}' + N_{m-1}' t_{m-1})} b_m^{N_m (t_m' + N_m' t_m)} \\
&\;\;\vdots \\
&= \pi(w_{j+1} y_{j+1}^{r_{j+1}'} \cdots w_m y_m^{r_m'}) b_j^{N N_j'} b_{j+1}^{N(t_{j+1}' + N_{j+1}' t_{j+1})} \cdots b_m^{N(t_m' + N_m' t_m)}.
\end{aligned} \tag{3.10}$$

Therefore,

$$\begin{aligned}
&\pi(z') \\
&\quad = \pi(z_1 y_j^{M_j (t_j+1)} z_2) \quad \text{by (3.8)} \\
&\quad = \pi(z_1 y_j^{M_j t_j} w_{j+1} y_{j+1}^{r_{j+1}'} \cdots w_m y_m^{r_m'}) b_j^{N N_j'} b_{j+1}^{N(t_{j+1}' + N_{j+1}' t_{j+1})} \cdots b_m^{N(t_m' + N_m' t_m)} \quad \text{by (3.10)} \\
&\quad = \pi(z_1 y_j^{M_j t_j} w_{j+1} y_{j+1}^{r_{j+1}'} \cdots w_m y_m^{r_m'}) b_{j+1}^{N(t_{j+1}' + N_{j+1}' t_{j+1})} \cdots b_m^{N(t_m' + N_m' t_m)} b_j^{N N_j'} \quad \text{by (3.3)} \\
&\quad = \pi(z_1 y_j^{M_j t_j} w_{j+1} y_{j+1}^{r_{j+1}} y_{j+1}^{M_{j+1} t_{j+1}} \cdots w_m y_m^{r_m} y_m^{M_m t_m}) b_j^{N N_j'} \\
&\quad = \pi(z) b_j^{N N_j'} \quad \text{by (3.5) and (3.6)} \\
&\quad = \pi(uv) b_j^{N N_j'}
\end{aligned}$$

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(where the fourth equality follows as in (3.10), by applying (3.9) several times in the reverse order). Since $u < z_1$, we have that the normal form of $\pi(z')$ is not z'. Hence, $z' \notin N(S)$ and thus $(t_1, \dots, t_{j-1}, t_j + 1, t_{j+1}, \dots, t_m) \in I$ in this case.

Case 2 ($u = z_1$). In this case, since $uv < z$, we have that $v < y_j^{M_j t_j} z_2$. Let r'_i, t'_i be non-negative integers such that $r_i = r'_i + N'_i t'_i$ and $0 \leqslant r'_i < N'_i$, for $i = 1, \dots, m$. By (3.4), there exist $d_1, d_2, \dots, d_j \in S$ such that

$$\pi(w_1 y_1^{r'_1} \cdots w_{i-1} y_{i-1}^{r'_{i-1}} w_i y_i^{N'_i}) = d_i^N \pi(w_1 y_1^{r'_1} \cdots w_{i-1} y_{i-1}^{r'_{i-1}} w_i) \tag{3.11}$$

for all $i = 1, 2, \dots, j$. Hence, in view of (3.6) and using (3.11) several times, we get

$$\begin{aligned}
\pi(z_1 y_j^{M_j}) &= \pi(u y_j^{M_j}) \\
&= \pi(w_1 y_1^{r_1} y_1^{M_1 t_1} \cdots w_{j-1} y_{j-1}^{r_{j-1}} y_{j-1}^{M_{j-1} t_{j-1}} w_j y_j^{M_j} y_j^{r_j}) \\
&= \pi(w_1 y_1^{r'_1} y_1^{N'_1(t'_1 + N_1 t_1)} \cdots w_{j-1} y_{j-1}^{r'_{j-1}} y_{j-1}^{N'_{j-1}(t'_{j-1} + N_{j-1} t_{j-1})} w_j y_j^{M_j} y_j^{r'_j} y_j^{N'_j t'_j}) \\
&= d_1^{N(t'_1 + N_1 t_1)} \pi(w_1 y_1^{r'_1} w_2 y_2^{r'_2} y_2^{N'_2(t'_2 + N_2 t_2)} \cdots w_{j-1} y_{j-1}^{r'_{j-1}} y_{j-1}^{N'_{j-1}(t'_{j-1} + N_{j-1} t_{j-1})} \\
&\qquad\qquad \times w_j y_j^{M_j} y_j^{r'_j} y_j^{N'_j t'_j}) \\
&\;\;\vdots \\
&= d_1^{N(t'_1 + N_1 t_1)} \cdots d_{j-1}^{N(t'_{j-1} + N_{j-1} t_{j-1})} d_j^{N N_j} \pi(w_1 y_1^{r'_1} \cdots w_{j-1} y_{j-1}^{r'_{j-1}} w_j y_j^{r_j}).
\end{aligned} \tag{3.12}$$

Therefore,

$$\begin{aligned}
\pi(z') &= \pi(z_1 y_j^{M_j(t_j+1)} z_2) \quad \text{by (3.8)} \\
&= d_1^{N(t'_1 + N_1 t_1)} \cdots d_{j-1}^{N(t'_{j-1} + N_{j-1} t_{j-1})} d_j^{N N_j} \\
&\qquad \times \pi(w_1 y_1^{r'_1} \cdots w_{j-1} y_{j-1}^{r'_{j-1}} w_j y_j^{r_j} y_j^{M_j t_j} z_2) \quad \text{by (3.12)} \\
&= d_j^{N N_j} d_1^{N(t'_1 + N_1 t_1)} \cdots d_{j-1}^{N(t'_{j-1} + N_{j-1} t_{j-1})} \\
&\qquad \times \pi(w_1 y_1^{r'_1} \cdots w_{j-1} y_{j-1}^{r'_{j-1}} w_j y_j^{r_j} y_j^{M_j t_j} z_2) \quad \text{by (3.3)} \\
&= d_j^{N N_j} \pi(w_1 y_1^{r_1} y_1^{M_1 t_1} \cdots w_{j-1} y_{j-1}^{r_{j-1}} y_{j-1}^{M_{j-1} t_{j-1}} w_j y_j^{r_j} y_j^{M_j t_j} z_2) \\
&= d_j^{N N_j} \pi(z) = d_j^{N N_j} \pi(uv) \quad \text{by (3.5)} \\
&= d_j^{N N_j} \pi(w_1 y_1^{r_1} y_1^{M_1 t_1} \cdots w_{j-1} y_{j-1}^{r_{j-1}} y_{j-1}^{M_{j-1} t_{j-1}} w_j y_j^{r_j} v) \quad \text{by (3.6)} \\
&= d_j^{N N_j} d_1^{N(t'_1 + N_1 t_1)} \cdots d_{j-1}^{N(t'_{j-1} + N_{j-1} t_{j-1})} \pi(w_1 y_1^{r'_1} \cdots w_{j-1} y_{j-1}^{r'_{j-1}} w_j y_j^{r_j} v) \\
&= d_1^{N(t'_1 + N_1 t_1)} \cdots d_{j-1}^{N(t'_{j-1} + N_{j-1} t_{j-1})} d_j^{N N_j} \pi(w_1 y_1^{r'_1} \cdots w_{j-1} y_{j-1}^{r'_{j-1}} w_j y_j^{r_j} v) \quad \text{by (3.3)} \\
&= \pi(u y_j^{M_j} v)
\end{aligned}$$

(the fourth, eighth and last equalities follow as in (3.12) by applying (3.11) several times). Since $v < y_j^{M_j t_j} z_2$, we know that the normal form of $\pi(z')$ is not z'. Hence, $z' \notin N(S)$ and thus $(t_1, \dots, t_{j-1}, t_j + 1, t_{j+1}, \dots, t_m) \in I$ in this case.

Therefore, I is an ideal of $\mathbb{N}^m$, and the claim follows. □

It is well known that ideals of $\mathbb{N}^m$ are finitely generated [**6**, Theorems 5.1 and 7.8]. Therefore, there exist $(t_{1,1},\dots,t_{1,m}),\dots,(t_{s,1},\dots,t_{s,m})\in I$ such that

$$I=\bigcup_{i=1}^{s}((t_{i,1},\dots,t_{i,m})+\mathbb{N}^m).$$

Therefore,

$$T_{(r_1,\dots,r_m)}\setminus N(S)=\bigcup_{i=1}^{s} w_1 y_1^{r_1} y_1^{M_1 t_{i,1}}(y_1^{M_1})^*\cdots w_m y_m^{r_m} y_m^{M_m t_{i,m}}(y_m^{M_m})^*$$

is a regular language. Note that if L is a regular language in FM_n, then $\mathrm{FM}_n\setminus L$ is a regular language [**8**, Chapter 6]. In particular, since $T_{(r_1,\dots,r_m)}$ is a regular language, $\mathrm{FM}_n\setminus T_{(r_1,\dots,r_m)}$ is also a regular language. Since $T_{(r_1,\dots,r_m)}\setminus N(S)$ is a regular language, it follows that

$$\mathrm{FM}_n\setminus(T_{(r_1,\dots,r_m)}\cap N(S))=(\mathrm{FM}_n\setminus T_{(r_1,\dots,r_m)})\cup(T_{(r_1,\dots,r_m)}\setminus N(S))$$

is a regular language. Therefore, $N(S)\cap T_{(r_1,\dots,r_m)}$ is a regular language. Hence, $N(S)$ is a regular language and the result is proved. □

Remark 3.7. If S is a cancellative right non-degenerate monoid of skew type, then $N(S)$ is a regular language. This is proved via a modification of the argument used in the proof of Theorem 3.5. Namely, we can use Lemma 2.3 and, since the group of quotients of S is abelian-by-finite, assertion (iv) of Theorem 2.2 also holds. Then cancellativity of S can now be used in the part of the proof that originally required the left non-degenerate assumption. Namely, case 2 of the proof of Claim 3.6 can be proved easily using cancellativity.

Corollary 3.8. *If S is a non-degenerate monoid of skew type, then there exists $N\geqslant 1$ such that the monoid $A=\langle s^N\mid s\in S\rangle$ is commutative and finitely generated and*

$$S=\bigcup_{f\in F} fA=\bigcup_{f\in F} Af$$

for a finite set $F\subseteq S$. Moreover, A is a disjoint union of cancellative subsemigroups of S.

Proof. We use the notation of the proof of Theorem 3.5. Let $A=\langle s^N\mid s\in S\rangle$. Let m be a positive integer such that $m\leqslant n$. For $y_1,\dots,y_m\in X$ and $w_1,\dots,w_m\in\mathrm{FM}_n$ such that $\deg(w_1),\dots,\deg(w_m)\leqslant 2^n$, define

$$F(y_1,\dots,y_m,w_1,\dots,w_m)=\{\pi(w_1y_1^{r_1}\cdots w_my_m^{r_m})\mid 0\leqslant r_j<M_j\},$$

where the integers M_j are defined as in the proof of Theorem 3.5. Define

$$F=\bigcup_{m=1}^{n}\left(\bigcup_{y_1,\dots,y_m\in X}\left(\bigcup_{\substack{w_1,\dots,w_m\in\mathrm{FM}_n,\\ \deg(w_j)\leqslant 2^n}} F(y_1,\dots,y_m,w_1,\dots,w_m)\right)\right).$$

Clearly, $1 = \pi(x_1^0) \in F(x_1, 1) \subseteq F$. From the proof of Theorem 3.5 it follows that

$$S = \bigcup_{f \in F} fA = \bigcup_{f \in F} Af.$$

Hence, $K[S]$ is a finitely generated (right and left) module over the commutative subalgebra $K[A]$. For every $f_i, f_j \in F$ we choose $a_{i,j} \in A$, $f_{i,j} \in F$ such that $f_i f_j = f_{i,j} a_{i,j}$. Let $C = \langle a_{i,j} \mid f_i, f_j \in F \rangle$. Note that $a_k = \pi(x_k) \in F$ for all $k = 1, \dots, n$. We shall see that $S = \bigcup_{f \in F} fC$.

Suppose that $S \neq \bigcup_{f \in F} fC$. Let $s \in S \setminus (\bigcup_{f \in F} fC)$ be of minimal degree. Since $a_k \in F$ for all $k = 1, \dots, n$, we have that $\deg(s) > 1$. Hence, there exist $s' \in S$ and $1 \leqslant k \leqslant n$ such that $s = a_k s'$. Since $\deg(s') < \deg(s)$, $s' \in \bigcup_{f \in F} fC$. Thus, there exist $c \in C$ and $f' \in F$ such that $s' = f'c$. Then

$$s = a_k f' c \in \bigcup_{f \in F} fC,$$

because $a_k \in F$ and all $a_{i,j} \in C$, which is a contradiction. Therefore, $S = \bigcup_{f \in F} fC$.

Then $K[S]$ is a finitely generated right module over $K[C]$. Clearly, $K[C]$ is a commutative Noetherian algebra. Then $K[S]$ is a Noetherian $K[C]$-module; hence, its submodule $K[A]$ also is a Noetherian $K[C]$-module. Then $K[A]$ is a Noetherian algebra, so from [**7**, Theorem 5.1.5] we know that A is a finitely generated monoid. The proof of Theorem 3.5 also shows that A is a disjoint union of cancellative semigroups, because the integer N is chosen as in the proof of Theorem 2.2, implying that every s^N lies in a maximal subgroup of the corresponding monoid $M_m(L)$. □

The above is a natural extension of the results known earlier in the special case of monoids S satisfying the cyclic condition [**7**, Proposition 9.4.4], and hence, in particular, the results in the case of monoids of I-type with square-free defining relations [**7**, Chapter 8].

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References

1. A. Z. Anan′in, An intriguing story about representable algebras, in *Ring theory*, pp. 31–38 (Weizmann Science Press, Jerusalem, 1989).
2. J. P. Bell and P. Colak, Primitivity of finitely presented monomial algebras, *J. Pure Appl. Alg.* **213** (2009), 1299–1305.
3. T. Gateva-Ivanova, Skew polynomial rings with binomial relations, *J. Alg.* **185** (1996), 710–753.
4. T. Gateva-Ivanova, Quadratic algebras, Yang–Baxter equation, and Artin–Schelter regularity, preprint (arXiv:1011.6520).
5. T. Gateva-Ivanova and M. Van den Bergh, Semigroups of I-type, *J. Alg.* **206** (1998), 97–112.
6. R. Gilmer, *Commutative semigroup rings* (University of Chicago Press, 1984).

7. E. Jespers and J. Okniński, *Noetherian semigroup algebras* (Springer, 2007).
8. G. Lallement, *Semigroups and combinatorial applications* (Wiley, 1979).
9. J. Månsson and P. Nordbeck, Regular Gröbner bases, *J. Symb. Computat.* **33** (2002), 163–281.
10. J. Månsson and P. Nordbeck, A generalized Ufnarovski graph, in *Applied algebra, algebraic algorithms and error-correcting codes*, Lecture Notes in Computer Science, Volume 3857, pp. 293–306 (Springer, 2005).
11. J. Okniński, *Semigroup algebras* (Marcel Dekker, New York, 1991).
12. J. Okniński, *Semigroups of matrices* (World Scientific, Singapore, 1998).
13. J. Okniński, Structure of prime finitely presented monomial algebras, *J. Alg.* **320** (2008), 3199–3205.
14. D. S. Passman, *The algebraic structure of group rings* (Wiley-Interscience, New York, 1977).
15. W. Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang–Baxter equation, *Adv. Math.* **193** (2005), 40–55.
16. V. A. Ufnarovskii, Combinatorial and asymptotic methods in algebra, in *Encyclopedia of Mathematical Sciences*, Volume 57, pp. 1–196 (Springer, 1995).