Glasgow Math. J. **52A** (2010) 61–67. © Glasgow Mathematical Journal Trust 2010.
doi:10.1017/S0017089510000169.

ON THE OSOFSKY–SMITH THEOREM*

SEPTIMIU CRIVEI
Faculty of Mathematics and Computer Science, "Babeş-Bolyai" University,
Str. M. Kogălniceanu 1, 400084 Cluj-Napoca, Romania
e-mail: crivei@math.ubbcluj.ro

CONSTANTIN NĂSTĂSESCU
Faculty of Mathematics and Computer Science, University of Bucharest,
Str. Academiei 14, 010014 Bucharest, Romania
e-mail: cnastase@al.math.unibuc.ro

and BLAS TORRECILLAS
Departamento de Álgebra y Análisis, Universidad de Almería, 04071 Almería, Spain
e-mail: btorreci@ual.es

Abstract. We recall a version of the Osofsky–Smith theorem in the context of a Grothendieck category and derive several consequences of this result. For example, it is deduced that every locally finitely generated Grothendieck category with a family of completely injective finitely generated generators is semi-simple. We also discuss the torsion-theoretic version of the classical Osofsky theorem which characterizes semi-simple rings as those rings whose every cyclic module is injective.

2002 *Mathematics Subject Classification.* 16D50, 16S90.

1. Introduction. In the late 1960s, Osofsky showed her classical result which asserts that a ring is semi-simple if and only if every cyclic module is injective [**8**, Theorem], [**9**, Corollary]. Among the categorical generalizations of the Osofsky theorem, we mention the version established by Gómez Pardo et al. [**5**]. They showed that if $\mathcal{C}$ is a locally finitely generated Grothendieck category and M is a finitely presented object of $\mathcal{C}$ which is completely (pure-)injective and has a von Neumann regular endomorphism ring S, then S is a semi-simple ring [**5**, Theorem 1]. In the early 1990s, Osofsky and Smith established a module counterpart of the original Osofsky theorem. They proved that if M is a cyclic module with the property that every cyclic submodule of M is completely extending, then M is a finite direct sum of uniform modules [**10**]. As a consequence, if M is a module with every quotient of a cyclic submodule injective, then M is semi-simple. In the same paper, Osofsky and Smith noted that their result still holds in a more general categorical setting.

The purpose of this paper is to discuss some categorical version of the Osofsky–Smith theorem and give several applications. We first consider the setting of a locally finitely generated Grothendieck category $\mathcal{C}$ and deduce that if $\mathcal{C}$ has a family of completely injective finitely generated generators, then $\mathcal{C}$ is semi-simple. As an application, we give a positive partial answer to the following question raised by

*To Professor Patrick F. Smith on the occasion of his 65th birthday.

M. Teply: Does the torsion-theoretic version of the Osofsky theorem hold? In other words, if τ is a hereditary torsion theory such that every cyclic module is τ-injective, does it follow that every module is τ-injective? Finally, we show that a ring is semisimple if and only if every cyclic module is τ-injective τ-complemented.

2. Locally finitely generated Grothendieck categories.

DEFINITION 2.1. Let $\mathcal{C}$ be a Grothendieck category. Then an object C of $\mathcal{C}$ is called *completely injective* if for every object M of $\mathcal{C}$ and every morphism $f : C \to M$, $\mathrm{Im}(f)$ is an injective object.

REMARK. As an immediate consequence of the existence of an injective hull for every object in $\mathcal{C}$, an object C of $\mathcal{C}$ is completely injective if and only if for every injective object M of $\mathcal{C}$ and every morphism $f : C \to M$, $\mathrm{Im}(f)$ is an injective object.

We begin with a property that will be needed later.

PROPOSITION 2.2. *Let $\mathcal{C}$ be a Grothendieck category and $(U_i)_{i\in I}$ a family of completely injective objects of $\mathcal{C}$. Then every finite direct sum of U_i's is completely injective.*

Proof. Consider a finite direct sum of U_i's, say $U_1 \oplus \cdots \oplus U_n$, and let $f : U_1 \oplus \cdots \oplus U_n \to M$ be a morphism in $\mathcal{C}$. We show that $\mathrm{Im}(f)$ is an injective object. We prove it for $n = 2$, the general case that follows by induction. Let $f : U_1 \oplus U_2 \to M$ be a morphism in $\mathcal{C}$. Denote by $i_1 : U_1 \to U_1 \oplus U_2$ and $i_2 : U_2 \to U_1 \oplus U_2$ the inclusion morphisms. Also, put $f_1 = f \circ i_1$ and $f_2 = f \circ i_2$. Then it is easy to see that $\mathrm{Im}(f) = \mathrm{Im}(f_1) + \mathrm{Im}(f_2)$. Let $X = \mathrm{Im}(f_1)$, $Y = \mathrm{Im}(f_2)$, and let $g : U_1 \to X/(X \cap Y)$ be the composition of the natural epimorphisms $U_1 \to X$ and $X \to X/(X \cap Y)$. Then $(X + Y)/Y \cong X/(X \cap Y) \cong \mathrm{Im}(g)$ is an injective object by hypothesis. But Y is also injective, and so $\mathrm{Im}(f) = X + Y$ is an injective object. □

Recall that a Grothendieck category $\mathcal{C}$ is called *locally finitely generated* if it has a family of finitely generated generators [**12**].

COROLLARY 2.3. *Let $\mathcal{C}$ be a locally finitely generated Grothendieck category with a family of completely injective finitely generated generators. Then every finitely generated object in $\mathcal{C}$ is injective.*

EXAMPLE 2.4. The conclusion of Proposition 2.2 does not hold for an infinite family. Indeed, let us consider an infinite family of fields $(K_i)_{i\in I}$ and let $R = \prod_{i\in I} K_i$. Then R is a commutative von Neumann regular ring, that is, a V-ring, and so every simple R-module is injective. Now let $(e_i)_{i\in I}$ be the family of primitive orthogonal idempotents in R. Clearly, each $S_i = Re_i$ is a simple R-module, and so injective. Then each S_i is actually completely injective. Also, we have $\bigoplus_{i\in I} S_i = \mathrm{Soc}(R)$. Clearly, $\bigoplus_{i\in I} S_i$ is not injective, because otherwise this would imply that $R = \mathrm{Soc}(R)$. Now if we take $M = \bigoplus_{i\in I} S_i$ and f to be the identity homomorphism, it follows that $C = M$ is not completely injective.

EXAMPLE 2.5. If R is a right hereditary ring, then it is clear that the class of completely injective objects in the category Mod-R of right R-modules coincides with the class of injective objects in Mod-R.

In order to be able to state the Osofsky–Smith theorem, we need the definition of an extending object in a Grothendieck category, which is the same as for modules.

Definition 2.6. Let $\mathcal{C}$ be a Grothendieck category. An object M of $\mathcal{C}$ is called *extending* if every subobject of M is essential in a direct summand of M. Equivalently, M is extending if and only if every essentially closed subobject of M is a direct summand of M.

An object M of $\mathcal{C}$ is called *completely extending* if for every object M of $\mathcal{C}$ and every morphism $f : C \to M$, $\mathrm{Im}(f)$ is an extending object.

Let $\mathcal{C}$ be a Grothendieck category. For a class $\mathcal{P}$ of objects of $\mathcal{C}$, by a $\mathcal{P}$-subobject we mean a subobject belonging to $\mathcal{P}$. Let $\mathcal{P}$ be a class of finitely generated objects in $\mathcal{C}$ with the following properties:

(P_1) $\mathcal{P}$ is closed under quotients.

(P_2) If $X \in \mathcal{P}$ and Y is a $\mathcal{P}$-subobject of a quotient object of X, then there is a $\mathcal{P}$-subobject Z of X that projects onto Y.

Some examples of such classes $\mathcal{P}$ in $\mathcal{C}$ are the following: the class of all finitely generated objects, the class of finitely generated semi-simple objects and any class of finitely generated objects closed under subobjects and quotients.

Now basically the same proof of the basic theorem for modules (see [**7**] or [**10**]) works in our categorical context. This has also been noted in the original paper of Osofsky and Smith [**10**].

Theorem 2.7. *Let $\mathcal{C}$ be a Grothendieck category. Let $\mathcal{P}$ be a class of finitely generated objects in $\mathcal{C}$ satisfying (P_1) and (P_2) and let $M \in \mathcal{P}$ be such that every $\mathcal{P}$-subobject of M is completely extending. Then M is a finite direct sum of uniform objects.*

The next two corollaries are obtained as [**10**, Corollaries 1 and 2].

Corollary 2.8. *Let $\mathcal{C}$ be a Grothendieck category such that every finitely generated object is extending. Then every finitely generated object is a finite direct sum of uniform objects.*

Corollary 2.9. *Let $\mathcal{C}$ be a Grothendieck category. Let M be an object of $\mathcal{C}$ such that every quotient of every finitely generated subobject of M is injective. Then M is semi-simple.*

Recall that a Grothendieck category $\mathcal{C}$ is called *semi-simple* if every object of $\mathcal{C}$ is semi-simple [**12**]. Now Corollaries 2.3 and 2.9 yield the Osofsky–Smith theorem in locally finitely generated Grothendieck categories, stated as follows.

Theorem 2.10. *Let $\mathcal{C}$ be a locally finitely generated Grothendieck category with a family of completely injective finitely generated generators. Then $\mathcal{C}$ is semi-simple.*

By Corollary 2.3, the property of complete injectivity of the finitely generated generators of a locally finitely generated Grothendieck category passes to each finitely generated object. Now we immediately have the following consequences of Theorem 2.10.

Corollary 2.11 [**8**, Theorem]. *Let R be a ring with identity such that every cyclic (finitely generated) module is injective. Then R is semi-simple.*

Corollary 2.12 [**3**, Corollary 7.14]. *Let R be a ring with identity, M a module and $\sigma[M]$ the category of M-subgenerated modules. Suppose that every cyclic (finitely generated) module in $\sigma[M]$ is M-injective. Then M is semi-simple.*

COROLLARY 2.13. *Let R be a ring with enough idempotents such that every cyclic (finitely generated) module is injective. Then R is semi-simple.*

Recall that a Grothendieck category $\mathcal{C}$ is called *spectral* if every object of $\mathcal{C}$ is injective. It is well known that $\mathcal{C}$ is semi-simple if and only if it is locally finitely generated and spectral [**12**]. This suggests us to raise the following natural question, whose positive answer would generalize the Osofsky–Smith theorem 2.10.

QUESTION 1. *If $\mathcal{C}$ is a Grothendieck category with a family of completely injective generators, does it follow that $\mathcal{C}$ is spectral?*

3. Applications to torsion theories. Throughout this section, R is a ring with identity, all modules are unitary right R-modules and M is a module. Also, Mod-R denotes the category of unitary right R-modules, $\sigma[M]$ denotes the full subcategory of Mod-R consisting of M-subgenerated modules and $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory in Mod-R. Recall that a submodule B of a module A is called τ-*dense* (respectively τ-*closed*) in A if A/B is τ-torsion (respectively τ-torsion free). Also, a module M is called τ-*injective* if for every module B and every τ-dense submodule A of B, every homomorphism $A \to M$ extends to a homomorphism $B \to M$. For further background on torsion theories the reader is referred to [**4**] or [**12**].

Now we have the following consequence of the categorical Osofsky–Smith theorem for torsion theories.

COROLLARY 3.1. *Suppose that every cyclic τ-torsion module is τ-injective. Then every τ-torsion module is τ-injective.*

Proof. Note that $\mathcal{T}$ is generated by the modules of the form R/I for the τ-dense right ideals I of R. Each factor of such an R/I is cyclic τ-torsion, and hence, τ-torsion τ-injective by hypothesis, and so injective in $\mathcal{T}$. Thus, each such generator R/I is completely injective in $\mathcal{T}$. Now by Theorem 2.10, $\mathcal{T}$ is semi-simple, and so spectral. Then every τ-torsion module is injective in $\mathcal{T}$, that is, every τ-torsion module is τ-injective. □

A related question is the following one, which was raised by M. Teply:

QUESTION 2. *If every cyclic module is τ-injective, does it follow that every module is τ-injective?*

REMARK. Note that, by Corollary 3.1, if every cyclic τ-torsion module is τ-injective, then every τ-torsion module is τ-injective, and so every τ-torsion module is semi-simple by [**4**, Proposition 8.15]. Hence, Question 2 reduces to the case of a specialization of the Dickson torsion theory [**2**]. Recall that the Dickson torsion theory is the hereditary torsion theory generated by all simple modules. Its torsion class consists of all semi-artinian modules, whereas its torsion-free class consists of all modules with zero socle.

In the following we shall obtain a positive answer in case τ is of finite type. Recall that a torsion theory is called *of finite type* if its Gabriel filter contains a cofinal subset of finitely generated left ideals. A module is called τ-*finitely generated* if it has a finitely generated τ-dense submodule. We need the following lemma.

LEMMA 3.2. *Suppose that every cyclic module is τ-injective. Then every τ-finitely generated module is τ-injective.*

Proof. First we show that every finitely generated module is τ-injective. Let M be a finitely generated module, say $M = Rx_1 + \cdots + Rx_n$. Use induction on n. For $n = 1$ it is clear. Suppose that every module generated by $n - 1$ elements is τ-injective. Then $M/(Rx_1 + \cdots + Rx_{n-1}) \cong Rx_n/((Rx_1 + \ldots + Rx_{n-1}) \cap Rx_n)$ is cyclic, and so τ-injective. But $Rx_1 + \cdots + Rx_{n-1}$ is also τ-injective, so that M is τ-injective.

Now let M be a τ-finitely generated module; hence, M has some τ-dense finitely generated submodule N. Then N is τ-injective by the argument given in the previous paragraph. Clearly, M/N is τ-torsion, and hence, τ-injective by Corollary 3.1. Thus, it follows that M is τ-injective. □

THEOREM 3.3. *Let τ be of finite type and suppose that every cyclic module is τ-injective. Then every module is τ-injective.*

Proof. Let I be a τ-dense left ideal of R. Then there exists a finitely generated left ideal $J \subseteq I$ and we have I/J τ-torsion. Then J is τ-injective by Lemma 3.2; hence, it is a direct summand of R, and so a direct summand of I, say $I = J \oplus J'$. But $J' \cong I/J$ is τ-torsion, and hence, τ-injective. It follows that I is τ-injective, and hence, I is a direct summand of R. Therefore, every module is τ-injective by [**4**, Proposition 8.10]. □

There are situations when the condition that every cyclic τ-torsion module is τ-injective assures that every module is τ-injective. We present one based on the recent result stating that every Baer module over a commutative domain is projective [**6**, Theorem 3.4]. Recall that a module M is called *τ-projective* if $\mathrm{Ext}^1_R(M, T) = 0$ for every τ-torsion module T. If R is a commutative domain and τ is the usual torsion theory in Mod-R, then a τ-projective module is called *Baer*. We need the following easy lemma.

LEMMA 3.4. *Every τ-torsion module is τ-injective if and only if every τ-torsion module is τ-projective.*

COROLLARY 3.5. *Let R be a commutative domain and τ the usual torsion theory in* Mod-*R. The following are equivalent:*

(i) Every cyclic τ-torsion module is injective.
(ii) Every τ-torsion module is injective.
(iii) Every τ-torsion module is Baer.
(iv) Every module is injective.
(v) R is a field.

Proof. Recall that a module is τ-torsion if and only if every non-zero element $x \in M$ is annihilated by a non-zero ideal. Since R/I is τ-torsion for every non-zero ideal of R, τ-injectivity coincides with usual injectivity.

(i)⇒(ii) By Corollary 3.1.

(ii)⇒(iii) By Lemma 3.4.

(iii)⇒(iv) By Lemma 3.4, every τ-torsion module is Baer, and so projective by [**6**, Theorem 3.4]. Then every module is τ-injective [**4**, Proposition 8.10], and so injective.

(iv)⇒(v) In this case R is semi-simple, and so R must be a field.

(v)⇒(i) Clear. □

In the following, we establish a characterization of semi-simple modules using certain relative injective modules. Let τ be a hereditary torsion theory in the category $\sigma[M]$. Recall that a module $N \in \sigma[M]$ is called *(M, τ)-injective* if N is injective with respect to every exact sequence $0 \to K \to L$ in $\sigma[M]$ with L/K τ-torsion. We consider the following notion which generalizes that of complemented module with respect to a hereditary torsion theory in Mod-R from [**11**]. A module $N \in \sigma[M]$ is called (M, τ)-*complemented* if every submodule of N is τ-dense in a direct summand of N.

THEOREM 3.6. *The following are equivalent:*
(i) M is semi-simple.
(ii) Every module in $\sigma[M]$ is (M, τ)-injective (M, τ)-complemented.
(iii) Every cyclic module in $\sigma[M]$ is (M, τ)-injective (M, τ)-complemented.
(iv) Every cyclic module in $\sigma[M]$ is injective in $\sigma[M]$.

Proof. (*i*) $\Rightarrow$ (*ii*) Suppose that M is semi-simple. Then every module in $\sigma[M]$ is injective in $\sigma[M]$ [**14**, 20.3], and hence, (M, τ)-injective. Also, every module in $\sigma[M]$ is semi-simple in $\sigma[M]$ [**14**, 20.3], and hence, (M, τ)-complemented.

(*ii*) $\Rightarrow$ (*iii*) Clear.

(*iii*) $\Rightarrow$ (*iv*) Let $\mathcal{C}$ be the smallest closed subcategory of $\sigma[M]$ containing the (M, τ)-complemented modules. Then $\mathcal{C} = \sigma[N]$ for some module $N \in \sigma[M]$, and a family of finitely generated generators for $\mathcal{C}$ consists of the modules R/I with $R/I \in \sigma[N]$. Each such R/I is (M, τ)-complemented, and so an object of $\mathcal{C}$. Thus, $\mathcal{C} = \sigma[M]$. By an easy adaptation of [**13**, Lemma 2] in $\sigma[M]$, it follows that τ is a generalization of the Goldie torsion theory; hence, (M, τ)-injectivity coincides with injectivity.

(iv)$\Rightarrow$(i) By Corollary 2.12. □

Now we have the following characterization of semi-simple rings.

COROLLARY 3.7. *R is semi-simple if and only if every cyclic module is τ-injective τ-complemented.*

The classical Osofsky theorem is obtained by taking $\tau = \tau_G$, i.e. the Goldie torsion theory, or $\tau = \chi$, i.e. the torsion theory with all modules torsion. Note that a module is τ_G-injective τ_G-complemented if and only if it is injective. Also, every module is χ-complemented.

In [**1**] it has been shown that the class of τ-injective τ-complemented modules is strictly contained in the class of quasi-injective modules. Now recall the following result.

THEOREM 3.8 [**7**, Theorem 6.83]. *The following are equivalent:*
(i) R is semi-simple.
(ii) Every module is quasi-injective.
(iii) Every finitely generated module is quasi-injective.

The condition that every cyclic module is quasi-injective is, in general, weaker than that in the previous theorem. For instance, $R = \mathbb{Q}[x]/(x^2)$ is self-injective, and every cyclic module is quasi-injective, but R is not semi-simple [**7**]. Hence, Corollary 3.7 may be seen as a refinement of Theorem 3.8 for cyclic modules.

ACKNOWLEDGEMENTS. This work was partially supported by the Romanian grants PN-II-ID-PCE-2008-2 project ID_2271, PN-II-ID-PCE-2007-1 project ID_1005 and MEC of Spain.

REFERENCES

1. S. Crivei, On τ-complemented modules, *Mathematica (Cluj)* **45**(68) (2003), 127–136.

2. S. E. Dickson, A torsion theory for abelian categories, *Trans. Amer. Math. Soc.* **121** (1966), 223–235.

3. N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, *Extending modules*, Pitman Research Notes in Mathematics Series, vol. 313 (Longman Scientific & Technical, Harlow, UK, 1994).

4. J. S. Golan, *Torsion theories*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 29 (Longman Scientific & Technical, Harlow, UK, 1986).

5. J. L. Gómez Pardo, N. V. Dung and R. Wisbauer, Complete pure injectivity and endomorphism rings, *Proc. Amer. Math. Soc.* **118** (1993), 1029–1034.

6. L. A. Hügel, S. Bazzoni and D. Herbera, A solution to the Baer splitting problem, *Trans. Amer. Math. Soc.* **360** (2008), 2409–2421.

7. T. Y. Lam, *Lectures on modules and rings* (Springer, New York, 1999).

8. B. L. Osofsky, Rings all of whose finitely generated modules are injective, *Pacific J. Math.* **14** (1964), 645–650.

9. B. L. Osofsky, Noninjective cyclic modules, *Proc. Amer. Math. Soc.* **19** (1968), 1383–1384.

10. B. L. Osofsky and P. F. Smith, Cyclic modules whose quotients have all complement submodules direct summands, *J. Algebra* **139** (1991), 342–354.

11. P. F. Smith, A. M. Viola-Prioli and J. E. Viola-Prioli, Modules complemented with respect to a torsion theory, *Comm. Algebra* **25** (1997), 1307–1326.

12. B. Stenström, *Rings of quotients* (Springer-Verlag, Berlin, 1975).

13. A. M. de Viola-Prioli and J. E. Viola-Prioli, The smallest closed subcategory containing the μ-complemented modules, *Comm. Algebra* **28** (2000), 4971–4980.

14. R. Wisbauer, *Foundations of module and ring theory* (Gordon and Breach, Reading, UK, 1991).