ANZIAM J. 56(2014), 179–191
doi:10.1017/S1446181114000352

SOLVING INFINITE-HORIZON OPTIMAL CONTROL PROBLEMS USING THE HAAR WAVELET COLLOCATION METHOD

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(Received 4 August, 2013; revised 30 June, 2014; first published online 15 December 2014)

Abstract

We consider infinite-horizon optimal control problems. The main idea is to convert the problem into an equivalent finite-horizon nonlinear optimal control problem. The resulting problem is then solved by means of a direct method using Haar wavelets. A local property of Haar wavelets is applied to simplify the calculation process. The accuracy of the present method is demonstrated by two illustrative examples.

2010 *Mathematics subject classification*: primary 49J15; secondary 65T60, 90C30.

Keywords and phrases: infinite horizon, optimal control problem, Haar wavelet, nonlinear programming.

1. Introduction

The study of the existence and structure of solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research (see, for example, the articles [4, 10–12] and the references therein). These problems arise in several areas, such as engineering [1], models of economic growth [26], infinite discrete models of solid-state physics related to dislocations of one-dimensional crystals [2] and the theory of thermodynamical equilibrium for materials [18]. The necessary conditions of optimality for an infinite-horizon optimal control problem were studied by Blot and Michel [5]. The maximum principle for this problem without transversality conditions at infinity was considered by Carlson and Haurie [7]. Transversality conditions were derived by Smirnov [24], and a nonsmooth version of this result was obtained by Ye [25] for some dynamical optimization problems in mathematical economics.

Recently, wavelet theory has attracted considerable attention due to the advantages wavelets have over traditional Fourier transforms in accurately approximating

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functions that have discontinuities and sharp peaks. Wavelets have been applied to solve surface integral equations, improve the finite-difference time-domain method, solve linear differential equations and nonlinear partial differential equations, solve optimal control problems, model nonlinear semiconductor devices, signal processing, multi-scale phenomenon modelling and pattern recognition [3, 9, 13, 14, 19, 20, 22].

The Haar wavelet function was introduced by Alfred Haar in 1910 in the form of a regular pulse pair [6]. Later, many other wavelet functions were generated and introduced. Those included the Shannon, Daubechies and Legendre wavelets. Among those functions, Haar wavelets have the simplest orthonormal series with compact support. Haar functions are also notable for their rapid convergence for the expansion of functions, which makes them very useful with regard to the wavelet theory. Due to these characteristics, Haar wavelets are often applied to solving optimal control problems.

Motivated by the above discussion, this paper is focused on the solutions of infinite-horizon optimal control problems via the Haar transform by taking advantage of the nice properties of Haar wavelets. First, we transform the infinite-horizon problem to a finite-horizon problem by reducing the interval $[0, \infty)$ to $[0, 1)$. Then, we approximate the control variables and derivatives of the state variables in the optimal control problems by linear combinations of Haar wavelets with unknown coefficients. The state variables are calculated by using the Haar operational integration matrix. Therefore, all variables in the nonlinear system of equations are expressed as series of the Haar family and its operational matrix. Finally, the task of finding the unknown parameters that optimize the designated performance while satisfying all constraints is performed by a nonlinear programming method. The proposed method, in this paper, is similar to the control parametrization method discussed in the articles [8, 15–17], where the main idea is to discretize the control space by approximating the control function by a linear combination of basis functions. Under this approximation, the optimal control problem is reduced to an approximate nonlinear optimization problem with a finite number of decision variables. This problem is then solved by using nonlinear programming methods, such as a gradient-based optimization technique and an exact penalty method. Some advantages of the proposed method compared to the existing ones are reported at the end of Section 5.

2. Statement of the problem and its transformation

We consider the following optimal control problem:

$$\text{minimize} \quad \int_0^{\infty} g(t, x(t), u(t))\, dt \tag{2.1}$$

subject to

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{2.2}$$

$$x(0) = x^0 \quad \text{and} \quad \lim_{t\to\infty} x(t) = x^1. \tag{2.3}$$

Here $(\dot{\cdot})$ denotes differentiation with respect to the independent variable t, $x(t) = (x_1(t), \ldots, x_n(t)) \in A \subseteq \mathbb{R}^n$ and $u(t) = (u_1(t), \ldots, u_m(t)) \in U \subseteq \mathbb{R}^m$ are, respectively, the values of the phase vector of the control system (2.2) and of the control vector at time $t \geq 0$. Also, A and U are nonempty compact rectangular boxes in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and $x^0 \in \mathbb{R}^n$ is a given initial state of the system. Next, we assume that the vector function $f : [0, \infty) \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and the scalar function $g : [0, \infty) \times \mathbb{R}^n \times U \to \mathbb{R}$ are continuously differentiable in x. The class of admissible controls (for system (2.2)) is the set of all *Lebesgue measurable* functions $u : [0, \infty) \to U$. An admissible trajectory (of system (2.2)) corresponding to an admissible control u is defined to be a *Carathéodory solution* [23] x of the differential equation (2.2) which is defined on $[0, \infty)$, satisfies the initial and final conditions (2.3) and takes values in A. We assume that for every admissible control u there exists an admissible trajectory x corresponding to u and the *Lebesgue integral* in (2.1) converges absolutely. We also say that an admissible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ is an *optimal solution* of problem (2.1)–(2.3) if

$$\int_0^\infty g(t, x(t), u(t))\, dt \geq \int_0^\infty g(t, \bar{x}(t), \bar{u}(t))\, dt$$

for any admissible pair $(x(\cdot), u(\cdot))$.

A time transformation is introduced in order to use Haar wavelet functions [11] defined on $\tau \in [0, 1)$ as

$$t = \tan\left(\frac{\pi}{2}\tau\right), \quad t \in [0, \infty). \tag{2.4}$$

Then the above problem is transformed into a variational nonlinear optimal control problem,

$$\begin{aligned}
&\text{minimize} && J = \int_{[0,1)} \frac{\pi}{2} g\left(\tan\left(\frac{\pi}{2}\tau\right), x\left(\tan\left(\frac{\pi}{2}\tau\right)\right), u\left(\tan\left(\frac{\pi}{2}\tau\right)\right)\right) \sec^2\left(\frac{\pi}{2}\tau\right) d\tau \\
&\text{subject to} && \\
& && \dot{x}\left(\tan\left(\frac{\pi}{2}\tau\right)\right) = \frac{\pi}{2} f\left(\tan\left(\frac{\pi}{2}\tau\right), x\left(\tan\left(\frac{\pi}{2}\tau\right)\right), u\left(\tan\left(\frac{\pi}{2}\tau\right)\right)\right) \sec^2\left(\frac{\pi}{2}\tau\right), \tau \in [0, 1), \\
& && x(0) = x^0, \quad \lim_{\tau \longrightarrow 1^-} x\left(\tan\left(\frac{\pi}{2}\tau\right)\right) = x^1 \quad \text{and} \\
& && \left(x\left(\tan\left(\frac{\pi}{2}\tau\right)\right), u\left(\tan\left(\frac{\pi}{2}\tau\right)\right)\right) \in A \times U, \quad \tau \in [0, 1).
\end{aligned}$$

Assume that

$$\begin{cases} y(\tau) = x\left(\tan\left(\frac{\pi}{2}\tau\right)\right), \\ v(\tau) = u\left(\tan\left(\frac{\pi}{2}\tau\right)\right). \end{cases}$$

Using this transformation, the optimal control problem of the nonlinear system in (2.2) and (2.3) with performance index (2.1) is replaced by the following optimization

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problem [10]:

$$\text{minimize} \quad J = \int_{[0,1)} \frac{\pi}{2} g\Big(\tan\Big(\frac{\pi}{2}\tau\Big), y(\tau), v(\tau)\Big) \sec^2\Big(\frac{\pi}{2}\tau\Big)\, d\tau \tag{2.5}$$

subject to

$$\dot{y}(\tau) = \frac{\pi}{2} f\Big(\tan\Big(\frac{\pi}{2}\tau\Big), y(\tau), v(\tau)\Big) \sec^2\Big(\frac{\pi}{2}\tau\Big), \tag{2.6}$$

$$y(0) = y^0 = x^0, \quad \lim_{\tau \to 1^-} y(\tau) = y^1 = x^1, \tag{2.7}$$

$$(y(\tau), v(\tau)) \in A \times U. \tag{2.8}$$

In the next section, we will discuss the properties of a direct collocation method based on Haar functions, and use it for solving the finite time horizon problem (2.5)–(2.8).

3. Haar wavelets

3.1. Rationalized Haar (RH) functions For $r = 1, 2, \ldots,$ the RH functions [19] are defined on the interval $[0, 1)$ as

$$RH(r,t) = \begin{cases} 1 & \text{if } J_1 \leqslant t < J_{1/2}, \\ -1 & \text{if } J_{1/2} \leqslant t < J_0, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$J_u = \frac{j-u}{2^i}, \quad u = 0, \frac{1}{2}, 1.$$

The positive integer r is expressed in terms of the two parameters i and j as follows:

$$r = 2^i + j - 1, \quad \text{where } i = 0, 1, 2, 3, \ldots, j = 1, 2, 3, \ldots, 2^i.$$

For $i = j = 0$, we define

$$RH(0,t) = 1, \quad 0 \leqslant t < 1.$$

A set of the first eight RH functions is shown in Figure 1 (see [22]), where $r = 0, 1, 2, \ldots, 7$. The orthogonality property is given by

$$\int_0^1 RH(r,t) RH(\nu,t)\, dt = \begin{cases} 2^{-i} & r = \nu, \\ 0 & r \neq \nu, \end{cases}$$

where

$$\nu = 2^n + m - 1, \quad n = 0, 1, 2, 3, \ldots, \quad m = 1, 2, 3, \ldots, 2^n.$$

It should be noted that the set of RH functions is a complete orthogonal set in the Hilbert space $L^2[0, 1]$. Thus, we can expand any function in this space in terms of RH functions.

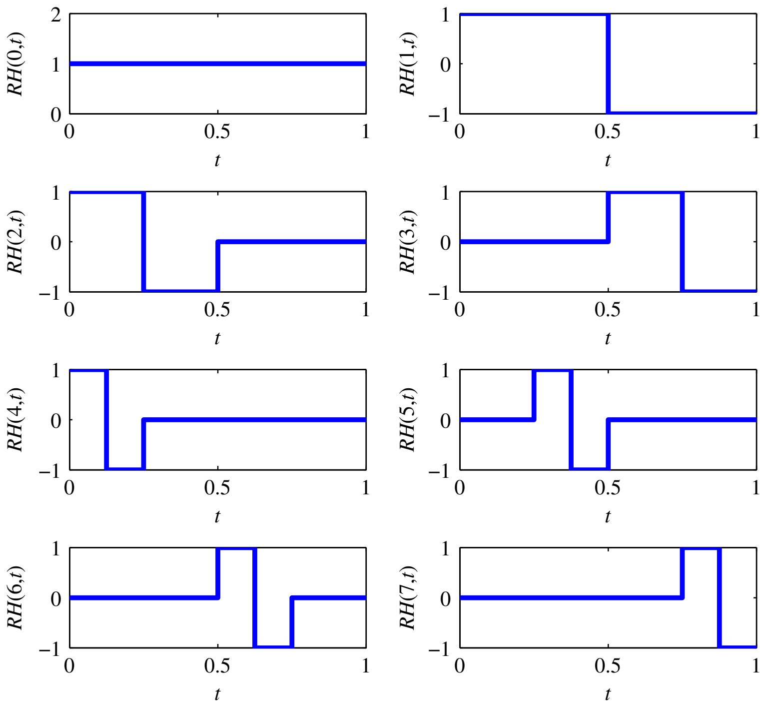

FIGURE 1. The first eight *RH* functions for $i = 0, 1$ and 2.

3.2. Function approximation A function $\mathcal{G}(t) \in L^2[0, 1]$, may be expanded as an infinite series of *RH* functions as

$$\mathcal{G}(t) = \sum_{r=0}^{\infty} a_r RH(r, t), \tag{3.1}$$

where a_r is given by

$$a_r = 2^i \int_0^1 \mathcal{G}(t) RH(r, t)\, dt, \qquad r = 0, 1, 2, \ldots$$

with $r = 2^i + j - 1, i = 0, 1, 2, 3, \ldots, j = 1, 2, 3, \ldots, 2^i$, and $r = 0$ for $i = j = 0$. If we let $i = 0, 1, 2, \ldots, \alpha$, then the infinite series in (3.1) is truncated into its first K terms as

$$\mathcal{G}(t) \simeq \sum_{r=0}^{K-1} a_r RH(r, t) = A^T \Phi(t), \tag{3.2}$$

where $K = 2^{\alpha+1}, \alpha = 0, 1, 2, 3, \ldots$. The vector function $\Phi(t)$ and the coefficient vector A for *RH* functions are defined as

$$A = [a_0, a_1, \ldots, a_{K-1}]^T,$$
$$\Phi(t) = [\phi_0(t), \phi_1(t), \ldots, \phi_{K-1}(t)]^T,$$

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where

$$\phi_r(t) = RH(r,t), \quad r = 0, 1, 2, \ldots, K-1.$$

If we set all the collocation points t_l at the middle of each respective wavelet, then t_l is defined as

$$t_l = \frac{l-0.5}{K}, \quad l = 1, 2, \ldots, K.$$

With these collocation points, the function is discretized over a series of equally spaced nodes. The vector $\Phi(t)$ can also be determined at these collocation points. Let the Haar matrix $\hat{\Phi}_{K\times K}$ be the combination of $\Phi(t)$ at all the collocation points, so we get

$$\hat{\Phi}_{K\times K} = [\Phi(t_1), \Phi(t_2), \ldots, \Phi(t_K)]. \tag{3.3}$$

For example, if each waveform is divided into eight intervals, the magnitude of the waveform can be represented as (see, for example, Ohkita and Kobayashi [20])

$$\hat{\Phi}_{8\times 8} = [\Phi(t_1), \Phi(t_2), \ldots, \Phi(t_8)] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Using (3.2) and (3.3), we have

$$[\mathcal{G}(t_1), \mathcal{G}(t_2), \ldots, \mathcal{G}(t_K)] = A^T \hat{\Phi}_{K\times K},$$

which yields

$$A^T = [\mathcal{G}(t_1), \mathcal{G}(t_2), \ldots, \mathcal{G}(t_K)] \hat{\Phi}^{-1}_{K\times K},$$

where (as in [20]),

$$\hat{\Phi}^{-1}_{K\times K} = \left(\frac{1}{K}\right) \hat{\Phi}^T_{K\times K} \operatorname{diag}\Big(1, 1, 2, 2, \underbrace{2^2, \ldots, 2^2}_{2^2}, \underbrace{2^3, \ldots, 2^3}_{2^3}, \ldots, \underbrace{\frac{K}{2}, \ldots, \frac{K}{2}}_{K/2}\Big). \tag{3.4}$$

Therefore, the function $\mathcal{G}(t)$ is approximated as

$$\mathcal{G}(t_l) \approx A^T_{1\times K} \hat{\Phi}_{K\times K}, \quad l = 1, 2, \ldots, K.$$

It is also expected that the function $\mathcal{G}(t)$ is approximated with minimum mean integral square error

$$\varepsilon = \int_0^1 [\mathcal{G}(t) - A^T \Phi(t)]^2 \, dt.$$

Obviously, ε should reduce when the level K gets larger, and it should go close to zero when K approaches infinity.

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3.3. Operational integration matrix In the solution of optimal control problems, we always need to deal with equations involving differentiation and integration. If the system function is expressed in Haar wavelets, the integration or differentiation operation of Haar series cannot be avoided. The differentiation of step waves generates pulse signals which are difficult to handle, while the integration of step waves results in constant slope functions which can be calculated by using

$$\int_0^t \Phi(t')\,dt' \simeq P\Phi(t),$$

where

$$P = P_{K\times K} = \frac{1}{2K}\begin{bmatrix} 2KP_{K/2\times K/2} & -\hat{\Phi}_{K/2\times K/2} \\ \hat{\Phi}^{-1}_{K/2\times K/2} & 0 \end{bmatrix} \tag{3.5}$$

is the same as given by Razzaghi and Ordokhani [22] with $\hat{\Phi}_{1\times 1} = [1]$, $P_{1\times 1} = [1/2]$, and $\hat{\Phi}_{K/2\times K/2}$, $\hat{\Phi}^{-1}_{K/2\times K/2}$ can be obtained from (3.3) and (3.4), respectively. They have also given the integration of the cross product of the two RH vectors as

$$\int_0^1 \Phi(t)\Phi^T(t)\,dt = \hat{D},$$

where

$$\hat{D} = \mathrm{diag}\Big(1, 1, \frac{1}{2}, \frac{1}{2}, \underbrace{\frac{1}{2^2}, \ldots, \frac{1}{2^2}}_{2^2}, \ldots, \underbrace{\frac{1}{2^\alpha}, \ldots, \frac{1}{2^\alpha}}_{2^\alpha}\Big) \tag{3.6}$$

is the diagonal matrix.

4. Direct collocation

4.1. Haar discretization method In Section 3 we discussed how to approximate a function via Haar wavelets and its corresponding operational integration matrix. We will apply this method to optimal control problems, so that Haar discretization is used in direct collocation [9]. Thus, the continuous solution to a problem will be represented by state and control variables in terms of Haar series and its operational matrix to satisfy the differential equations. Here, the standard interval considered for τ is $[0, 1)$ with collocation points

$$\tau_l = \frac{l - 0.5}{K}, \quad l = 1, 2, \ldots, K,$$

where K is the number of nodes used in the discretization as well as the maximum wavelet index number. Note that the magnitude of K is a power of two, so that the number of collocation points is also increasing by the same power. All the collocation points are equally distributed over the entire time interval $[0, 1)$ with $1/K$ as the time distance between the adjacent nodes. We assume that the derivative of the state

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variables, $\dot{y}(\tau)$, and control variables, $v(\tau)$, can be approximated by Haar wavelets with K collocation points, that is,

$$\dot{y}(\tau) \approx C_y^T \Phi(\tau),$$
$$v(\tau) \approx C_v^T \Phi(\tau),$$

where

$$C_y^T = [C_{y1}, C_{y2}, \ldots, C_{yK}] \quad \text{and} \quad C_v^T = [C_{v1}, C_{v2}, \ldots, C_{vK}].$$

By using the operational integration matrix P defined in (3.5), the state variables $y(\tau)$ can be expressed as

$$y(\tau) = \int_0^\tau \dot{y}(\tau')\, d\tau' + y_0 = \int_0^\tau C_y^T \Phi(\tau')\, d\tau' + y_0 = C_y^T P \Phi(\tau) + y_0.$$

As stated in (3.3), the expansion of the matrix $\Phi(\tau)$ at the K collocation points will yield the $K \times K$ Haar matrix $\hat{\Phi}$. So it follows that

$$\dot{y}(\tau_l) = C_y^T \Phi(\tau_l), \quad v(\tau_l) = C_v^T \Phi(\tau_l), \quad y(\tau_l) = C_y^T P \Phi(\tau_l) + y_0, \quad l = 1, \ldots, K. \tag{4.1}$$

From the above expression, we can evaluate the variables at any collocation point by using the product of its coefficient vectors and the corresponding column vector in the Haar matrix.

4.2. Nonlinear programming When the Haar collocation method is applied in the optimal control problems, the nonlinear programming variables can be set as the unknown coefficient vector of the derivative of the state variables and control variables, that is,

$$\tilde{y} = [C_{y1}, C_{y2}, \ldots, C_{yK}, C_{v1}, C_{v2}, \ldots, C_{vK}]^T.$$

The performance index in (2.5) is then restated as

$$J = \int_{[0,1)} \frac{\pi}{2} g\Big(\tan\Big(\frac{\pi}{2}\tau\Big), (C_y^T P \Phi(\tau) + y_0), C_v^T \Phi(\tau)\Big) \sec^2\Big(\frac{\pi}{2}\tau\Big)\, d\tau. \tag{4.2}$$

Since the Haar wavelets are expected to be constant steps at each time interval, equation (4.2) can be simplified as

$$J = \frac{\pi}{2K} \sum_{l=1}^{K} g\Big(\tan\Big(\frac{\pi}{2}\tau_l\Big), (C_y^T P \Phi(\tau_l) + y_0), C_v^T \Phi(\tau_l)\Big) \sec^2\Big(\frac{\pi}{2}\tau_l\Big). \tag{4.3}$$

Substituting $\dot{y}$, v and y of (2.6) into the Haar wavelet expressions (4.1) separately, we get

$$C_y^T \Phi(\tau_l) = \frac{\pi}{2} f\Big(\tan\Big(\frac{\pi}{2}\tau_l\Big), (C_y^T P \Phi(\tau_l) + y_0), C_v^T \Phi(\tau_l)\Big) \sec^2\Big(\frac{\pi}{2}\tau_l\Big).$$

The constraints for the system of equations are all treated as nonlinear constraints in a nonlinear programming solver. The boundary constraints need more attention. Since

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the first and last collocation points are not set as the initial and final time, the initial and final state variables are calculated according to

$$y_0 = y(\tau_1) - \frac{\dot{y}(\tau_1)}{2K}, \quad y_1 = y(\tau_K) + \frac{\dot{y}(\tau_K)}{2K}.$$

In this way, the optimal control problems are transformed into nonlinear programming problems in a structured form which is solved by the LINGO 11 software [21].

5. Numerical examples

We present two numerical examples to illustrate the proposed method.

EXAMPLE 5.1. Consider the following problem:

$$\text{minimize} \quad \frac{1}{2}\int_0^\infty \{x^2(t) + 4u^2(t)\}\,dt, \tag{5.1a}$$

$$\text{subject to}$$

$$\ddot{x}(t) = -\dot{x}(t) + u(t), \tag{5.1b}$$

$$x(0) = \dot{x}(0) = 0.1, \quad \lim_{t\to\infty} x(t) = \lim_{t\to\infty} \dot{x}(t) = 0. \tag{5.1c}$$

Suppose that $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$. Then problem (5.1a)–(5.1c) is converted to the following form:

$$\begin{aligned}
&\text{minimize} && \frac{1}{2}\int_0^\infty \{x_1^2(t) + 4u^2(t)\}\,dt \\
&\text{subject to} && \\
& && \dot{x}_1(t) = x_2(t), \\
& && \dot{x}_2(t) = -x_2(t) + u(t), \\
& && x_1(0) = x_2(0) = 0.1, \quad \lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = 0.
\end{aligned}$$

Effati et al. [10] have given the optimal trajectory of the problem as

$$x_1(t) = \left[0.1 + \left(0.1 + \frac{0.1}{\sqrt{2}}\right)t\right]\exp\left(\frac{-t}{\sqrt{2}}\right),$$

$$x_2(t) = \left[0.1 - \left(0.1 + \frac{0.1}{\sqrt{2}}\right)\frac{t}{\sqrt{2}}\right]\exp\left(\frac{-t}{\sqrt{2}}\right).$$

Now by the change of variable (2.4), the above problem is transformed to:

$$\begin{aligned}
&\text{minimize} && \int_{[0,1)} \frac{\pi}{4}\{y_1^2(\tau) + 4v^2(\tau)\}\sec^2\left(\frac{\pi}{2}\tau\right)d\tau \\
&\text{subject to} && \\
& && \dot{y}_1(\tau) = \frac{\pi}{2}y_2(\tau)\sec^2\left(\frac{\pi}{2}\tau\right), \\
& && \dot{y}_2(\tau) = \frac{\pi}{2}\{-y_2(\tau) + v(\tau)\}\sec^2\left(\frac{\pi}{2}\tau\right), \\
& && y_1(0) = y_2(0) = 0.1, \quad \lim_{\tau\to 1^-} y_1(\tau) = \lim_{\tau\to 1^-} y_2(\tau) = 0.
\end{aligned}$$

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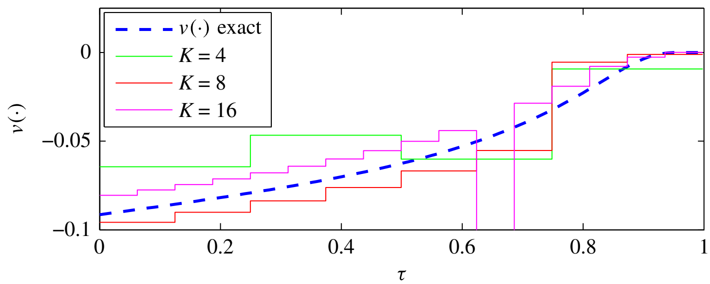

FIGURE 2. Exact solution of $v(\cdot)$ and Haar wavelet solutions for $K = 4, 8$ and 16.

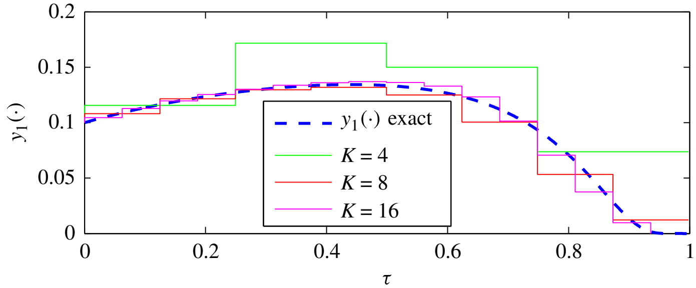

FIGURE 3. Exact solution of $y_1(\cdot)$ and Haar wavelet solutions for $K = 4, 8$ and 16.

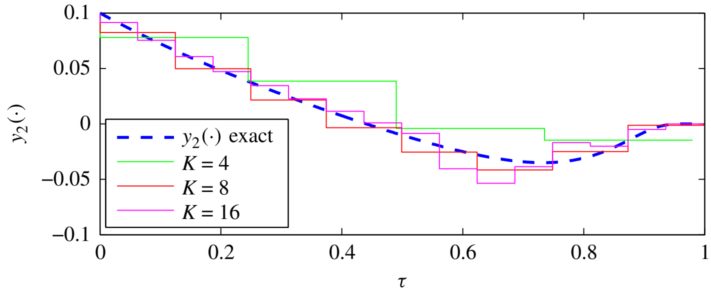

FIGURE 4. Exact solution of $y_2(\cdot)$ and Haar wavelet solutions for $K = 4, 8$ and 16.

To see how accurate the solution is, we solve the problem for an increasing number of collocation points. The optimal value of J for $K = 4, 8$ and 16 is 0.0450, 0.0443 and 0.0427, respectively, and the analytical optimal value using Pontryagin's maximum principle [7] is derived as $J = 0.0424$. The optimal control $v(\tau)$ and the corresponding trajectories, $y_1(\tau)$ and $y_2(\tau)$, depicted in Figures 2–4, also compare very well with the exact solution. It is further verified that the convergence improves with an increasing number of collocation points.

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EXAMPLE 5.2. Consider the following problem:

$$\begin{aligned} &\text{minimize} && \frac{1}{2}\int_0^\infty \{\log^2(x(t)) + u^2(t)\}\,dt, && (5.2a)\\ &\text{subject to} && \\ & && \dot{x}(t) = x(t)\log(x(t)) + x(t)u(t), && (5.2b)\\ & && x(0) = \exp(2), \quad \lim_{t\to\infty} x(t) = 1. && (5.2c)\end{aligned}$$

Using (2.4), problem (5.2a)–(5.2c) is transformed to:

$$\begin{aligned} &\text{minimize} && \frac{\pi}{4}\int_{[0,1)} \{\log^2 y(\tau) + v^2(\tau)\}\sec^2\Big(\frac{\pi}{2}\tau\Big)\,d\tau,\\ &\text{subject to} && \\ & && \dot{y}(\tau) = \frac{\pi}{2}(y(\tau)\log y(\tau) + y(\tau)v(\tau))\sec^2\Big(\frac{\pi}{2}\tau\Big),\\ & && y(0) = \exp(2), \quad \lim_{\tau\to 1^-} y(\tau) = 1.\end{aligned}$$

The exact optimal solution of the problem is given by Garg et al. [12]:

$$\begin{aligned} x(t) &= \exp(2\exp(-t\sqrt{2})),\\ u(t) &= -2(1+\sqrt{2})\exp(-t\sqrt{2}).\end{aligned}$$

To see how accurate the solution is, we solve the problem for an increasing number of collocation points. The optimal value of J for $K = 4$ and 8 is 4.86 and 4.84, respectively, and the analytical optimal value, $J = 4.83$, is derived using Pontryagin's maximum principle [7]. The optimal control $v(\tau)$ and the corresponding trajectory $y(\tau)$, depicted in Figures 5 and 6, also compare very well with the exact solution.

Finally, a natural question arises: are there advantages of the proposed collocation method compared to the existing ones? To answer this, we summarize what we have observed from numerical experiments and theoretical results as follows.

- One of the main advantages of using Haar wavelets is that the matrices $\hat{\Phi}_{K\times K}, \hat{\Phi}^{-1}_{K\times K}$ and $\hat{D}$ introduced in (3.3), (3.4) and (3.6), respectively, have a large number of zero elements, and they are sparse. Hence, the present method is very attractive, and reduces CPU time and computer memory, at the same time as giving an accurate solution.
- The simple implementation of Haar wavelet-based optimal control in real applications is interesting.
- Haar functions are also notable for their rapid convergence for the expansion of functions, which is very useful in Haar function theory.
- The proposed method also produces results similar to some other collocation methods [8, 15–17] for the continuous optimal control problem, and shows advantages in discrete optimal control problems when the switching time is unknown.
- The proposed orthogonal collocation method leads to rapid convergence as the number of collocation points increases.

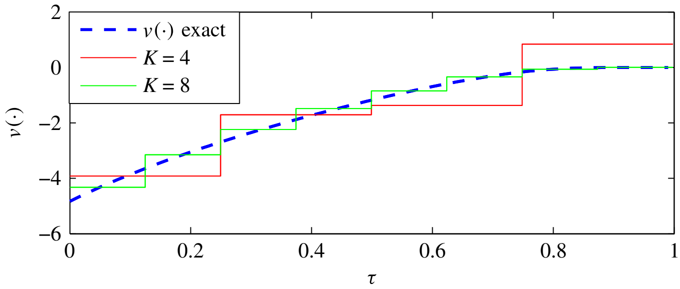

FIGURE 5. Exact solution of $v(\cdot)$ and Haar wavelet solutions for $K = 4$ and $K = 8$.

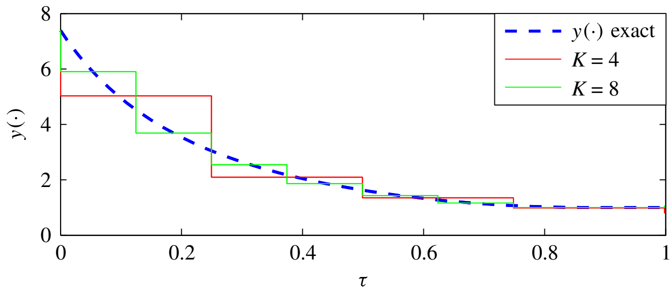

FIGURE 6. Exact solution of $y(\cdot)$ and Haar wavelet solutions for $K = 4$ and $K = 8$.

- With $\tau_l = (l - 0.5)/K \neq 1,\ l = 1, 2, \ldots, K$, there is no numerical difficulty. In fact, we do not apply numerical integration methods such as Simpson's rule for calculation of integral (2.5), since it leads to some problems at the right-hand end-point. Instead, we use formula (4.3) to calculate the integral in (2.5) that does not require $\tau = 1$. Thus, the integration on the finite-time interval is convergent.

6. Conclusion

Approximate solutions to infinite-horizon optimal control problems are obtained by a combined algorithm of parameters and function optimization. To this end, a suitable transformation is used to obtain a corresponding finite-horizon problem. According to the approximation of dynamic systems and performance index into Haar series, an efficient and accurate method is then applied to solve finite-horizon optimal control problems. The feasibility and effectiveness of this method are proved through two simulation experiments.

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