GLUING AN INFINITE NUMBER OF INSTANTONS

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Abstract. This paper is one step toward infinite energy gauge theory and the geometry of infinite dimensional moduli spaces. We generalize a gluing construction in the usual Yang-Mills gauge theory to an "infinite energy" situation. We show that we can glue an infinite number of instantons, and that the resulting ASD connections have infinite energy in general. Moreover they have an infinite dimensional parameter space. Our construction is a generalization of Donaldson's "alternating method".

§1. Introduction

Since the remarkable paper of S. K. Donaldson [D1] appeared in 1983, Yang-Mills gauge theory has been one of central themes in geometry and mathematical physics. There are plenty of papers on this subject, and many interesting theories have been developed.

But most of them study only *finite energy* ASD connections and their *finite dimensional* moduli spaces. We know little about *infinite energy* ASD connections and their moduli spaces. In fact I don't know even whether such connections exist in general situations. This paper is one step in this direction. We will show that we can construct a huge amount of infinite energy ASD connections by using a "gluing construction".

Gluing ASD connections is a well-known technique in Yang-Mills theory [D1], [D2], [DFK], [DK], [FU], [T1], [T2], but we usually study a gluing of two (or a finite number of) instantons. In this paper we study a gluing of infinitely many instantons, and show that this new gluing technique produces the desired infinite energy ASD connections. Moreover we show that the construction of "gluing an infinite number of instantons" has the following interesting new feature; To define the gluing of ASD connections, we need to introduce a "gluing parameter space", which is isomorphic to

$SU(2)$ in the case of gluing two $SU(2)$ instantons. In the case of gluing infinitely many $SU(2)$ instantons, the gluing parameter space becomes the *infinite product* of the copies of $SU(2)$. This gigantic parameter space is the interesting new feature of our approach.

Although the above "infinite dimensional gluing parameter space" is the source of the richness of the construction, it also causes several serious difficulties in analysis. In particular, we cannot use the usual Fredholm deformation theory because of infinite degrees of freedom of the parameter space. We overcome this difficulty by adapting Donaldson's "alternating method" in [D2] to this situation. This decomposes the problem into an infinite number of small pieces where we can use the usual Fredholm deformation theory.

I think that the technique of "gluing an infinite number of instantons" probably can be applied to other equations*, e.g., the Seiberg-Witten equation, pseudo-holomorphic curves, etc. In fact the classical Mittag-Leffler theorem in complex analysis can be considered as a variant of this gluing technique. Singularities of a meromorphic function are instanton-like objects, and the Mittag-Leffler theorem constructs a meromorphic function which has an infinite number of singularities given in the complex plane. In other words the result of this paper is a Yang-Mills analogue of the Mittag-Leffler theorem.

The organization of the paper is as follows. In Section 2 we present notation and state main theorems. We establish basic estimates in Section 3. The details of the gluing construction are given in Section 4. We discuss a moduli problem in Section 5.

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§2. Main results

2.1. Brief review of Yang-Mills theory

To begin with, we review some basic notions of Yang-Mills theory. For the details, see [DK] or [FU]. Let G be a Lie group $SU(2)$ or $SO(3)$.

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*After I posted a preprint version of this paper to the arXiv in 2005, I was informed that there is a paper of M. Macrì, M. Nolasco and T. Ricciardi [MNR] which studies an "infinite energy" situation for selfdual vortices in $\mathbb{R}^2$. From [MNR] I learned that there is a paper of S. Angenent [A] which develops a gluing technique for infinitely many solutions of an elliptic PDE of the form $\Delta u + f(x, u) = 0$ in $\mathbb{R}^n$. [MNR] adapts this gluing technique to an elliptic equation with singular sources. I think that our alternating method is different from the analytic technique in [A].

Let $C(G)$ be the center of G, i.e., $C(SU(2)) = \{\pm 1\}$ and $C(SO(3)) = 1$. Suppose that X is a closed oriented Riemannian 4-manifold and that $E \to X$ is a principal G bundle over X. Let $\operatorname{ad} E$ be the Lie algebra bundle associated with E. A connection A on E is said to be irreducible if the isotropy group of A in the gauge group is equal to $C(G)$. ($C(G)$ acts trivially on all connections.)

A connection A is said to be anti-self-dual, or ASD, if the self-dual part of its curvature is zero:

$$F_A^+ = 0.$$

If A is an ASD connection, then we have the Atiyah-Hitchin-Singer complex for A:

$$0 \longrightarrow \Omega^0(\operatorname{ad} E) \xrightarrow{d_A} \Omega^1(\operatorname{ad} E) \xrightarrow{d_A^+} \Omega^+(\operatorname{ad} E) \longrightarrow 0,$$

where Ω^+ is the space of self-dual 2-forms and d_A^+ denotes the self-dual part of d_A. An ASD connection A is said to be acyclic if the above complex is exact (acyclic). The Atiyah-Hitchin-Singer complex is exact if and only if

$$d_A^* + d_A^+ : \Omega^1(\operatorname{ad} E) \longrightarrow \Omega^0(\operatorname{ad} E) \oplus \Omega^+(\operatorname{ad} E)$$

is isomorphism. Here d_A^* is the formal adjoint of $d_A : \Omega^0(\operatorname{ad} E) \to \Omega^1(\operatorname{ad} E)$. A is said to be regular if A is irreducible and $d_A^* + d_A^+$ is surjective.

EXAMPLE 2.1. Let X be the 4-sphere S^4 with the usual round metric. All non-flat G ASD connections on S^4 are regular. This follows from the Weitzenböck formula. For example, see [FU, Chapter 6].

EXAMPLE 2.2. $K3$ surface has a non-flat, irreducible, acyclic $SO(3)$ ASD connection. See [DK, Section 9.1.3].

2.2. Gluing construction
Next we introduce the following convenient definition.

DEFINITION 2.3. A 5-tuple (X, x^L, x^R, E, A) is called a gluing data if it satisfies the following (i)–(v):

(i) X is a closed oriented Riemannian 4-manifold.
(ii) x^L and x^R are two distinct points in X.
(iii) E is a principal G bundle over X.

(iv) A is an irreducible acyclic ASD connection on E. (We can replace
    this condition with the weaker condition that A is a regular ASD
    connection. See the remark at the end of Section 5.)

(v) The metric on X is flat in some neighborhoods of x^L and x^R. (See
    the remark below.)

Remark 2.4. The condition (v) in the Definition 2.3 is artificial and
just for simplicity. We can construct our gluing argument without this
condition; see [D2, (4.48) Proposition]. Actually, this condition is allowable
in the following sense. Let g be a Riemannian metric on X which does *not*
satisfy the condition (v). Let A be an irreducible acyclic (resp. regular)
g-ASD connection on E. Then small perturbation gives a new metric g'
satisfying the condition (v) and an irreducible acyclic (resp. regular) g'-ASD
connection A'. This fact follows from the usual implicit function theorem.

DEFINITION 2.5. Let $(X_i, x_i^L, x_i^R, E_i, A_i)$, $(i \in \mathbb{Z})$, be a sequence of
gluing data indexed by integers. This sequence is said to be *of finite type* if
it satisfies the following:

$$\{(X_i, x_i^L, x_i^R, E_i, A_i) \mid i \in \mathbb{Z}\} \text{ is a finite set.}$$

(Of course, the sequence itself is an *infinite* sequence.) This condition as-
sures the uniformity of analysis and geometry on each data.

We will define a connected sum of gluing data. This construction is
a generalization of the "conformal connected sum" construction in [D2,
pp. 306–307] and [DK, Section 7.2.1].

Let (X, x^L, x^R, E, A) be a gluing data. By the condition (v) in Def-
inition 2.3, there are an oriented Euclidean coordinate ξ centered on x^R
and an oriented Euclidean coordinate η centered on x^L. We set $k := 0.9$,
and we define the annular region Ω^R and the "shells" R^- and R^+ in the
neighborhood of x^R by

$$\Omega^R := \{\xi \mid kN^{-1}\sqrt{\lambda} < |\xi| < k^{-1}N\sqrt{\lambda}\},$$
$$R^- := \{\xi \mid kN^{-1}\sqrt{\lambda} < |\xi| < N^{-1}\sqrt{\lambda}\},$$
$$R^+ := \{\xi \mid N\sqrt{\lambda} < |\xi| < k^{-1}N\sqrt{\lambda}\}.$$

Here N and λ are positive parameters. We define Ω^L, L^- and L^+ in the
same way by using the Euclidean coordinate η centered on x^L. So L^- is the

inner "shell" of Ω^L and L^+ is the outer "shell"; see Figure 1. (Figure 1 is a variant of [D2, Diagram (4.13)].) We define the open set U by

$$U := X \setminus (\bar{B}(x^R, kN^{-1}\sqrt{\lambda}) \cup \bar{B}(x^L, kN^{-1}\sqrt{\lambda})).$$

Here $\bar{B}(x^R, kN^{-1}\sqrt{\lambda})$ and $\bar{B}(x^L, kN^{-1}\sqrt{\lambda})$ are the balls $|\xi| \leq kN^{-1}\sqrt{\lambda}$ and $|\eta| \leq kN^{-1}\sqrt{\lambda}$.

In the later gluing argument, "error terms" will be supported in the shells $R^\pm$ and $L^\pm$. The volumes of these shells are determined by the parameters λ and N. We will choose λ small and N large. The value of N will be fixed in the end of Section 3. We choose λ so small that

$$(1) \qquad\qquad\qquad\qquad \lambda N^{100} \ll 1.$$

Here "100" does not have a particular meaning. The point is that λ is very small. (In the above, we set $k = 0.9$. This value "0.9" does not have a particular meaning, either. The point is that k is smaller than 1.)
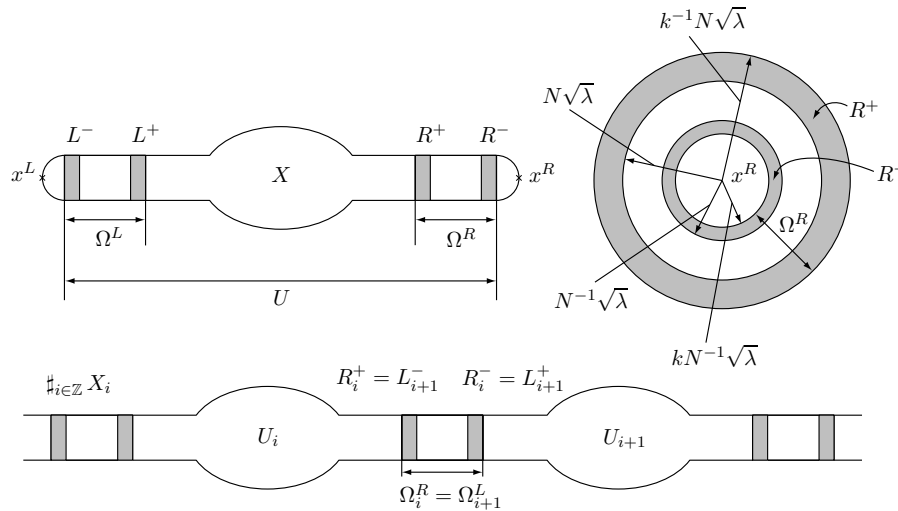

Figure 1: A gluing data and a connected sum.

Let $(X_i, x_i^L, x_i^R, E_i, A_i)_{i \in \mathbb{Z}}$ be a sequence of gluing data of finite type. We have Ω_i^R, R_i^-, R_i^+, and Ω_i^L, L_i^-, L_i^+, and U_i for each $i \in \mathbb{Z}$. We define the connected sum

$$\sharp_{i \in \mathbb{Z}} X_i := \cdots \sharp X_{-1} \sharp X_0 \sharp X_1 \sharp \cdots .$$

by identifying Ω_i^R and Ω_{i+1}^L. The identification map between Ω_i^R and Ω_{i+1}^L is defined by

$$(2) \qquad\qquad\qquad \eta = \lambda\bar{\xi}/|\xi|^2.$$

Here ξ is an oriented Euclidean coordinate centered on x_i^R, and η is an oriented Euclidean coordinate centered on x_{i+1}^L. $\xi \mapsto \bar{\xi}$ is any reflection. R_i^- and R_i^+ are identified with L_{i+1}^+ and L_{i+1}^-, respectively. The connected sum space $\sharp_{i\in\mathbb{Z}} X_i$ is covered by the open sets U_i:

$$\sharp_{i\in\mathbb{Z}} X_i = \bigcup_{i\in\mathbb{Z}} U_i.$$

Since the identification map (2) is conformal, $\sharp_{i\in\mathbb{Z}} X_i$ has a natural conformal structure which is compatible with the metrics of the given manifolds X_i. (Note that ASD condition is conformally invariant.)

Next we will define a gluing of the principal G bundles E_i.

DEFINITION 2.6. We define the *gluing parameter space* GlP and the *effective gluing parameter space* EGlP by

$$\mathrm{GlP} := \prod_{i\in\mathbb{Z}} \mathrm{Hom}_G((E_i)_{x_i^R}, (E_{i+1})_{x_{i+1}^L}) \cong G^{\mathbb{Z}},$$

$$\mathrm{EGlP} := \prod_{i\in\mathbb{Z}} (\mathrm{Hom}_G((E_i)_{x_i^R}, (E_{i+1})_{x_{i+1}^L})/C(G)) \cong (G/C(G))^{\mathbb{Z}}.$$

Here $\mathrm{Hom}_G((E_i)_{x_i^R}, (E_{i+1})_{x_{i+1}^L})$ is the set of G-isomorphisms between the fibres $(E_i)_{x_i^R}$ and $(E_{i+1})_{x_{i+1}^L}$, and it is isomorphic to G.

If $x \in X_i$ is a point in a small neighborhood of x_i^R, then the fibre $(E_i)_x$ can be identified with $(E_i)_{x_i^R}$ by using the parallel transport along the radial line from x_i^R to x. This "exponential gauge" centered on x_i^R defines a local trivialization of E_i over Ω_i^R. In the same way we define a trivialization of E_i over Ω_i^L by the exponential gauge centered on x_i^L. (For the detail of exponential gauge, see [FU, Chapter 9] or [DK, Section 2.3.1].) Let $\rho = (\rho_i)_{i\in\mathbb{Z}} \in \mathrm{GlP}$ be a gluing parameter. ρ_i is an isomorphism between the fibres $(E_i)_{x_i^R}$ and $(E_{i+1})_{x_{i+1}^L}$. Since we have the identification $\Omega_i^R \cong \Omega_{i+1}^L$ in (2) and the bundle trivializations $E_i|_{\Omega_i^R} \cong \Omega_i^R \times (E_i)_{x_i^R}$ and $E_{i+1}|_{\Omega_{i+1}^L} \cong \Omega_{i+1}^L \times (E_{i+1})_{x_{i+1}^L}$, ρ_i defines an identification map between

$E_i|_{\Omega_i^R}$ and $E_{i+1}|_{\Omega_{i+1}^L}$ covering the base space identification (2). (See the diagram below.) We define a principal G bundle $\boldsymbol{E}(\rho)$ over $\natural_{i\in\mathbb{Z}} X_i$ by using this identification map.

(3)
$$
\begin{array}{ccc}
E_i|_{\Omega_i^R} & \xrightarrow{\text{exp. gauge}} & \Omega_i^R \times (E_i)_{x_i^R} \\
\downarrow & & \downarrow{\scriptstyle \rho_i} \\
E_{i+1}|_{\Omega_{i+1}^L} & \xrightarrow{\text{exp. gauge}} & \Omega_{i+1}^L \times (E_{i+1})_{x_{i+1}^L}
\end{array}
$$

The gluing construction in Section 4 will give an ASD connection $\boldsymbol{A}(\rho)$ on $\boldsymbol{E}(\rho)$ which is "close" to A_i over each U_i:

THEOREM 2.7. *For any positive number $\varepsilon > 0$ and $p \geq 1$, we can choose the parameters N and λ so that there exists an ASD connection $\boldsymbol{A}(\rho)$ on $\boldsymbol{E}(\rho)$ for all $\rho \in \mathrm{GlP}$ such that*

$$\|\boldsymbol{A}(\rho) - A_i\|_{L^p(U_i)} < \varepsilon \quad \text{for all } i \in \mathbb{Z}.$$

(Since there is a natural identification between $\boldsymbol{E}(\rho)$ and E_i over U_i, $\boldsymbol{A}(\rho) - A_i$ is well defined over U_i.) Moreover, if there are infinitely many non-flat ASD connections A_i in the given sequence of gluing data, then the ASD connection $\boldsymbol{A}(\rho)$ has infinite L^2-energy:

$$\|F(\boldsymbol{A}(\rho))\|_{L^2} = +\infty.$$

In the above, the norm $\|\cdot\|_{L^p(U_i)}$ is defined by using the Riemannian metric on X_i. Since the L^2-norm of a 2-form is conformally invariant, the L^2-norm of the curvature $\|F(\boldsymbol{A}(\rho))\|_{L^2}$ is well-defined. ($\natural_{i\in\mathbb{Z}} X_i$ has the natural conformal structure.) We will prove a more precise result in Section 4.

In Section 5 we will discuss a moduli problem. Our main result is the following. (Here the parameters N and λ are chosen appropriately.)

THEOREM 2.8. *Let ρ and ρ' be two gluing parameters. Then $\boldsymbol{A}(\rho)$ is gauge equivalent to $\boldsymbol{A}(\rho')$ if and only if $[\rho] = [\rho']$ in EGlP. Here the gauge equivalence is defined by usual smooth bundle maps between the principal G bundles $\boldsymbol{E}(\rho)$ and $\boldsymbol{E}(\rho')$.*

The effective gluing parameter space $\mathrm{EGlP} = (G/C(G))^{\mathbb{Z}}$ has infinite degrees of freedom. Hence we can consider that our ASD connections $\boldsymbol{A}(\rho)$ have infinite dimensional parameter space.

2.3. Outline of the alternating method

We will construct the ASD connection $\boldsymbol{A}(\rho)$ by applying "Donaldson's alternating method" in [D2]. Here we explain the basic idea of it. The alternating method is an iterative process for solving the ASD equation. First we construct the initial approximate solution $\boldsymbol{A}^{(0)}(\rho)$ on $\boldsymbol{E}(\rho)$ by using cut-off functions. Of course, this $\boldsymbol{A}^{(0)}(\rho)$ is not ASD in general, i.e., $F^+(\boldsymbol{A}^{(0)}(\rho)) \neq 0$. The alternating method iteratively improves this $\boldsymbol{A}^{(0)}(\rho)$ as follows;

Step 0: To cancel the "error term" $F^+(\boldsymbol{A}^{(0)}(\rho))$, we slightly perturb $\boldsymbol{A}^{(0)}(\rho)$ over each X_{2i} and construct $\boldsymbol{A}^{(1)}(\rho)$ on $\boldsymbol{E}(\rho)$ by using these perturbations and cut-off functions. The new error term $F^+(\boldsymbol{A}^{(1)}(\rho))$ becomes smaller than the old error term $F^+(\boldsymbol{A}^{(0)}(\rho))$.

Step 1: To cancel the error term $F^+(\boldsymbol{A}^{(1)}(\rho))$, we slightly perturb $\boldsymbol{A}^{(1)}(\rho)$ over each X_{2i+1} and construct $\boldsymbol{A}^{(2)}(\rho)$ on $\boldsymbol{E}(\rho)$ by using these perturbations and cut-off functions. The new error term $F^+(\boldsymbol{A}^{(2)}(\rho))$ becomes smaller than the old error term $F^+(\boldsymbol{A}^{(1)}(\rho))$.

Step 2 is the same as Step 0, and Step 3 is the same as Step 1. In general,

Step $(2n)$: We slightly perturb $\boldsymbol{A}^{(2n)}(\rho)$ over each "even" manifold X_{2i} and construct $\boldsymbol{A}^{(2n+1)}(\rho)$.

Step $(2n+1)$: We slightly perturb $\boldsymbol{A}^{(2n+1)}(\rho)$ over each "odd" manifold X_{2i+1} and construct $\boldsymbol{A}^{(2n+2)}(\rho)$.

The error terms $F^+(\boldsymbol{A}^{(n)}(\rho))$ go to 0 as n goes to infinity, and the sequence of the approximate solutions $\boldsymbol{A}^{(n)}(\rho)$ converges to the desired ASD connection $\boldsymbol{A}(\rho)$. This is the outline of the alternating method in Section 4.

§3. Main estimates

In this section we establish basic estimates for the proofs of Theorems 2.7 and 2.8. These are essentially the reproduction of the estimates in [D2, (4.22) Lemma and (4.24) Remarks]. We follow the arguments of [D2, IV (iii)].

Fix $p > 6$ and let (X, x^L, x^R, E, A) be a gluing data. We have open sets $L^\pm$, $R^\pm$ and U on X. Let us consider the following map over X:

$$
(4) \quad
\begin{aligned}
L_1^p(\Omega^1(\mathrm{ad}\,E)) \times L^{2p}(\Omega^1(\mathrm{ad}\,E)) &\longrightarrow L^p(\Omega^0(\mathrm{ad}\,E) \oplus \Omega^+(\mathrm{ad}\,E)), \\
(b, a) &\longmapsto D_A b + [a \wedge b]^+ + (b \wedge b)^+.
\end{aligned}
$$

Here D_A is the differential operator $d_A^* + d_A^+ : \Omega^1(\mathrm{ad}\, E) \to \Omega^0(\mathrm{ad}\, E) \oplus \Omega^+(\mathrm{ad}\, E)$. Since we suppose that A is acyclic in Definition 2.3, D_A becomes an invertible map from L_1^p to L^p. Hence if $a \in \Omega^1(\mathrm{ad}\, E)$ is small in L^{2p} and $\sigma \in \Omega^+(\mathrm{ad}\, E)$ is small in L^p, then we can solve the following equation for $b \in \Omega^1(\mathrm{ad}\, E)$:

$$(5) \qquad D_A b + [a \wedge b]^+ + (b \wedge b)^+ = -\sigma,$$

and we get a solution b which satisfies

$$(6) \qquad \|b\|_{L_1^p(X)} \le \mathrm{const} \cdot \|\sigma\|_{L^p(X)}.$$

(This is because $b = 0$ is a solution to (5) if $\sigma = 0$.)

Suppose that a is a smooth form defined on U and that the self-dual part of the curvature $\sigma := F^+(A + a)$ is supported in $L^+ \cup R^+$. (σ is also defined on U.) We extend a and σ to X by zero, and set

$$\delta := \|\sigma\|_{L^\infty(X)}.$$

The volume of $(L^+ \cup R^+)$ which contains the support of σ is bounded by $\mathrm{const} \cdot N^4 \lambda^2$. Hence if a is small in L^{2p} and δ is bounded by some constant, then we have a small solution b to (5) for sufficiently small λ. In the later gluing argument σ will be the "error term" and b will be the "perturbation" described in Section 2.3.

Let us introduce a cut-off function ψ which satisfies $\mathrm{supp}(d\psi) \subset L^- \cup R^-$ and $\psi = 0$ on $X \setminus U$ and $\psi = 1$ on $U \setminus (L^- \cup R^-)$. We can choose ψ such that

$$(7) \qquad |d\psi| \le \frac{20N}{\sqrt{\lambda}}.$$

(Note that $20N/\sqrt{\lambda}$ is a large number because we choose λ very small.) We define the "new error term" τ by $\tau := F^+(A + a + \psi b)$ on U.

$$\begin{aligned}
\tau &= F^+(A + a) + d_A^+(\psi b) + \psi[a \wedge b]^+ + \psi^2(b \wedge b)^+ \\
&= \sigma + (d\psi \wedge b)^+ + \psi(d_A^+ b + [a \wedge b]^+ + (b \wedge b)^+) + \psi(\psi - 1)(b \wedge b)^+ \\
&= (1 - \psi)\sigma + (d\psi \wedge b)^+ + \psi(\psi - 1)(b \wedge b)^+ \\
&= (d\psi \wedge b)^+ + \psi(\psi - 1)(b \wedge b)^+, \quad (\mathrm{supp}(\sigma) \cap \mathrm{supp}(1 - \psi) = \emptyset).
\end{aligned}$$

Hence τ is supported in $L^- \cup R^-$, and we have

$$(8) \qquad \|\tau\|_{L^\infty} \le \frac{20N}{\sqrt{\lambda}} \|b|_{L^- \cup R^-}\|_{L^\infty} + \|b|_{L^- \cup R^-}\|_{L^\infty}^2.$$

Note that the "old" error term σ is supported in $L^+ \cup R^+$ and that the "new" error term τ is supported in $L^- \cup R^-$.

PROPOSITION 3.1. *For any $\kappa > 0$, there is $\eta > 0$ such that if $\delta \leq \kappa$, $\|a\|_{L^{2p}} \leq \eta$ and $\lambda \ll 1$, then we have the following*:

(i) $\|b\|_{L^{2p}} \leq C_1(N) \cdot \lambda^{(p+2)/2p} \cdot \delta$.

(ii) $\|\tau\|_{L^\infty} \leq C_2 \cdot K_N \cdot N^4 \cdot \delta$.   *Here we put $K_N := N/(N - N^{-1})^3$.*

(iii) $\|\psi \cdot b\|_{L_1^p} \leq C_3(N) \cdot \lambda^{2/p} \cdot \delta$.

(iv) $\|F(A + a + \psi \cdot b) - F(A + a)\|_{L^2(U)} \leq C_4(N) \cdot \lambda \cdot \delta$.

Here $C_1(N)$, $C_3(N)$, $C_4(N)$ are constants depending on N, and C_2 is a constant independent of N. (The fact that C_2 is independent of N is essential to the later argument.)

Proof. The proof follows [D2, (4.22) Lemma]. In our case, some care must be paid to interactions between $L^\pm$ and $R^\mp$.

(i) This follows from (6) and the Sobolev embedding $L_1^{4p/(p+2)} \hookrightarrow L^{2p}$. (Note that this Sobolev embedding *does not* hold at $p = \infty$.)

$$\|b\|_{L^{2p}} \leq \text{const} \cdot \|b\|_{L_1^{4p/(p+2)}}$$
$$\leq \text{const} \cdot \|\sigma\|_{L^{4p/(p+2)}}$$
$$\leq C_1(N) \cdot \lambda^{(p+2)/2p} \cdot \delta.$$

(ii) The estimate of the L^∞ norm of τ can be derived from the L^∞ norm of b over $L^- \cup R^-$. (See (8).) First we will estimate a "leading term" of b. Let $\tilde{b}$ be the solution to

$$D_A \tilde{b} = -\sigma.$$

Since $D_A = d_A^* \oplus d_A^+$ is invertible, we have the Green kernel $G(x, y)$ for D_A:

$$\tilde{b}(x) = -\int_X G(x, y) \cdot \sigma(y) \, dy.$$

$G(x, y)$ has a singularity along the diagonal such that

$$|G(x, y)| \leq \text{const} \cdot \frac{1}{d(x, y)^3}.$$

Here $d(x, y)$ is the distance between x and y. This is because D_A is a *first order* elliptic differential operator. (For the details, see [D2, (4.22) Lemma].) In the complement of a neighborhood of the diagonal, $G(x, y)$ is bounded.

Let us consider the case $x \in R^-$. Since σ is supported in $L^+ \cup R^+$,

$$\tilde{b}(x) = -\int_{L^+} G(x,y) \cdot \sigma(y)\, dy - \int_{R^+} G(x,y) \cdot \sigma(y)\, dy.$$

Because the distance between R^- and L^+ is sufficiently large, the first term can be easily estimated:

$$\left| \int_{L^+} G(x,y) \cdot \sigma(y)\, dy \right| \le \mathrm{const} \cdot \mathrm{vol}(L^+) \cdot \delta \le \mathrm{const} \cdot N^4 \cdot \lambda^2 \cdot \delta.$$

This implies that the interaction between R^- and L^+ is comparatively small. The second term is more delicate. Since $d(R^+, R^-) = N\sqrt{\lambda} - N^{-1}\sqrt{\lambda}$, we have

$$\left| \int_{R^+} G(x,y) \cdot \sigma(y)\, dy \right| \le \mathrm{const} \cdot \mathrm{vol}(R^+) \cdot \frac{\delta}{(N\sqrt{\lambda} - N^{-1}\sqrt{\lambda})^3}$$

$$\le \mathrm{const} \cdot K_N \cdot N^3 \cdot \sqrt{\lambda} \cdot \delta, \quad \left(K_N = \frac{N}{(N - N^{-1})^3} \right).$$

Using (1) in Section 2.2, we get

$$|\tilde{b}(x)| \le \mathrm{const} \cdot K_N \cdot N^3 \cdot \sqrt{\lambda} \cdot \delta.$$

We have the same estimate in the case $x \in L^-$.

Next we will estimate the "lower term". Set $b = \tilde{b} + \beta$. Then β satisfies

$$D_A\beta + [a \wedge b]^+ + (b \wedge b)^+ = 0.$$

From the Sobolev embeddings,

$$\|\beta\|_{L^\infty} \le \mathrm{const} \cdot \|D_A\beta\|_{L^6}, \quad (L_1^6 \hookrightarrow L^\infty),$$
$$\le \mathrm{const} \cdot (\|a\|_{L^{12}} \cdot \|b\|_{L^{12}} + \|b\|_{L^{12}}^2)$$
$$\le \mathrm{const} \cdot (\|a\|_{L^{2p}} \cdot \|b\|_{L_1^3} + \|b\|_{L_1^3}^2), \quad (L_1^3 \hookrightarrow L^{12}),$$
$$\le \mathrm{const} \cdot (\eta \cdot \|b\|_{L_1^3} + \|b\|_{L_1^3}^2).$$

From (6) we have

$$\|b\|_{L_1^3} \le \mathrm{const} \cdot \|\sigma\|_{L^3} \le \mathrm{const} \cdot N^{4/3} \cdot \lambda^{2/3} \cdot \delta.$$

Hence

$$\|\beta\|_{L^\infty} \le C(N) \cdot \lambda^{2/3} \cdot \delta, \quad (C(N) \text{ is a constant depending on } N).$$

Since $C(N) \cdot \lambda^{2/3} \ll \sqrt{\lambda}$, $\tilde{b}$ is the leading term of b over $R^- \cup L^-$:

(9) $$\|b|_{R^- \cup L^-}\|_{L^\infty} \le \text{const} \cdot K_N \cdot N^3 \cdot \sqrt{\lambda} \cdot \delta.$$

Substituting this into (8), we conclude that

$$\|\tau\|_{L^\infty} \le \frac{20N}{\sqrt{\lambda}} \cdot \|b|_{R^- \cup L^-}\|_{L^\infty} + \|b|_{R^- \cup L^-}\|_{L^\infty}^2 \le C_2 \cdot K_N \cdot N^4 \cdot \delta.$$

(iii) Since $\nabla_A(\psi \cdot b) = d\psi \otimes b + \psi \cdot \nabla_A b$ and $|d\psi| \le 20N/\sqrt{\lambda}$,

$$\|\psi \cdot b\|_{L_1^p} \le \frac{20N}{\sqrt{\lambda}} \cdot \text{vol}(L^- \cup R^-)^{1/p} \cdot \|b|_{L^- \cup R^-}\|_{L^\infty} + \|b\|_{L_1^p}$$

$$\le \text{const} \cdot N^{1+4/p} \cdot \lambda^{-1/2+2/p} \cdot \|b|_{L^- \cup R^-}\|_{L^\infty} + \|b\|_{L_1^p}.$$

From (6) and (9)

$$\|\psi \cdot b\|_{L_1^p} \le C_3(N) \cdot \lambda^{2/p} \cdot \delta.$$

(iv)

$$F(A + a + \psi b) = F(A + a) + d_A(\psi b) + \psi[a \wedge b] + \psi^2(b \wedge b).$$

Hence

$$\|F(A + a + \psi b) - F(A + a)\|_{L^2(U)} \le \|\psi b\|_{L_1^2} + \sqrt{2}\|a\|_{L^4} \cdot \|b\|_{L^4} + \sqrt{2}\|b\|_{L^4}^2.$$

The arguments in (i) and (iii) show

$$\|\psi b\|_{L_1^2} \le C(N) \cdot \lambda \cdot \delta, \quad \|b\|_{L^4} \le C'(N) \cdot \lambda \cdot \delta.$$

It follows that

$$\|F(A + a + \psi b) - F(A + a)\|_{L^2(U)} \le C_4(N) \cdot \lambda \cdot \delta. \qquad \square$$

Proposition 3.1 is sufficient to prove Theorem 2.7, but Theorem 2.8 needs more. Next we will establish a "parametrized version" of Proposition 3.1, which is the key estimate for the proof of Theorem 2.8. Here we follow the arguments in [D2, (4.24) Remarks].

Suppose that a and σ smoothly depend on a parameter $t \in [0, 1]$: $a = a(t)$, $\sigma = \sigma(t)$. (To be precise, it is enough to assume that $a : [0, 1] \to L^{2p}$ and $\sigma : [0, 1] \to L^\infty$ are smooth maps.) Set $\delta = \delta(t) = \|\sigma(t)\|_{L^\infty(X)}$. If $\delta \le \kappa$ and $\|a\|_{L^{2p}(X)} \le \eta$ for all t, (κ and η are the constants in Proposition 3.1), then we have the following proposition.

PROPOSITION 3.2. *If we choose η and λ sufficiently small, then we have the following estimates*:

(i) $\left\|\dfrac{\partial b}{\partial t}\right\|_{L^{2p}(X)} \le C_5(N) \cdot \lambda^{(p+2)/2p}\left(\left\|\dfrac{\partial\sigma}{\partial t}\right\|_{L^\infty(X)} + \delta \cdot \left\|\dfrac{\partial a}{\partial t}\right\|_{L^{2p}(X)}\right).$

(ii) $\left\|\dfrac{\partial\tau}{\partial t}\right\|_{L^\infty(X)} \le C_6 \cdot K_N \cdot N^4 \cdot \left\|\dfrac{\partial\sigma}{\partial t}\right\|_{L^\infty(X)} + C_7(N) \cdot \lambda^{1/p} \cdot \delta \cdot \left\|\dfrac{\partial a}{\partial t}\right\|_{L^{2p}(X)}.$

Here $C_5(N)$ and $C_7(N)$ are constants depending on N and C_6 is a constant independent of N.

Proof. (i) It is clear that $b : [0,1] \to L_1^p$ is smooth. Differentiating the equation $D_A b + [a \wedge b]^+ + (b \wedge b)^+ = -\sigma$ with respect to t, we get

$$D_A\left(\frac{\partial b}{\partial t}\right) + \left[\frac{\partial a}{\partial t} \wedge b\right]^+ + \left[a \wedge \frac{\partial b}{\partial t}\right]^+ + \left(\frac{\partial b}{\partial t} \wedge b\right)^+ + \left(b \wedge \frac{\partial b}{\partial t}\right)^+ = -\frac{\partial\sigma}{\partial t}.$$

Put $D'_A(\,\cdot\,) = D_A(\,\cdot\,) + [a \wedge \cdot\,]^+ + (\,\cdot \wedge b)^+ + (b \wedge \cdot\,)^+$. Then

$$D'_A\left(\frac{\partial b}{\partial t}\right) = -\left[\frac{\partial a}{\partial t} \wedge b\right]^+ - \frac{\partial\sigma}{\partial t}.$$

Since D_A is invertible and $\|a\|_{L^{2p}} < \eta$ and $\|b\|_{L^{2p}} < C_1(N) \cdot \lambda^{(2+p)/2p}\delta$ are sufficiently small,

$$\begin{aligned}
\left\|\frac{\partial b}{\partial t}\right\|_{L^{2p}} &\le \mathrm{const} \cdot \left\|D'_A\left(\frac{\partial b}{\partial t}\right)\right\|_{L^{4p/(p+2)}} \\
&\le \mathrm{const}\left(\left\|\frac{\partial\sigma}{\partial t}\right\|_{L^{4p/(p+2)}} + \left\|\left[\frac{\partial a}{\partial t} \wedge b\right]^+\right\|_{L^{4p/(p+2)}}\right) \\
&\le C(N)\left(\lambda^{(2+p)/2p}\left\|\frac{\partial\sigma}{\partial t}\right\|_{L^\infty} + \left\|\frac{\partial a}{\partial t}\right\|_{L^{2p}}\|b\|_{L^{2p}}\right) \\
&\le C_5(N) \cdot \lambda^{(2+p)/2p}\left(\left\|\frac{\partial\sigma}{\partial t}\right\|_{L^\infty} + \delta \cdot \left\|\frac{\partial a}{\partial t}\right\|_{L^{2p}}\right).
\end{aligned}$$

Here we have used the fact that $\partial\sigma/\partial t$ is supported in $L^+ \cup R^+$ whose volume is $O(\lambda^2)$.

(ii) Differentiating $\tau = (d\psi \wedge b)^+ + \psi(\psi - 1)(b \wedge b)^+$ with respect to t, we have

$$\frac{\partial\tau}{\partial t} = \left(d\psi \wedge \frac{\partial b}{\partial t}\right)^+ + \psi(\psi - 1)\left\{\left(\frac{\partial b}{\partial t} \wedge b\right)^+ + \left(b \wedge \frac{\partial b}{\partial t}\right)^+\right\}.$$

From (7) and (9)

(10)
$$
\begin{aligned}
\left\|\frac{\partial \tau}{\partial t}\right\|_{L^\infty} &\leq \frac{20N}{\sqrt{\lambda}} \left\|\left.\frac{\partial b}{\partial t}\right|_{L^-\cup R^-}\right\|_{L^\infty} + 2\sqrt{2}\cdot \|b|_{L^-\cup R^-}\|_{L^\infty} \left\|\left.\frac{\partial b}{\partial t}\right|_{L^-\cup R^-}\right\|_{L^\infty} \\
&\leq \frac{20N}{\sqrt{\lambda}} \left\|\left.\frac{\partial b}{\partial t}\right|_{L^-\cup R^-}\right\|_{L^\infty} + C(N)\cdot \sqrt{\lambda}\cdot \delta \cdot \left\|\left.\frac{\partial b}{\partial t}\right|_{L^-\cup R^-}\right\|_{L^\infty}.
\end{aligned}
$$

Set $b = \tilde{b} + \beta$ with $D_A \tilde{b} = -\sigma$ as in the proof of Proposition 3.1 (ii). Then

$$
\begin{aligned}
\frac{\partial \tilde{b}}{\partial t}(x) &= -\int_X G(x,y)\cdot \frac{\partial \sigma}{\partial t}(y)\,dy, \quad (G(x,y) \text{ is the Green kernel}), \\
&= -\int_{L^+} G(x,y)\cdot \frac{\partial \sigma}{\partial t}(y)\,dy - \int_{R^+} G(x,y)\cdot \frac{\partial \sigma}{\partial t}(y)\,dy.
\end{aligned}
$$

This can be estimated in the same way as before and we get

$$
\left\|\left.\frac{\partial \tilde{b}}{\partial t}\right|_{L^-\cup R^-}\right\|_{L^\infty} \leq \text{const}\cdot K_N \cdot N^3 \cdot \sqrt{\lambda}\cdot \left\|\frac{\partial \sigma}{\partial t}\right\|_{L^\infty}.
$$

We have $\partial b/\partial t = \partial \tilde{b}/\partial t + \partial \beta/\partial t$ and $\partial \beta/\partial t$ satisfies

$$
D_A\left(\frac{\partial \beta}{\partial t}\right) + \left[\frac{\partial a}{\partial t}\wedge b\right]^+ + \left[a \wedge \frac{\partial b}{\partial t}\right]^+ + \left(\frac{\partial b}{\partial t}\wedge b\right)^+ + \left(b \wedge \frac{\partial b}{\partial t}\right)^+ = 0.
$$

Using the above (i) and the Sobolev embedding, we get

$$
\begin{aligned}
\left\|\frac{\partial \beta}{\partial t}\right\|_{L^\infty} &\leq \text{const}\cdot \left\|D_A\left(\frac{\partial \beta}{\partial t}\right)\right\|_{L^6}, \quad (L_1^6 \hookrightarrow L^\infty), \\
&\leq C(N)\cdot \lambda^{(2+p)/2p}\left(\left\|\frac{\partial \sigma}{\partial t}\right\|_{L^\infty} + \delta \cdot \left\|\frac{\partial a}{\partial t}\right\|_{L^{2p}}\right).
\end{aligned}
$$

Hence

$$
\left\|\left.\frac{\partial b}{\partial t}\right|_{L^-\cup R^-}\right\|_{L^\infty} \leq C\cdot K_N N^3 \sqrt{\lambda}\left\|\frac{\partial \sigma}{\partial t}\right\|_{L^\infty} + C(N)\cdot \lambda^{(2+p)/2p}\cdot \delta \left\|\frac{\partial a}{\partial t}\right\|_{L^{2p}}.
$$

Substituting this into (10), we conclude that

$$
\left\|\frac{\partial \tau}{\partial t}\right\|_{L^\infty} \leq C_6 \cdot K_N N^4 \left\|\frac{\partial \sigma}{\partial t}\right\|_{L^\infty} + C_7(N)\cdot \lambda^{1/p}\cdot \delta \left\|\frac{\partial a}{\partial t}\right\|_{L^{2p}}. \qquad \square
$$

Here we fix the value of N. Any value of N with

$$C_2 \cdot K_N \leq \frac{1}{2}, \quad C_6 \cdot K_N \leq \frac{1}{2},$$

will do. (Note that $K_N = N/(N - N^{-1})^3$ converges to zero as $N \to \infty$.) So let N be a large positive number which satisfies the above. (This choice depends on the gluing data (X, x^L, x^R, E, A).) Then Proposition 3.1 (ii) and Proposition 3.2 (ii) become:

(ii)$'$ $\|\tau\|_{L^\infty} \leq \dfrac{1}{2} \cdot N^4 \cdot \delta$,

(ii)$'$ $\left\|\dfrac{\partial \tau}{\partial t}\right\|_{L^\infty} \leq \dfrac{1}{2} \cdot N^4 \cdot \left\|\dfrac{\partial \sigma}{\partial t}\right\|_{L^\infty} + C_7(N) \cdot \lambda^{1/p} \cdot \delta \cdot \left\|\dfrac{\partial a}{\partial t}\right\|_{L^{2p}}$.

Remark 3.3. In fact, the parameter N can be chosen independent of the gluing data (X, x^L, x^R, E, A). See [D2, p. 308, (4.14)] and the proof of [D2, (4.22) Lemma]. But we don't need this fact in this paper.

§4. Donaldson's alternating method

In this section we prove Theorem 2.7. Let $(X_i, x_i^L, x_i^R, E_i, A_i)_{i \in \mathbb{Z}}$ be a sequence of gluing data of finite type as in Section 2. The finite type condition means that we can apply the estimates in Section 3 to all $(X_i, x_i^L, x_i^R, E_i, A_i)$ with the *same* constants. (This is the reason for introducing this notion.)

To begin with, we make one remark on the conformal transformation (2) which identifies Ω_i^R and Ω_{i+1}^L:

Remark 4.1. The coordinate transformation (2) is only conformal and not an isometry. Hence the metric of X_i is different from the metric of X_{i+1} over the identified region $\Omega_i^R \cong \Omega_{i+1}^L$. The derivative of (2) over $\Omega_i^R = \{kN^{-1}\sqrt{\lambda} < |\xi| < k^{-1}N\sqrt{\lambda}\}$ has the following bound independent of λ:

$$\left\|\frac{\partial \eta}{\partial \xi}\right\| = \frac{\lambda}{|\xi|^2} \leq \frac{N^2}{k^2}, \quad \left\|\frac{\partial \xi}{\partial \eta}\right\| = \frac{\lambda}{|\eta|^2} \leq \frac{N^2}{k^2}.$$

Here $\|\cdot\|$ is the operator norm. In particular, if $\xi \in R_i^- = L_{i+1}^+$, then

$$\left\|\frac{\partial \xi}{\partial \eta}\right\| \leq \frac{1}{N^2}.$$

It follows that

$$\|\omega\|_{L^\infty(L_{i+1}^+)} \leq \frac{1}{N^4} \|\omega\|_{L^\infty(R_i^-)}$$

for a 2-form ω under the identification $R_i^- = L_{i+1}^+$.

Let $\rho = (\rho_i)_{i \in \mathbb{Z}} \in \mathrm{GlP}$ be a gluing parameter, $(\rho_i \in \mathrm{Hom}_G((E_i)_{x_i^R},$ $(E_{i+1})_{x_{i+1}^L}))$, and $\boldsymbol{E}(\rho)$ be the principal G bundle over $\sharp_{i \in \mathbb{Z}} X_i$ defined in Section 2.2. As we explained in Section 2.3, we will inductively construct a sequence of connections $\boldsymbol{A}^{(0)}(\rho), \boldsymbol{A}^{(1)}(\rho), \boldsymbol{A}^{(2)}(\rho), \dots$, on $\boldsymbol{E}(\rho)$ which converges to an ASD connection $\boldsymbol{A}(\rho)$. This is a generalization of the alternating method in [D2, IV (iv)]. At the n-th stage of the iteration we will have a connection $\boldsymbol{A}^{(n)}(\rho)$ which is expressed by

$$\boldsymbol{A}^{(n)}(\rho) = (A_i + a_i^{(n)}(\rho))_{i \in \mathbb{Z}}.$$

Here $a_i^{(n)}(\rho)$ is an element of $\Omega^1(\mathrm{ad}\, E_i)$ over U_i and we have $A_i + a_i^{(n)}(\rho) = A_{i+1} + a_{i+1}^{(n)}(\rho)$ over $U_i \cap U_{i+1}$ under the identification ρ_i. Then they compatibly define the connection $\boldsymbol{A}^{(n)}(\rho)$ on $\boldsymbol{E}(\rho)$.

Let us consider the following "inductive hypotheses":

HYPOTHESES 4.2. (i) Let $\sigma^{(n)} := F^+(\boldsymbol{A}^{(n)}(\rho))$ be the self-dual curvature of $\boldsymbol{A}^n(\rho)$, and set

$$\delta_n := \begin{cases} \sup_{i \in \mathbb{Z}} \|\sigma_{2i}^{(n)}\|_{L^\infty(X_{2i})} & \text{if } n \text{ is even.} \\ \sup_{i \in \mathbb{Z}} \|\sigma_{2i+1}^{(n)}\|_{L^\infty(X_{2i+1})} & \text{if } n \text{ is odd.} \end{cases}$$

Here $\sigma_i^{(n)}$ is the restriction of $\sigma^{(n)}$ on U_i. (We extend $\sigma_i^{(n)}$ to X_i by zero.) We suppose that $\sigma^{(n)}$ is supported as follows:

$$\mathrm{supp}(\sigma^{(n)}) \subset \begin{cases} \coprod_{i \in \mathbb{Z}} (L_{2i}^+ \cup R_{2i}^+) & \text{if } n \text{ is even.} \\ \coprod_{i \in \mathbb{Z}} (L_{2i+1}^+ \cup R_{2i+1}^+) & \text{if } n \text{ is odd.} \end{cases}$$

And we suppose that δ_n has the following bound:

$$\delta_n \leq \kappa, \quad (\kappa \text{ is the constant in Proposition 3.1 and will be given later}).$$

(ii) We suppose that $a_i^{(n)}$ has the following bound:

$$\|a_i^{(n)}\|_{L^{2p}(U_i)} \leq \eta, \quad (\eta \text{ is also the constant in Proposition 3.1}).$$

If these conditions hold at the n-th stage of the iteration and $\lambda \ll 1$, then we can pass to the $(n+1)$-th stage as follows. If n is even, then we use

the construction in Section 3 on each X_{2i} and get $b_{2i}^{(n)}$. We define $a_i^{(n)}(\rho)$ over U_i by

(11)
$$a_{2i}^{(n+1)}(\rho) := a_{2i}^{(n)}(\rho) + \psi_{2i} \cdot b_{2i}^{(n)},$$
$$a_{2i+1}^{(n+1)}(\rho) := a_{2i+1}^{(n)}(\rho) + \psi_{2i} \cdot \rho_{2i} b_{2i}^{(n)} \rho_{2i}^{-1} + \psi_{2i+2} \cdot \rho_{2i+1}^{-1} b_{2i+2}^{(n)} \rho_{2i+1}.$$

Here ψ_i is a cut-off over X_i as in Section 3. Note that $\psi_{2i} \cdot \rho_{2i} b_{2i}^{(n)} \rho_{2i}^{-1}$ and $\psi_{2i+2} \cdot \rho_{2i+1}^{-1} b_{2i+2}^{(n)} \rho_{2i+1}$ are well-defined over U_{2i+1}. From the arguments in Section 3 and Proposition 3.1 (ii)$'$ at the end of Section 3, we have

$$\text{supp}(\sigma^{(n+1)}) \subset \coprod_{i\in\mathbb{Z}} (R_{2i}^- \cup L_{2i}^-) = \coprod_{i\in\mathbb{Z}} (L_{2i+1}^+ \cup R_{2i+1}^+),$$
$$\|\sigma_{2i}^{(n+1)}\|_{L^\infty(X_{2i})} \leq \frac{1}{2} \cdot N^4 \delta_n.$$

Taking account of Remark 4.1, we have

$$\|\sigma_{2i+1}^{(n+1)}\|_{L^\infty(X_{2i+1})} \leq \frac{1}{2}\delta_n.$$

Thus

(12)
$$\delta_{n+1} \leq \frac{1}{2}\delta_n.$$

If n is odd, then we use Proposition 3.1 over each X_{2i+1} and construct a similar procedure to pass to the $(n+1)$-th stage. The estimate (12) holds as long as we can continue this iteration. Hence

(13)
$$\delta_n \leq \frac{1}{2^n}\delta_0.$$

We start the iteration by defining $\boldsymbol{A}^{(0)}(\rho) = (A_i + a_i^{(0)}(\rho))_{i\in\mathbb{Z}}$ by

(14)
$$a_{2i+1}^{(0)}(\rho) := (\psi_{2i+1} - 1)(A_{2i+1} - \rho_{2i} A_{2i} \rho_{2i}^{-1} - \rho_{2i+1}^{-1} A_{2i+2} \rho_{2i+1}),$$
$$a_{2i}^{(0)}(\rho) := \psi_{2i+1}(-A_{2i} + \rho_{2i}^{-1} A_{2i+1} \rho_{2i}) + \psi_{2i-1}(-A_{2i} + \rho_{2i-1} A_{2i-1} \rho_{2i-1}^{-1}).$$

Here each A_i is the connection matrix in the exponential gauge centered at x_i^L and x_i^R. These expressions are well-defined on U_{2i+1} or U_{2i} and compatible over the overlap regions. The self-dual curvature of $\boldsymbol{A}^{(0)}(\rho)$ is supported in $\coprod(L_{2i+1}^- \cup R_{2i+1}^-) = \coprod(L_{2i}^+ \cup R_{2i}^+)$. Since each $|A_i|$ is $O(\sqrt{\lambda})$

and $|d\psi_i|$ is $O(1/\sqrt{\lambda})$ (cf. (7)), we have a constant κ independent of λ such that

$$(15) \qquad\qquad \|\sigma_i^{(0)}\|_{L^\infty(X_i)} \le \kappa.$$

We take this κ as the constant in Proposition 3.1. The condition (i) in Hypotheses 4.2 is satisfied at $n = 0$. Since $a_i^{(0)}$ is supported in the overlap whose volume is $O(\lambda^2)$, we have

$$(16) \qquad\qquad \|a_i^{(0)}\|_{L^{2p}(U_i)} \le \text{const} \cdot \lambda^{(2+p)/2p}.$$

Therefore the condition (ii) is also satisfied at $n = 0$ if we choose $\lambda \ll 1$. Then we can start the iteration.

The geometric decay (13) shows that the condition (i) in Hypotheses 4.2 is always satisfied as long as the condition (ii) is valid. At the n-th stage of the iteration, if n is even, Proposition 3.1 (i) shows

$$\|a_{2i}^{(n+1)} - a_{2i}^{(n)}\|_{L^{2p}(U_{2i})} \le \text{const} \cdot \lambda^{(2+p)/2p}\delta_n \le \text{const} \cdot \lambda^{(2+p)/2p}\frac{\delta_0}{2^n}.$$

From Remark 4.1,

$$\|a_{2i+1}^{(n+1)} - a_{2i+1}^{(n)}\|_{L^{2p}(U_{2i+1})} \le \text{const} \cdot \lambda^{(2+p)/2p}\frac{\delta_0}{2^n}.$$

We have similar estimates when n is odd, and

$$(17) \quad \text{const} \cdot \lambda^{(2+p)/2p} + \text{const} \cdot \lambda^{(2+p)/2p} \sum_{n\ge 0}\frac{\delta_0}{2^n} \le \text{const} \cdot \lambda^{(2+p)/2p} \ll 1.$$

Hence the condition (ii) in Hypotheses 4.2 is satisfied for any n if we choose λ small enough. Thus we can continue the iteration indefinitely and the sequence $\{a_i^{(n)}(\rho)\}_{n=0}^\infty$ has a limit $a_i(\rho)$ in $L^{2p}(U_i)$.

Proposition 3.1 (iii) and an argument similar to the above show that $\{a_i^{(n)}(\rho)\}_{n=0}^\infty$ converges to $a_i(\rho)$ also in $L_1^p(U_i)$. From the geometric decay (13), the self-dual curvature of $A_i + a_i(\rho)$ becomes 0. Thus we get an ASD connection

$$\boldsymbol{A}(\rho) = (A_i + a_i(\rho))_{i\in\mathbb{Z}}.$$

(17) shows that this is close to A_i over each U_i:

$$(18) \qquad\qquad \|a_i\|_{L^{2p}(U_i)} \le \text{const} \cdot \lambda^{(2+p)/2p}.$$

So we get a conclusion:

THEOREM 4.3. *For all sufficiently small λ, $\boldsymbol{A}(\rho)$ is an ASD connection on $\boldsymbol{E}(\rho)$ such that*

$$\|\boldsymbol{A}(\rho) - A_i\|_{L^{2p}(U_i)} \leq C_8 \cdot \lambda^{(2+p)/2p} \quad \text{for all } i \in \mathbb{Z}.$$

Here C_8 is a constant independent of i, ρ, λ.

In general this ASD connection $\boldsymbol{A}(\rho)$ has infinite energy:

THEOREM 4.4. *If there are infinitely many non-flat ASD connections A_i in the given sequence of gluing data, then the L^2-energy of $\boldsymbol{A}(\rho)$ is infinite: $\|F(\boldsymbol{A}(\rho))\|_{L^2} = +\infty$.*

Proof. We will show that the loss of energy over each U_i during the iteration is sufficiently small. At the n-th stage of the iteration, if n is even, we can apply Proposition 3.1 (iv) to (11) and get

$$\|F(A_{2i} + a_{2i}^{(n+1)}) - F(A_{2i} + a_{2i}^{(n)})\|_{L^2(U_{2i})} \leq \text{const} \cdot \lambda \cdot \delta_n \leq \text{const} \cdot \frac{\lambda}{2^n}.$$

Since $F(A_{2i+1} + a_{2i+1}^{(n+1)}) - F(A_{2i+1} + a_{2i+1}^{(n)})$ is supported in the overlaps and the L^2-norm of a 2-form is conformally invariant,

$$\|F(A_{2i+1} + a_{2i+1}^{(n+1)}) - F(A_{2i+1} + a_{2i+1}^{(n)})\|_{L^2(U_{2i+1})} \leq \text{const} \cdot \frac{\lambda}{2^n}.$$

We have the same estimates in the case that n is odd. Hence

$$\|F(A_i + a_i) - F(A_i + a_i^{(0)})\|_{L^2(U_i)} \leq \text{const} \cdot \lambda \sum_{n \geq 0} \frac{1}{2^n} \leq \text{const} \cdot \lambda.$$

Next we will show that $\|F(A_i + a_i^{(0)}) - F(A_i)\|_{L^2(U_i)}$ is also small. An estimate similar to (15) shows

$$|F(A_i + a_i^{(0)}) - F(A_i)| \leq \text{const}, \quad \text{(independent of } \lambda\text{)}.$$

Since this is supported in the overlaps, we get

$$\|F(A_i + a_i^{(0)}) - F(A_i)\|_{L^2(U_i)} \leq \text{const} \cdot \lambda.$$

Hence

$$\|F(A_i + a_i) - F(A_i)\|_{L^2(U_i)} \leq \text{const} \cdot \lambda.$$

From the finite type condition, we have a positive constant M independent of λ such that $\|F(A_i)\|_{L^2(U_i)} \geq M$ if A_i is not flat and λ is sufficiently small. Therefore if we take $\lambda \ll 1$, then

$$\|F(A_i + a_i)\|_{L^2(U_i)} \geq \frac{1}{2}M$$

for non-flat A_i. Since there are infinitely many non-flat A_i, we conclude that

$$\|F(\boldsymbol{A}(\rho))\|_{L^2} = +\infty. \qquad \square$$

§5. Moduli problem

We will prove Theorem 2.8 in this section. The proof is a generalization of [D2, (4.31) Lemma]. We always suppose that λ is sufficiently small.

We need the following elementary result.

LEMMA 5.1. *Let K be a positive number and ε be a positive number with $\varepsilon \leq 1/100$. Let $\{\alpha_n\}, \{s_n\}, (n = 0, 1, 2, \dots)$, be sequences of non-negative numbers which satisfy the following recurrence inequalities*:

$$\alpha_{n+1} - \alpha_n \leq \varepsilon\Big(s_n + \frac{1}{2^n}\alpha_n + \frac{1}{2^n}K\Big),$$
$$s_{n+1} \leq \frac{1}{2}s_n + \frac{1}{2^n}\alpha_n + \frac{1}{2^{n+1}}K.$$

Suppose that

$$\alpha_0 \leq \varepsilon \cdot K, \quad s_0 \leq K.$$

Then we get

$$\alpha_n \leq 10\varepsilon \cdot K \quad \text{for all } n.$$

Proof. Set $t_n := 2^n s_n$. Then we have $t_0 \leq K$, and the above recurrence inequalities become

$$\alpha_{n+1} - \alpha_n \leq \frac{\varepsilon}{2^n}(t_n + \alpha_n + K),$$
$$t_{n+1} \leq t_n + 2\alpha_n + K.$$

We will inductively prove $\alpha_n \leq 10\varepsilon \cdot K$ for all n. Suppose that this is true if $n \leq n_0$ for some n_0. Then

$$t_n \leq K + 2(10\varepsilon K)n + Kn = K\{1 + (1 + 20\varepsilon)n\}, \quad (n \leq n_0 + 1).$$

Hence

$$\begin{aligned}
\alpha_{n_0+1} - \alpha_0 &= \sum_{n=0}^{n_0} (\alpha_{n+1} - \alpha_n) \\
&\leq \sum_{n \geq 0} \frac{\varepsilon}{2^n} \left[K\{1 + (1 + 20\varepsilon)n\} + 10\varepsilon K + K \right] \\
&= \varepsilon K(6 + 60\varepsilon) \leq 7\varepsilon K.
\end{aligned}$$

Thus

$$\alpha_{n_0+1} \leq \varepsilon K + 7\varepsilon K \leq 10\varepsilon K. \qquad \square$$

PROPOSITION 5.2. *For two gluing parameters* $\rho = (\rho_i)_{i \in \mathbb{Z}}$ *and* $\rho' = (\rho'_i)_{i \in \mathbb{Z}}$, *let* $K := \sup_{i \in \mathbb{Z}} |\rho_i - \rho'_i|$. *Then we have*

$$\|a_i(\rho) - a_i(\rho')\|_{L^{2p}(U_i)} \leq C_9 \cdot \lambda^{(2+p)/2p} K \quad \text{for all } i \in \mathbb{Z}.$$

Here C_9 *is a constant independent of* i, λ, ρ, ρ'.

Proof. Let $\bar{\rho}(t) = (\bar{\rho}_i(t))_{i \in \mathbb{Z}}$, $(0 \leq t \leq 1)$, be a path in GlP such that each $\bar{\rho}_i(t)$ is the geodesic from ρ_i to ρ'_i in G. Since the distance $d(\rho_i, \rho'_i)$ is equivalent to $|\rho_i - \rho'_i|$:

$$C^{-1}|\rho_i - \rho'_i| \leq d(\rho_i, \rho'_i) \leq C|\rho_i - \rho'_i| \quad \text{for some constant } C,$$

we have $|d\bar{\rho}_i/dt| = d(\rho_i, \rho'_i) \leq C \cdot K$.

Let $\boldsymbol{A}^{(n)}(\bar{\rho}(t)) = (A_i + a_i^{(n)}(\bar{\rho}(t)))_{i \in \mathbb{Z}}$ be the connection on $\boldsymbol{E}(\bar{\rho}(t))$ constructed by the iterative process in Section 4; this is parametrized by t and we can apply Proposition 3.2. We define $\alpha_n = \alpha_n(t)$ and $s_n = s_n(t)$ by

$$\alpha_n := \sup_{i \in \mathbb{Z}} \left\| \frac{\partial}{\partial t} a_i^{(n)} \right\|_{L^{2p}(X_i)},$$

$$s_n := \begin{cases} \sup_{i \in \mathbb{Z}} \left\| \frac{\partial}{\partial t} \sigma_{2i}^{(n)} \right\|_{L^{\infty}(X_{2i})} & \text{if } n \text{ is even.} \\ \sup_{i \in \mathbb{Z}} \left\| \frac{\partial}{\partial t} \sigma_{2i+1}^{(n)} \right\|_{L^{\infty}(X_{2i+1})} & \text{if } n \text{ is odd.} \end{cases}$$

From the definition of $a_i^{(0)}$ in (14),

$$\left| \frac{\partial}{\partial t} a_i^{(0)} \right| \leq \text{const} \cdot \sqrt{\lambda} \cdot K.$$

Here we used the fact that the connection matrices in the exponential gauges are $O(\sqrt{\lambda})$. Since $\partial a_i^{(0)}/\partial t$ is supported in the overlaps, we get

$$\left\|\frac{\partial}{\partial t}a_i^{(0)}\right\|_{L^{2p}(X_i)} \leq \mathrm{const}\cdot\lambda^{(2+p)/2p}K.$$

In a similar way, we have

$$\left\|\frac{\partial}{\partial t}\sigma_i^{(0)}\right\|_{L^{\infty}(X_i)} \leq \mathrm{const}\cdot K.$$

Hence

(19) $$\alpha_0 \leq \mathrm{const}\cdot\lambda^{(2+p)/2p}K,\quad s_0 \leq \mathrm{const}\cdot K.$$

Next we will establish recurrence inequalities on α_n and s_n. If n is even, then Propositions 3.1 and 3.2 give bounds for $\|b_{2i}^{(n)}\|_{L^{2p}(X_{2i})}$ and $\|\partial b_{2i}^{(n)}/\partial t\|_{L^{2p}(X_{2i})}$. Then, from the inductive definitions (11) and the geometric decay (13),

$$\left\|\frac{\partial}{\partial t}a_{2i}^{(n+1)}\right\|_{L^{2p}(X_{2i})} \leq \alpha_n + \mathrm{const}\cdot\lambda^{(2+p)/2p}\left(s_n + \frac{1}{2^n}\alpha_n\right),$$

$$\left\|\frac{\partial}{\partial t}a_{2i+1}^{(n+1)}\right\|_{L^{2p}(X_{2i+1})} \leq \alpha_n + \mathrm{const}\cdot\frac{\lambda^{(2+p)/2p}}{2^n}K$$
$$+ \mathrm{const}\cdot\lambda^{(2+p)/2p}\left(s_n + \frac{1}{2^n}\alpha_n\right).$$

It follows that

(20) $$\alpha_{n+1} - \alpha_n \leq \mathrm{const}\cdot\lambda^{(2+p)/2p}\left(s_n + \frac{1}{2^n}\alpha_n + \frac{K}{2^n}\right).$$

Similarly, from Proposition 3.2 (ii)$'$ at the end of Section 3,

$$\left\|\frac{\partial}{\partial t}\sigma_{2i}^{(n+1)}\right\|_{L^{\infty}(X_{2i})} \leq \frac{N^4}{2}s_n + \mathrm{const}\cdot\frac{\lambda^{1/p}}{2^n}\alpha_n.$$

Since $\sigma_{2i+1}^{(n+1)} = \bar{\rho}_{2i}\sigma_{2i}^{(n+1)}\bar{\rho}_{2i}^{-1}$ on $U_{2i}\cap U_{2i+1}$ and $\sigma_{2i+1}^{(n+1)} = \bar{\rho}_{2i+1}^{-1}\sigma_{2i+2}^{(n+1)}\bar{\rho}_{2i+1}$ on $U_{2i+1}\cap U_{2i+2}$, we get

$$\left\|\frac{\partial}{\partial t}\sigma_{2i+1}^{(n+1)}\right\|_{L^{\infty}(X_{2i+1})} \leq \mathrm{const}\cdot\delta_{n+1}\cdot K + \frac{1}{2}s_n + \mathrm{const}\cdot\frac{\lambda^{1/p}}{2^n}\alpha_n$$
$$\leq \mathrm{const}\cdot\frac{K}{2^{n+1}} + \frac{1}{2}s_n + \mathrm{const}\cdot\frac{\lambda^{1/p}}{2^n}\alpha_n.$$

Here we have used Remark 4.1. This shows

$$(21) \qquad s_{n+1} \le \frac{1}{2}s_n + \text{const} \cdot \frac{\lambda^{1/p}}{2^n}\alpha_n + \text{const} \cdot \frac{K}{2^{n+1}}.$$

The inequalities (20), (21) hold also in the case that n is odd. So we can apply Lemma 5.1 with initial values (19) and $\varepsilon = \text{const} \cdot \lambda^{(2+p)/2p}$. Then we get

$$\alpha_n \le \text{const} \cdot \lambda^{(2+p)/2p}K.$$

Therefore

$$\|a_i^{(n)}(\rho) - a_i^{(n)}(\rho')\|_{L^{2p}(U_i)} = \left\| \int_0^1 \frac{\partial}{\partial t}a_i^{(n)}\,dt \right\|_{L^{2p}(U_i)} \le \text{const} \cdot \lambda^{(2+p)/2p}K,$$

for all i and n. So we get a conclusion:

$$\|a_i(\rho) - a_i(\rho')\|_{L^{2p}(U_i)} \le \text{const} \cdot \lambda^{(2+p)/2p}K. \qquad \square$$

The following is Theorem 2.8.

THEOREM 5.3. *For two gluing parameters ρ and ρ', $\boldsymbol{A}(\rho)$ is gauge equivalent to $\boldsymbol{A}(\rho')$ if and only if $[\rho] = [\rho']$ in EGlP.*

Proof. If $[\rho] = [\rho']$ in EGlP, we have $\varepsilon_i \in C(G)$ with $\rho_i' = \varepsilon_i\rho_i$ for all $i \in \mathbb{Z}$. Since the center $C(G)$ acts trivially on connections, it is clear from the construction in Section 4 that we have $a_i(\rho) = a_i(\rho')$. And we can choose $\gamma_i \in C(G)$ such that $\gamma_{i+1} = \varepsilon_i\gamma_i$. Then these γ_i give a gauge transformation $h = (\gamma_i)_{i\in\mathbb{Z}} : \boldsymbol{E}(\rho) \to \boldsymbol{E}(\rho')$ with $h.\boldsymbol{A}(\rho) = \boldsymbol{A}(\rho')$.

Conversely, suppose that there is a gauge transformation $g : \boldsymbol{E}(\rho) \to \boldsymbol{E}(\rho')$ with $g.\boldsymbol{A}(\rho) = \boldsymbol{A}(\rho')$. This means that there is a gauge transformation $g_i : E_i \to E_i$ over each U_i such that

$$d_{A_i}g_i = g_ia_i(\rho) - a_i(\rho')g_i,$$
$$g_{i+1}\rho_i = \rho_i'g_i.$$

Because A_i is irreducible, we have a constant C independent of λ such that

$$\min_{\gamma\in C(G)}\|\gamma g_i - 1\|_{C^0(U_i)} \le C \cdot \|d_{A_i}g_i\|_{L^{2p}(U_i)}.$$

The proof of this inequality is standard. For example, see [D2, (4.31) Lemma]. We can suppose that $\|g_i - 1\|_{C^0(U_i)} \le C \cdot \|d_{A_i} g_i\|_{L^{2p}(U_i)}$ without loss of generality. From (18) and Proposition 5.2,

$$
\begin{aligned}
\|g_i - 1\|_{C^0} &\le C \cdot \|g_i a_i(\rho) - a_i(\rho') g_i\|_{L^{2p}} \\
&\le C \cdot \|g_i - 1\|_{C^0}(\|a_i(\rho)\|_{L^{2p}} + \|a_i(\rho')\|_{L^{2p}}) + C \cdot \|a_i(\rho) - a_i(\rho')\|_{L^{2p}} \\
&\le \mathrm{const} \cdot \lambda^{(2+p)/2p} \|g_i - 1\|_{C^0} + \mathrm{const} \cdot \lambda^{(2+p)/2p} K.
\end{aligned}
$$

Since λ is sufficiently small, we get

$$
\|g_i - 1\|_{C^0} \le \mathrm{const} \cdot \lambda^{(2+p)/2p} K.
$$

On the other hand, the "compatibility condition" $g_{i+1}\rho_i = \rho_i' g_i$ means

$$
\rho_i' - \rho_i = (g_{i+1} - 1)\rho_i g_i^{-1} + \rho_i g_i^{-1}(1 - g_i).
$$

Hence

$$
\begin{aligned}
|\rho_i' - \rho_i| &\le \mathrm{const} \cdot (\|g_{i+1} - 1\|_{C^0(U_{i+1})} + \|g_i - 1\|_{C^0(U_i)}) \\
&\le \mathrm{const} \cdot \lambda^{(2+p)/2p} K.
\end{aligned}
$$

Since this holds for all i, it follows

$$
K \le \mathrm{const} \cdot \lambda^{(2+p)/2p} K.
$$

If we choose λ small enough, then this shows $K = 0$ and $\rho = \rho'$. □

Remark 5.4. So far we have studied only acyclic connections. But this is just for simplicity. The condition (iv) in Definition 2.3 can be replaced with

(iv)$'$ A is a regular ASD connection on E.

If we glue an infinite number of regular ASD connections, then the resulting ASD connections are parametrized by

$$
\mathrm{EGlP} \times \prod_{i \in \mathbb{Z}} B_i \subset \mathrm{EGlP} \times \prod_{i \in \mathbb{Z}} H^1_{A_i}.
$$

Here B_i is a small ball centered at the origin in $H^1_{A_i} := \ker D_{A_i}$. This generalization is a simple application of the methods in this paper and [D2, pp. 324–325]. The proof is routine, so I omit the details. □

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