ON THE PRODUCT OF VECTOR MEASURES

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Let μ and ν be measures defined on some σ-algebras with values in locally convex topological vector spaces X and Y, respectively. It is possible [1] to construct their product $\lambda = \mu \times \nu$ as a measure on a σ-algebra if λ is allowed to take its values in $X \otimes_\varepsilon Y$, the completion of $X \otimes Y$ in the topology of bi-equicontinuous convergence. The reason is, roughly speaking, that the topology of bi-equicontinuous convergence on $X \otimes Y$ is coarse enough to make λ σ-additive and the completion $X \otimes_\varepsilon Y$ is big enough to accommodate all values of λ. Here we are going to improve the result by introducing a finer topology on $X \otimes Y$ in which λ will be σ-additive and such that all values of λ will belong to the completion of $X \otimes Y$ under that topology. The topology in question is obtained by a slight modification from a topology considered for the first time in the work [3] of Jacobs. Curiously enough, the proof of the improved result is simpler than that of [1] and reduces almost to a direct observation avoiding duality arguments.

Let X and Y be locally convex topological vector spaces. Let the topology of X be given by a family P of semi-norms and that of Y by a family Q of semi-norms. For every u in $X \otimes Y$ and for $p \in P$, $q \in Q$ put

$$p\sigma_l q(u) = \inf\sup p\left(\sum_{i=1}^{k} \alpha_i q(y_i) x_i\right)$$

$$p\sigma_r q(u) = \inf\sup q\left(\sum_{i=1}^{k} \alpha_i p(x_i) y_i\right),$$

where, in both formulas, the supremum is taken over all choices of numbers α_i such that $|\alpha_i| \leqq 1$, $i = 1, 2, \cdots, k$, and the infimum is taken over all expression of u in the form

$$u = \sum_{i=1}^{k} x_i \otimes y_i \tag{1}$$

with $x_i \in X$, $y_i \in Y$, $i = 1, 2, \cdots, k$; $k = 1, 2, \cdots$.

It is readily seen, that $p\sigma_r q$ and $p\sigma_l q$ are cross-products of seminorms p and q.

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The completion of $X \otimes Y$, equipped with the topology generated by the family $\{p\sigma_l q : p \in P, q \in Q\}$ of seminorms, is denoted by $X \otimes_{\sigma l} Y$. Similar meaning has $X \otimes_{\sigma r} Y$. These concepts were introduced by Jacobs in [3]. In fact, he considered only one of the spaces $X \otimes_{\sigma l} Y$, $X \otimes_{\sigma r} Y$ since it is obvious that $X \otimes_{\sigma l} Y$ is isomorphic to $Y \otimes_{\sigma r} X$. But it is convenient for our purposes to consider them both and we are going to introduce still another one.

For $p \in P$ and $q \in Q$ we define

$$p\tau q = \tfrac{1}{2}(p\sigma_l q + p\sigma_r q).$$

This formula obviously defines a cross-product of semi-norms p and q. The completion of $X \otimes Y$ equipped with the topology generated by the family $\{p\tau q : p \in P, q \in Q\}$ will be denoted by $X \otimes_\tau Y$.

As customary, for u in the form (1) and $p \in P$, $q \in Q$ we put

$$p\varepsilon q = \sup \left\{ \left| \sum_{i=1}^{k} \langle x_i, x' \rangle \langle y_i, y' \rangle \right| : x' \in U_p^\circ, y' \in U_q^\circ \right\}$$

where U_p° and U_q° are polars of sets $U_p = \{x : p(x) \leqq 1\}$ and $U_q = \{y : q(y) \leqq 1\}$, respectively. This is the cross-product of bi-equicontinuous convergence. The correspondencing completed tensor-product space is denoted by $X \in \otimes_\varepsilon Y$.

The projective cross-product of $p \in P$, $q \in Q$ is defined by

$$p\pi q(u) = \inf \sum_{i=1}^{k} p(x_i) q y_i),$$

where the infimum is taken for all expressions of u in the form (1). The completed projective tensor product of X and Y is denoted by $X \otimes_\pi Y$.

It is readily seen that, for any $p \in P$ and $q \in Q$,

$$p\varepsilon q \leqq \begin{matrix} p\sigma_l q \\ \\ p\sigma_r q \end{matrix} \leqq p\tau q \leqq p\pi q$$

on $X \otimes Y$. It follows that the identity map on $X \otimes Y$ extends uniquely to continuous inclusions

$$X \otimes_\pi Y \to X \otimes_\tau Y \begin{matrix} \nearrow & X \otimes_{\sigma l} Y & \searrow \\ \searrow & X \otimes_{\sigma r} Y & \nearrow \end{matrix} X \otimes_\varepsilon Y.$$

Moreover, examples can be exhibited showing that all indicated inclusions could be strict.

Let S and T be abstract sets. Let $\mathscr{S}$ be a ring (possibly σ-ring, σ-algebra, etc.) of subsets of S and $\mathscr{T}$ a ring of subsets of T. We denote

$$\mathscr{S} \times \mathscr{T} = \{A \times B : A \in \mathscr{S}, B \in \mathscr{T}\}.$$

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Further, $\mathscr{S} \otimes \mathscr{T}$ will stand for the ring generated by $\mathscr{S} \times \mathscr{T}$, $\mathscr{S} \otimes_\sigma \mathscr{T}$ will denote the σ-ring generated by $\mathscr{S} \times \mathscr{T}$ and, finally, the symbol $\mathscr{S} \otimes_\delta \mathscr{T}$ will be used for the δ-ring generated by $\mathscr{S} \times \mathscr{T}$.

The term measure or vector measure will mean a σ-additive measure, i.e. a σ-additive map from a ring of sets into a topological vector space.

THEOREM. *Let $\mathscr{S}$ and $\mathscr{T}$ be σ-rings and $\mu: \mathscr{S} \to X$ and $\nu: \mathscr{T} \to Y$ vector measures. Then there exists a vector measure $\lambda: \mathscr{S} \otimes_\sigma \mathscr{T} \to X \otimes_\tau Y$ such that*

$$\lambda(A \times B) = \mu(A) \otimes \nu(B), \ A \in \mathscr{S}, \ B \in \mathscr{T}. \tag{2}$$

PROOF. Let $p \in P$ and $q \in Q$. Let m be a real finite non-negative measure on $\mathscr{S}$ such that $m(A) \to 0$, $A \in \mathscr{S}$, implies $p(\mu(A)) \to 0$ and let n be a finite non-negative measure on $\mathscr{T}$ such that $n(B) \to 0$, $B \in \mathscr{T}$, implies $q(\nu(B)) \to 0$ The existence of such measures m and n follows e.g. from Theorem 4.2 in [4]. The same Theorem gives that it it now enough to prove that $m \times n(E) \to 0$, $E \in \mathscr{S} \otimes \mathscr{T}$, implies $p\tau q(\lambda(E)) \to 0$, where λ is the unique $X \otimes Y$-valued additive function on $\mathscr{S} \otimes \mathscr{T}$ satisfying (2). We are going to prove it. But it means that we are going to prove that $p\sigma_l q(\lambda(E)) \to 0$ and $p\sigma_r q(\lambda(E)) \to 0$ if $m \times n(E) \to 0$. Because of symmetry we will prove only the statement concerning σ_l.

It is known that, there is $A_0 \in \mathscr{S}$ and $B_0 \in \mathscr{T}$ such that $m(A) = m(A \cap A_0)$, for $A \in \mathscr{S}$, and $n(B) = n(B \cap B_0)$, for $B \in \mathscr{T}$. So, without loss of generality we can assume that $m(S) = 1$, $n(T) = 1$ and that

$$p\left(\sum_{i=1}^{k} \alpha_i \mu(A_i) \right) \leqq 1, \quad q(\nu(B)) \leqq 1$$

for all collections $A_1, A_2, \cdots, A_k$ of pairwise disjoint sets in $\mathscr{S}$, all numbers α_i with $|\alpha| \leqq 1$, $i = 1, 2, \cdots, k$, and all $B \in \mathscr{T}$.

Let $\varepsilon > 0$. Let $\delta_1 > 0$ be such that $m(A) < \delta_1$, $A \in \mathscr{S}$, implies $p(\mu(A)) < \varepsilon/8$. Let $\delta_2 > 0$ be such that $n(B) < \delta_2$, $B \in \mathscr{T}$, implies $q(\nu(B)) < \varepsilon/2$. Let $\delta = \delta_1 \delta_2$ Let $E \in \mathscr{S} \otimes \mathscr{T}$, $m \times n(E) < \delta$. Suppose E to be expressed in the form

$$E = \bigcup_{i=1}^{k} A_i \times B_i, \ A_i \in \mathscr{S}, \ B_i \in \mathscr{T},$$

with pairwise disjoint A_i, $i = 1, 2, \cdots, k$. Assume the notation arranged in such a way that $n(B_i) < \delta_2$, for $1 \leqq i \leqq k$, and $n(B_i) \geqq \delta_2$, for $l+1 \leqq i \leqq k$, where $0 \leqq l \leqq k$. Since $m \times n(E) < \delta_1 \delta_2$,

$$m(A_{l+1} \cup \cdots \cup A_k) < \delta_1 .$$

By definition of $p\sigma_l q$ we have (the supremum is taken everywhere over all choices of numbers α_i with $|\alpha_i| \leqq 1$):

$$p\sigma_l q(\lambda(E)) \leqq \sup p\left(\sum_{i=1}^{k} \alpha_i q(\nu(B_i))\mu(A_i)\right)$$
$$\leqq \sup p\left(\sum_{i=1}^{l} \alpha_i q(\nu(B_i))\mu(A_i)\right) + \sup p\left(\sum_{i=l+1}^{k} \alpha_i q(\nu(B_i))\mu(A_i)\right)$$
$$\leqq \tfrac{1}{2}\varepsilon \sup p\left(\sum_{i=1}^{l} \frac{2}{\varepsilon}\alpha_i q(\nu(B_i))\mu(A_i)\right) + \sup p\left(\sum_{i=l+1}^{k} \alpha_i \mu(A_i)\right)$$
$$\leqq \tfrac{1}{2}\varepsilon + 4 \cdot \frac{1}{8}\varepsilon = \varepsilon,$$

since

$$\left|\frac{2}{\varepsilon}\alpha_i q(\nu(B_i))\right| \leqq 1,\ i = 1,2,\cdots,l;\ \left|\alpha_i q(\nu(B_i))\right| \leqq 1,$$
$$i = l+1,\cdots,k.$$

COROLLARY. 1. *If* $\mathscr{S}$ *and* $\mathscr{T}$ *are* σ*-rings and* $\mu: \mathscr{S} \to X$ *and* $\nu: \mathscr{T} \to X$ *are vector measures, then there exists a vector measure* $\lambda: \mathscr{S} \otimes_\sigma \mathscr{T} \to X \otimes_\varepsilon Y$ *such that* (2) *holds.*

PROOF. $X \otimes_\tau Y$ is continuously included in $X \otimes_\varepsilon Y$.

Obviously in the Corollary the space $X \otimes_\varepsilon Y$ can be replaced by the completion $X \otimes_\eta Y$ of $X \otimes Y$ with respect to any cross-product topology η coarser than τ and finer than ε.

If $\mathscr{S}$ is a σ-algebra, then $ca(\mathscr{S})$ denotes the Banach space of all complex measures on $\mathscr{S}$ with the variation on the whole space serving as a norm.

COROLLARY 2. (*Cf.* [1], *Lemma.*) *Let* $\mathscr{S}$ *and* $\mathscr{T}$ *be* σ*-algebras. Let* $M \subset ca(\mathscr{S})$ $N \subset ca(\mathscr{T})$ *be relatively weakly compact sets. Then* $\{m \times n\colon m \in M,\ n \in N\}$ *is a relatively weakly compact subset of* $ca(\mathscr{S} \otimes_\sigma \mathscr{T})$.

PROOF. Let $l^\infty(M)$ be the Banach space of all bounded functions x on M written as vectors $x = (x_m)_{m \in M}$ equipped with sup-norm. By Theorem IV.9.1 in [2], M is relatively weakly compact if and only if M is bnunded and the σ-additivity is uniform in M. Hence, if, for every $A \in \mathscr{S}$, we define $\mu(A) = (x_m)_{m \in M}$ where $x_m = m(A)$, $m \in M$, then $A \to \mu(A)$, $A \in \mathscr{S}$, we define measure on $\mathscr{S}$ with values in $l^\infty(M)$. Similarly we define $\nu: \mathscr{T} \to l^\infty(N)$. By Corollary 1 there is a vector measure $\lambda: \mathscr{S} \otimes_\sigma \mathscr{T} \to l^\infty(M) \otimes_\varepsilon l^\infty(N)$ satisfying (2). By Corollary IV.10.2 in [2], the set of measure $\{z' \circ \lambda\colon z' \in U'\}$, where U' is the unit ball in the dual-space to $l^\infty(M) \otimes_\varepsilon l^\infty(N)$, is relatively weakly compact in $ca(\mathscr{S} \otimes_\varepsilon \mathscr{T})$. So it suffices to show that

$$\{m \times n\colon m \in M,\ n \in N\} \subset \{z' \circ \lambda\colon z' \in U'\}.$$

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But $m \times n = z' \circ \lambda$ if z' is the continuous linear functional on $l^\infty(M) \otimes_\varepsilon l^\infty(N)$ for which $z'(x \otimes y) = x_m y_n$, $x \in l^\infty(M)$, $y \in l^\infty(N)$. By the definition of the ε-product so defined z' belongs to U'.

COROLLARY 3. *Let $\mathscr{S}$ and $\mathscr{T}$ be δ-rings and $\mu: \mathscr{S}\ \mathscr{T} \to Y$ vector measures. Then there is a vector measure $\lambda: \mathscr{S} \otimes_\delta \mathscr{T} \to X \otimes Y$. satisfying* (2).

PROOF. Starting from $\lambda: \mathscr{S} \otimes \mathscr{T} \to X \otimes Y$ defined by the requirement (2) and that of additivity we have to show that λ can be extended as a σ-additive function onto whole of $\mathscr{S} \otimes_\delta \mathscr{T}$ taking values in $X \otimes_\tau Y$. To achieve this it is sufficient to show that if $E_j \in \mathscr{S} \otimes \mathscr{T}$, $E_j \supset E_{j+1}$, $j = 1, 2, \cdots$, then $\lim_j \lambda(E_j)$ exists in $X \otimes_\tau Y$ and if it happens that $\bigcap_{j=1}^\infty E_j = Q$ then this limit is 0. See theorem in [5]). Since there is $A_0 \in \mathscr{S}$ and $B_0 \in \mathscr{T}$ such that $E \subset A_0 \times B_0$ the result will follow from Theorem by applying it to the restriction of μ to the system $\{A: A \in \mathscr{S},\ A \subset A_0\}$ and to the restriction of ν to the system $\{B: B \in \mathscr{T}, B \subset B_0\}$. These systems are σ-algebras of subsets of A_0 and B_0 respectively.

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