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On Ulam Stability of a Functional Equation in Banach Modules

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Abstract. Let X and Y be Banach spaces and let $f: X \to Y$ be an odd mapping. For any rational number $r \neq 2$, C. Baak, D. H. Boo, and Th. M. Rassias proved the Hyers–Ulam stability of the functional equation

$$rf\left(\frac{\sum_{j=1}^{d} x_j}{r}\right) + \sum_{\substack{i(j)\in\{0,1\}\\ \sum_{j=1}^{d} i(j)=\ell}} rf\left(\frac{\sum_{j=1}^{d}(-1)^{i(j)}x_j}{r}\right) = (C_{d-1}^{\ell} - C_{d-1}^{\ell-1} + 1)\sum_{j=1}^{d} f(x_j),$$

where d and ℓ are positive integers so that $1 < \ell < \frac{d}{2}$, and $C_q^p := \frac{q!}{(q-p)!p!}$, $p, q \in \mathbb{N}$ with $p \leq q$.

In this note we solve this equation for arbitrary nonzero scalar r and show that it is actually Hyers–Ulam stable. We thus extend and generalize Baak et al.'s result. Other questions concerning the *-homomorphisms and the multipliers between C*-algebras are also considered.

1 Introduction and Preliminaries

The Ulam stability problem consists of whether an approximate solution of a functional equation must be approximated by an exact solution of the same equation. This problem was stated in the frame of groups by S. M. Ulam [25] as follows: if G_1 is a group, (G_2, d) is a metric group and $\varepsilon > 0$ is a scalar, does there exist a number $\delta > 0$ such that, whenever a function $f: G_1 \to G_2$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta, \quad \forall x, y \in G_1,$$

there exists a group homomorphism $T: G_1 \to G_2$ such that

$$d(f(x), T(x)) < \varepsilon, \quad \forall x \in G_1.$$

A first partial solution of Ulam's problem was given by D. H. Hyers [10] in the frame of real Banach spaces. Later, the approximation condition was first improved by Th. M. Rassias by allowing the Cauchy differences, in the Cauchy functional equation, to be unbounded [23]. Other improvements of the approximation conditions have also been made by K. W. Jun and H. M. Kim [11], by P. Găvruta [9], and by L. Cadariu and V. Radu in [5, 6]. Most of the proofs rely either on the direct method or on the fixed point method. Nowadays, many functional equations have been investigated either alone (see for example [1, 13, 16]) or in combination with other ones in order to cover, as stable mappings with respect to the so-obtained systems of equations, different familiar mappings such as algebra homomorphism, multipliers, derivations,

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C*-algebra homomorphisms, and so on (see [7, 17, 18, 20, 21]). For further details concerning the Ulam stability, we refer the reader to the books [12, 14, 24].

In this paper, we are concerned with the functional equation

$$rf\left(\frac{\sum_{j=1}^{d} x_j}{r}\right) + \sum_{\substack{i(j)\in\{0,1\} \\ \sum_{j=1}^{d} i(j)=\ell}} rf\left(\frac{\sum_{j=1}^{d}(-1)^{i(j)}x_j}{r}\right) = T\sum_{j=1}^{d} f(x_j), \tag{1.1}$$

where r is a nonzero scalar, f is an odd mapping from a vector space X into a Banach one Y, d and ℓ are positive integers so that $1 < \ell < \frac{d}{2}$, and $T := (C_{d-1}^{\ell} - C_{d-1}^{\ell-1} + 1)$. This equation was considered first by C. Baak, D. H. Boo, and Th. M. Rassias [2]. The authors showed there that, whenever r is a positive rational number, an odd mapping f satisfies (1.1) if and only if it is additive. They have also shown that, for all positive rational $r \neq 2$, the functional equation (1.1) is Hyers–Ulam stable.

In [19], C. Park investigated the stability of isomorphisms between JC*-algebras with respect to (1.1). The same equation has been also considered by J. R. Lee and D. Y. Shin [15]. The authors generalize there, in some technical respect, the results of C. Baak et al. The functional equation (1.1) has also been studied in the frame of multi-normed spaces by C. Park and R. Saadati [22]. The authors established similar results to those obtained by Baak et al. It is worth noting that all the authors mentioned above consider only the case where r is a positive rational number with $r \neq 2$.

Notice at this point that an additive mapping f automatically fulfils $f(sx) = sf(x)$ for all rational s and all $x \in X$. However if r fails to be rational, an additive mapping need not satisfy $f(rx) = rf(x)$ for all x. This is the main difference between the case where r is rational and when it is not. In this note, we solve the functional equation (1.1) whenever r is any arbitrary nonzero scalar. We show that an odd mapping f satisfies (1.1) if and only if it is additive and satisfies $f(rx) = rf(x)$, $x \in X$. We then extend and improve the results of [2] and several other previous results. In particular, we show the Hyers–Ulam stability of the C*-algebra homomorphisms with respect to (1.1).

In what follows, d and ℓ will be positive integers so that $1 < \ell < \frac{d}{2}$, while r will be a nonzero scalar. The vector spaces and algebras in consideration will have as basic field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Unless the contrary is expressly stated, we will assume that A is a unitary (complex) C*-algebra whose norm is denoted by $|\cdot|$, X a vector space and $(Y, \|\cdot\|)$ a Banach space. The unitary group of A will be denoted by $\mathcal{U}(A)$. This is $\mathcal{U}(A) = \{a \in A : a^*a = aa^* = e_A\}$, where obviously e_A stands for the unit of A. By $\overline{x}$, we will denote an arbitrary element $(x_1, x_2, \ldots, x_d)$ of X^d and, for any $1 \leq h < k \leq d$, by $X_{h,k}$ the subset of X^d consisting of all elements $\overline{x}$ of X^d such that $x_l = 0$ for all $l \neq h$ and $l \neq k$. For any $x, y \in X$, we will write $\overline{x}_{h,k}(x, y)$ to mean the element $\overline{x} \in X_{h,k}$ with $x_h = x$ and $x_k = y$. Similarly, $\overline{x}_h(x)$ will mean the element $\overline{x} \in X^d$ where $x_h = x$ and $x_l = 0$, for all $l \neq h$. If $t \in \mathbb{K}$ and $\overline{x} \in X^d$, we will set $t\overline{x} := \overline{tx} := (tx_1, tx_2, \ldots, tx_d)$.

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If $f: X \to Y$ is a mapping, $\mu \in \mathbb{T} := \{\mu \in \mathbb{K} : |\mu| = 1\}$, and $u \in \mathcal{U}(A)$, then we will set

$$D_\mu f(\overline{x}) := rf\Big(\frac{\sum_{j=1}^d \mu x_j}{r}\Big) + \sum_{\substack{i(j)\in\{0,1\}\\ \sum_{j=1}^d i(j)=\ell}} rf\Big(\frac{\sum_{j=1}^d (-1)^{i(j)} \mu x_j}{r}\Big) - (C_{d-1}^{\ell} - C_{d-1}^{\ell-1} + 1)\sum_{j=1}^{d} \mu f(x_j),$$

and similarly

$$D_u f(\overline{x}) := rf\Big(\frac{\sum_{j=1}^d u x_j}{r}\Big) + \sum_{\substack{i(j)\in\{0,1\}\\ \sum_{j=1}^d i(j)=\ell}} rf\Big(\frac{\sum_{j=1}^d (-1)^{i(j)} u x_j}{r}\Big) - (C_{d-1}^{\ell} - C_{d-1}^{\ell-1} + 1)\Big(\sum_{j=1}^{d} u f(x_j)\Big).$$

Whenever $\mu = 1$ and $u = e_A$, we will write $Df(\overline{x})$ instead of $D_1 f(\overline{x})$ and $D_{e_A} f(\overline{x})$ respectively.

We will designate the Kronecker symbol by $\delta_{h,k}$. This is $\delta_{h,k} = 0$ if $h \neq k$ and $\delta_{h,k} = 1$ if $h = k$.

We will use the following result due in its present form to J. Brzdęk [4]. It can also be deduced from [8].

Theorem 1.1 *Assume that (Y, d) is a complete metric space, K is a nonempty set, $f: K \to Y$, $\psi: Y \to Y$, $a: K \to K$, and $h: K \to [0, +\infty[$ are mappings, and λ is a nonnegative real number such that*

$$d(\psi \circ f \circ a(x), f(x)) \le h(x), \qquad \forall x \in K,$$
$$d(\psi(x), \psi(y)) \le \lambda d(x, y), \qquad \forall x, y \in Y$$

and

$$H(x) := \sum_{n=0}^{\infty} \lambda^n h(a^n(x)) < \infty, \qquad \forall x \in K.$$

Then for all $x \in K$, the limit $F(x) := \lim_{n\to\infty} \psi^n \circ f \circ a^n(x)$ exists and $F: K \to Y$ is the unique function such that $\psi \circ F \circ a = F$ and $d(F(x), f(x)) \le H(x)$ for all $x \in K$.

2 Solution and Stability of the Equation (1.1)

It is clear that, whenever a mapping $g: X \to Y$ is additive, it satisfies necessarily $g(sx) = sg(x)$ for all $x \in X$ and all rational s. However, if s is not rational, this identity need not hold. We first solve the equation (1.1), for arbitrary $r \neq 0$, in the following lemma which improves and extends [2, Lemma 2.1] and [15, Lemma 2.1].

Lemma 2.1 *For an arbitrary odd mapping $f: X \to Y$, the following assertions are equivalent:*

(i) *f is additive and fulfils $f(rx) = rf(x)$ for all $x \in X$.*

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(ii) *f satisfies* (1.1) *for all* $\overline{x} = (x_1, \dots, x_d) \in X^d$.
(iii) *For all* $1 \le h < k \le d$, *f satisfies* (1.1) *for all* $\overline{x} \in X_{h,k}$.
(iv) *There exist* $1 \le h < k \le d$ *such that f satisfies* (1.1) *for all* $\overline{x} \in X_{h,k}$.

Proof It is clear that (i)⇒ (ii) ⇒ (iii) ⇒ (iv). For the implication (iv) ⇒ (i), if we put in (1.1) $\overline{x} = \overline{x}_{h,k}(x, y)$, we get

$$r(C^{\ell}_{d-2} - C^{\ell-2}_{d-2} + 1) f\Big(\frac{x+y}{r}\Big) = T(f(x) + f(y)).$$

But $C^{\ell}_{d-2} - C^{\ell-2}_{d-2} + 1 = T$. Then

$$rf\Big(\frac{x+y}{r}\Big) = f(x) + f(y). \tag{2.1}$$

Letting $y = 0$, we get $f(rx) = rf(x)$ for all $x \in X$. Applying this in (2.1), we get the additivity of f and then (i) is satisfied. ∎

If f is assumed to satisfy, in addition to being odd, $f(t/r\,x) = t/r\, f(x)$ for all $x \in X$ and some nonzero scalar t, then we get the following result improving [15, Proposition 2.2].

Proposition 2.2 *Let $f: X \to Y$ be an odd mapping such that*

$$\exists \varepsilon \in \{-1, 1\},\ \exists t \in \mathbb{K} \setminus \{0\} : f\Big(\frac{t^{\varepsilon}}{r^{\varepsilon}} x\Big) = \frac{t^{\varepsilon}}{r^{\varepsilon}} f(x), \quad \forall x \in X. \tag{2.2}$$

Assume that there exist $1 \le h < k \le d$ *and a mapping* $\varphi: X_{h,k} \to [0, +\infty[$ *such that*

$$\lim_{n\to\infty} \frac{r^{\varepsilon n}}{t^{\varepsilon n}} \varphi\Big(\frac{t^{\varepsilon n}}{r^{\varepsilon n}} \overline{x}\Big) = 0, \quad \forall\, \overline{x} \in X_{h,k}, \tag{2.3}$$

$$\|Df(\overline{x})\| \le \varphi(\overline{x}), \quad \overline{x} \in X_{h,k}. \tag{2.4}$$

Then f is additive and satisfies

$$f(rx) = rf(x) \quad \text{and} \quad f(tx) = tf(x), \quad \forall x \in X.$$

Proof Since $f(\frac{t^{\varepsilon}}{r^{\varepsilon}} x) = \frac{t^{\varepsilon}}{r^{\varepsilon}} f(x)$ for all $x \in X$, we also have $\frac{r^{\varepsilon n}}{t^{\varepsilon n}} f\big(\frac{t^{\varepsilon n}}{r^{\varepsilon n}} x\big) = f(x)$ for all $x \in X$ and all $n \in \mathbb{N}$. Therefore,

$$Df(\overline{x}) = \frac{r^{\varepsilon n}}{t^{\varepsilon n}} Df\Big(\frac{t^{\varepsilon n}}{r^{\varepsilon n}} \overline{x}\Big), \quad \forall \overline{x} \in X_{h,k},\ n \in \mathbb{N}.$$

By (2.4), we get

$$\|Df(\overline{x})\| \le \frac{|r|^{\varepsilon n}}{|t|^{\varepsilon n}} \varphi\Big(\frac{t^{\varepsilon n}}{r^{\varepsilon n}} \overline{x}\Big), \quad \forall \overline{x} \in X_{h,k},\ n \in \mathbb{N}.$$

Letting n tend to ∞ we get, due to (2.3), $Df(\overline{x}) = 0$ for all $\overline{x} \in X_{h,k}$. By Lemma 2.1, f is additive and satisfies $f(rx) = rf(x)$ for all $x \in X$. Now, since, by (2.2),

$$f\Big(\frac{t^{\varepsilon}}{r^{\varepsilon}} x\Big) = \frac{t^{\varepsilon}}{r^{\varepsilon}} f(x), \quad \forall x \in X,$$

we get $f(x) = \frac{r^{\varepsilon}}{t^{\varepsilon}} f(\frac{t^{\varepsilon}}{r^{\varepsilon}} x) = \frac{1}{t^{\varepsilon}} f(t^{\varepsilon} x)$. Whereby, $f(tx) = tf(x)$ for all $x \in X$. ∎

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Since $X_{h,k}$ is isomorphic to X^2, the result still holds for any mapping $\varphi: X^2 \to [0, +\infty[$, with appropriate changes.

Our main result is the following theorem. It generalizes and improves [2, Theorem 3.1]. In order to prove it, we will henceforth put $s = r/2^{\delta^1_{|r|}}$ and again $T := C^{\ell}_{d-1} - C^{\ell-1}_{d-1} + 1$.

Theorem 2.3 *Let $f: X \to Y$ be an odd mapping. Assume that there exist $\varepsilon \in \{-1, 1\}$, $1 \le h < k \le d$, and a function $\varphi: X_{h,k} \to [0, +\infty[$ such that*

$$\lim_{n\to\infty} s^{\varepsilon n} \varphi\Big(\frac{1}{s^{\varepsilon n}} \overline{x}\Big) = 0, \qquad \forall \overline{x} \in X_{h,k}, \tag{2.5}$$

$$\widetilde{\varphi}(x) := \sum_{n=0}^{\infty} |s|^{\varepsilon n} \varphi\Big(\frac{1}{s^{\varepsilon n}} \overline{x}_{h,k}(x, \delta^1_{|r|} x)\Big) < \infty, \qquad \forall x \in X, \tag{2.6}$$

$$\|Df(\overline{x})\| \le \varphi(\overline{x}), \qquad \forall \overline{x} \in X_{h,k}. \tag{2.7}$$

Then there exists a unique mapping $L: X \to Y$ satisfying (1.1) (then L is additive and satisfies $L(rx) = rL(x)$, $\forall x \in X$) and

$$\|f(x) - L(x)\| \le \frac{|s|^{\frac{\varepsilon-1}{2}}}{2^{\delta^1_{|r|}} T} \widetilde{\varphi}(s^{\frac{1-\varepsilon}{2}} x), \quad \forall x \in X. \tag{2.8}$$

Proof Applying (2.7) to $\overline{x}_{h,k}(x, y)$, for all $x, y \in X$, we get

$$\Big\| Trf\Big(\frac{x+y}{r}\Big) - T\big(f(x) + f(y)\big)\Big\| \le \varphi\big(\overline{x}_{h,k}(x, y)\big).$$

Therefore, taking $y = \delta^1_{|r|} x$, we obtain

$$\Big\| Trf\Big(\frac{1}{s} x\Big) - T2^{\delta^1_{|r|}} f(x)\Big\| \le \varphi\big(\overline{x}_{h,k}(x, \delta^1_{|r|} x)\big), \quad \forall x \in X,$$

whence

$$\Big\| sf\Big(\frac{1}{s} x\Big) - f(x)\Big\| \le \frac{1}{2^{\delta^1_{|r|}} T} \varphi\big(\overline{x}_{h,k}(x, \delta^1_{|r|} x)\big), \quad \forall x \in X.$$

Equivalently,

$$\Big\| \frac{1}{s} f(sx) - f(x)\Big\| \le \frac{1}{2^{\delta^1_{|r|}} T} \frac{1}{|s|} \varphi\big(s\overline{x}_{h,k}(x, \delta^1_{|r|} x)\big), \quad \forall x \in X.$$

Hence, for $\varepsilon = \pm 1$, we have

$$\Big\| s^{\varepsilon} f\Big(\frac{1}{s^{\varepsilon}} x\Big) - f(x)\Big\| \le \frac{|s|^{\frac{\varepsilon-1}{2}}}{2^{\delta^1_{|r|}} T} \varphi\big(s^{\frac{1-\varepsilon}{2}} \overline{x}_{h,k}(x, \delta^1_{|r|} x)\big), \quad \forall x \in X.$$

We can apply Theorem 1.1 by taking $K = X$, $a(x) = \frac{1}{s^{\varepsilon}} x$, $\psi(y) = s^{\varepsilon} y$, $\lambda = |s|^{\varepsilon}$, and

$$h(x) = \frac{|s|^{\frac{\varepsilon-1}{2}}}{2^{\delta^1_{|r|}} T} \varphi\big(s^{\frac{1-\varepsilon}{2}} \overline{x}_{h,k}(x, \delta^1_{|r|} x)\big)$$

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to get a unique mapping $L\colon X \to Y$ such that $s^\varepsilon L(\frac{1}{s^\varepsilon}x) = L(x)$ and

$$\|L(x) - f(x)\| \le H(x) := \frac{|s|^{\frac{\varepsilon-1}{2}}}{2^{\delta^1_{|r|}}T} \sum_{n=0}^{\infty} |s|^{n\varepsilon} \varphi\Big(\frac{s^{\frac{1-\varepsilon}{2}}}{s^{n\varepsilon}} \overline{x}_{h,k}(x, \delta^1_{|r|}x)\Big), \quad x \in X.$$

But $H(x)$ is nothing but

$$\frac{|s|^{\frac{\varepsilon-1}{2}}}{2^{\delta^1_{|r|}}T} \widetilde{\varphi}(s^{\frac{1-\varepsilon}{2}}x),$$

as desired. Recall at this point that $L(x) = \lim_{n\to\infty} s^{\varepsilon n} f(\frac{x}{s^{\varepsilon n}})$, $x \in X$.

We claim that L is additive. Indeed, if we take $\frac{\overline{x}_{h,k}(x,y)}{s^{\varepsilon n}}$ in (2.7) instead of $\overline{x}_{h,k}(x, y)$, then multiply both sides by $|s|^{\varepsilon n}$, we get

$$\Big\| Ts^{\varepsilon n+1} f\Big(\frac{x+y}{s^{\varepsilon n+1}}\Big) - Ts^{\varepsilon n} f\Big(\frac{x}{s^{\varepsilon n}}\Big) - Ts^{\varepsilon n} f\Big(\frac{y}{s^{\varepsilon n}}\Big)\Big\| \le |s|^{\varepsilon n} \varphi\Big(\frac{1}{s^{\varepsilon n}}\overline{x}\Big).$$

Letting n tend to infinity, since $L(x) = \lim_{n\to\infty} s^{n\varepsilon} f(\frac{x}{s^{n\varepsilon}})$, $x \in X$, we get

$$TsL\Big(\frac{x+y}{s}\Big) - T(L(x) + L(y)) = 0.$$

Since $L(sz) = sL(z)$, $z \in X$, the additivity of L follows. Now, L being additive, it satisfies in particular $2L(x) = L(2x)$ for all $x \in X$. But L also satisfies $s^\varepsilon L(\frac{1}{s^\varepsilon}x) = L(x)$. Hence $L(rx) = rL(x)$, for all $x \in X$. ∎

Due to Lemma 2.1, one can use the same proof as for Theorem 2.3 for the following theorem.

Theorem 2.4 *Let $f\colon X \to Y$ be an odd mapping. Assume that there exist $\varepsilon \in \{-1, 1\}$ and a function $\varphi\colon X^2 \to [0, +\infty[$ such that*

$$\lim_{j\to\infty} s^{\varepsilon n} \varphi\Big(\frac{1}{s^{\varepsilon n}}x, \frac{1}{s^{\varepsilon n}}y\Big) = 0, \qquad \forall (x, y) \in X^2,$$

$$\widetilde{\varphi}(x) := \sum_{n=0}^{\infty} |s|^{\varepsilon n} \varphi\Big(\frac{x}{s^{\varepsilon n}}, \delta^1_{|r|}\frac{x}{s^{\varepsilon n}}\Big) < \infty, \qquad \forall x \in X,$$

$$\|Df(\overline{x})\| \le \varphi(x, y), \qquad \forall (x, y) \in X^2,$$

with $\overline{x} = \overline{x}_{h,k}(x, y)$ for some $0 \le h < k \le d$. Then there exists a unique mapping $L\colon X \to Y$ satisfying (1.1) and

$$\|f(x) - L(x)\| \le \frac{|s|^{\frac{\varepsilon-1}{2}}}{2^{\delta^1_{|r|}}T} \widetilde{\varphi}\big(s^{\frac{1-\varepsilon}{2}}x\big), \quad \forall x \in X.$$

If we take $\varepsilon = 1$ in Theorem 2.3, we get the following as a corollary.

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Corollary 2.5 *Let $f: X \to Y$ be an odd mapping. Assume that there exist $1 \le h < k \le d$ and a function $\varphi: X_{h,k} \to [0, +\infty[$ such that*

$$\lim_{n\to\infty} s^n \varphi\Big(\frac{1}{s^n}\overline{x}\Big) = 0, \qquad \forall \overline{x} \in X_{h,k},$$

$$\widetilde{\varphi}(x) := \sum_{n=0}^{\infty} |s|^n \varphi\Big(\frac{1}{s^n}\overline{x}_{h,k}(x, \delta^1_{|r|}x)\Big) < \infty, \qquad \forall x \in X,$$

$$\|Df(\overline{x})\| \le \varphi(\overline{x}), \qquad \forall \overline{x} \in X_{h,k}.$$

Then there exists a unique mapping $L: X \to Y$ satisfying (1.1) *and*

$$\|f(x) - L(x)\| \le \frac{1}{2^{\delta^1_{|r|}} T}\widetilde{\varphi}(x), \qquad \forall x \in X.$$

Notice at this point that, whenever $|r| \le 1$, every bounded function φ satisfies the two first conditions of Corollary 2.5. Therefore, we obtain the following corollary.

Corollary 2.6 *Let $f: X \to Y$ be an odd mapping. Assume that there exist $1 \le h < k \le d$ and a bounded function $\varphi: X^d \to [0, +\infty[$ such that*

$$\|Df(\overline{x})\| \le \varphi(\overline{x}), \qquad \forall \overline{x} \in X_{h,k}.$$

If $|r| \le 1$, then there exists a unique mapping $L: X \to Y$ satisfying (1.1) *and*

$$\|f(x) - L(x)\| \le \frac{1}{2^{\delta^1_{|r|}} T}\frac{\sup\{\varphi(\overline{x}),\ \overline{x} \in X^d\}}{1 - |s|}, \qquad \forall x \in X.$$

If φ is constant in Corollary 2.6, we obtain Hyers' classical theorem with (1.1) instead of the Cauchy equation.

Corresponding to $\varepsilon = -1$ in Theorem 2.3, we also get the following corollary.

Corollary 2.7 *Let $f: X \to Y$ be an odd mapping. Assume that there exist $1 \le h < k \le d$ and a function $\varphi: X_{h,k} \to [0, +\infty[$ such that*

$$\lim_{n\to\infty} \frac{1}{s^n}\varphi(s^n\overline{x}) = 0, \qquad \forall \overline{x} \in X_{h,k},$$

$$\widetilde{\varphi}(x) := \sum_{n=0}^{\infty} \frac{1}{|s|^n} \varphi\big(s^n\overline{x}_{h,k}(x, \delta^1_{|r|}x)\big) < \infty, \qquad \forall x \in X,$$

$$\|Df(\overline{x})\| \le \varphi(\overline{x}), \qquad \forall \overline{x} \in X_{h,k}.$$

Then there exists a unique mapping $L: X \to Y$ satisfying (1.1) *and*

$$\|f(x) - L(x)\| \le \frac{1}{|r|T}\widetilde{\varphi}(sx), \qquad \forall x \in X.$$

Again as for Corollary 2.5, if $|r| > 1$, then every bounded function φ satisfies the first two conditions of Corollary 2.7. Therefore, we have the following corollary.

Corollary 2.8 *Let $f: X \to Y$ be an odd mapping. Assume that there exist $1 \le h < k \le d$ and a bounded function $\varphi: X_{h,k} \to [0, +\infty[$ such that*

$$\|Df(\overline{x})\| \le \varphi(\overline{x}), \qquad \forall \overline{x} \in X_{h,k}.$$

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If $|r|>1$*, then there exists a unique mapping* $L: X \to Y$ *satisfying* (1.1) *and*

$$\|f(x)-L(x)\| \le \frac{\sup\{\varphi(\overline{x}),\ \overline{x}\in X_{h,k}\}}{T(|r|-1)}, \quad \forall x\in X.$$

Again, if φ is constant in Corollary 2.8, we obtain another version of the Hyers' theorem with respect to (1.1).

3 Stability of (1.1) in Modules on C^*-algebras

The following lemma gives conditions under which an approximate solution of (1.1) can be approximated by a linear exact solution of (1.1).

Lemma 3.1 *Let* $f: X\to Y$ *be an odd mapping. Assume that there exist* $\varepsilon\in\{-1,1\}$, $1\le h<k\le d$*, and a function* $\varphi: X_{h,k}\to[0,+\infty[$ *satisfying* (2.5)*,* (2.6)*, and*

$$\|D_\lambda f(\overline{x})\| \le \varphi(\overline{x}), \quad \forall \overline{x}\in X_{h,k}, \quad \forall\lambda\in\mathbb{T}. \tag{3.1}$$

Then there exists a unique linear mapping $L: X\to Y$ *satisfying* (2.8)*. In the real case we additionally assume that for all* $x\in X$*, the mappings* $f_x: t\mapsto f(tx)$ *and* $t\mapsto\widetilde{\varphi}(tx)$ *are bounded on some open interval centered at 0.*

Proof If we take $\lambda=1$ in (3.1), we get exactly (2.7). Hence by Theorem 2.3, there exists a unique function L satisfying (1.1) and (2.8). It remains to show that L is homogeneous. Taking $\frac{1}{s^{\varepsilon n}}\overline{x}_{h,k}(x,\delta^1_{|r|}x)$ in (3.1) then multiplying by $s^{\varepsilon n}$, we obtain

$$\Big\|s^{\varepsilon(n+1)}f\Big(\lambda\frac{x}{s^{\varepsilon(n+1)}}\Big)-\lambda s^{\varepsilon n}f\Big(\frac{x}{s^{\varepsilon n}}\Big)\Big\| \le \frac{|s|^{\frac{\varepsilon-1}{2}}}{2^{\delta^1_{|r|}}T}s^{\varepsilon n}\varphi\Big(s^{\frac{\varepsilon-1}{2}}\overline{x}_{h,k}\Big(\frac{x}{s^{\varepsilon n}},\delta^1_{|r|}x\Big)\Big).$$

Letting n tend to infinity, we obtain $L(\lambda x)=\lambda L(x)$ for all $x\in X$ and all $\lambda\in\mathbb{T}$. Now, for an arbitrary $z\in\mathbb{K}$, there exists $\lambda\in\mathbb{T}$ such that $z=|z|\lambda$. But also there are $n\in\mathbb{Z}$ and $0\le\alpha<1$ such that $|z|=n+\alpha$. Therefore, $L(zx)=n\lambda L(x)+\lambda L(\alpha x)$. The problem then reduces to $L(\alpha x)=\alpha L(x)$ for all $0\le\alpha<1$. But for such an α in the complex case, there are λ_1, $\lambda_2\in\mathbb{T}$ such that $\alpha=\frac{\lambda_1+\lambda_2}{2}$. Using the additivity of L, one immediately deduces its homogeneity. In the real case, let (u_n) be a sequence of rational numbers converging to α. Then there is some $M>0$ such that, for every positive integer p, some $n_p\in\mathbb{N}$ exists so that for $n\ge n_p$, we have

$$\frac{|s|^{\frac{\varepsilon-1}{2}}}{2^{\delta^1_{|r|}}T}\widetilde{\varphi}(s^{\frac{1-\varepsilon}{2}}p(\alpha-u_n)x)\le M \quad\text{and}\quad \|f\big(p(\alpha-u_n)x\big)\|\le M.$$

Hence,

$$\begin{aligned}p\|L(\alpha x)-u_nL(x)\| &= \|L(p(\alpha-u_n)x)\|\\ &\le \|L(p(\alpha-u_n)x)-f(p(\alpha-u_n)x)\|+\|f(p(\alpha-u_n)x)\|\\ &\le \frac{|s|^{\frac{\varepsilon-1}{2}}}{2^{\delta^1_{|r|}}T}\widetilde{\varphi}(s^{\frac{1-\varepsilon}{2}}p(\alpha-u_n)x)+\|f(p(\alpha-u_n)x)\|\le 2M.\end{aligned}$$

Therefore, when n tends to infinity, we get $\|L(\alpha x)-\alpha L(x)\|\le 2M/p$. Letting p tend to infinity, we come to $L(\alpha x)=\alpha L(x)$ and then to the homogeneity of L. ∎

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Notice that in the condition (3.1), one can replace $D_\lambda f(\overline{x})$ by $D'_\lambda f(\overline{x})$, where

$$D'_\lambda f(\overline{x}) := rf\Big(\frac{\lambda(x_h + x_k) + \sum_{j\notin\{h,k\}} x_j}{r}\Big) + \sum_{\substack{i(j)\in\{0,1\}\\ \sum_{j=1}^{d} i(j)=\ell}} rf\Big(\frac{\lambda((-1)^{i(h)}x_h + (-1)^{i(k)}x_k) + \sum_{j\notin\{h,k\}}(-1)^{i(j)}x_j}{r}\Big) - (C_{d-1}^{\ell} - C_{d-1}^{\ell-1} + 1)\Big(\lambda(f(x_h) + f(x_k)) + \sum_{j\notin\{h,k\}} f(x_j)\Big).$$

Now, we are able to investigate the A-linearity of the mapping L.

Theorem 3.2 *Assume that $\mathbb{K} = \mathbb{C}$, X is a left A-module, Y is a left Banach A-module, and let $f: X \to Y$ be an odd mapping. Suppose that there exist $\varepsilon \in \{-1,1\}$, $1 \le h < k \le d$, and a function $\varphi: X_{h,k} \to [0,+\infty[$ satisfying (2.5), (2.6), and*

$$\|D_u f(\overline{x})\| \le \varphi(\overline{x}), \ \forall \overline{x} \in X_{h,k}, \quad \forall u \in \mathcal{U}(A). \tag{3.2}$$

Then there exists a unique A-linear mapping $L: X \to Y$ satisfying (2.8).

Proof If f satisfies (3.2) for all $u \in \mathcal{U}(A)$, then it satisfies (3.1) for all $\lambda \in \mathbb{T}$. By Lemma 3.1, there exists a unique linear mapping $L: X \to Y$ satisfying (2.8). It remains to show that $L(ax) = aL(x)$ for all $a \in A$. As every element of a C^*-algebra is a finite linear combination of unital elements ([3, p. 70]), it suffices to show that $L(ux) = uL(x)$ for all unital element $u \in A$. But, for such an u, if we take in (3.2) $\overline{x}_{h,k}(x, \delta^1_{|r|}x)$, for arbitrary $x \in X$, we will get

$$\Big\| rf\Big(\frac{ux}{r}\Big) - uf(x)\Big\| \le \frac{1}{T}\varphi(\overline{x}_{h,k}(x, \delta^1_{|r|}x), \quad x \in X. \tag{3.3}$$

Taking in (3.3) $\frac{x}{s^{\varepsilon n}}$ instead of x, and then multiplying by $|s|^{\varepsilon n}$, we obtain:

$$\Big\| s^{\varepsilon n+1} f\Big(\frac{ux}{s^{\varepsilon n+1}}\Big) - us^{\varepsilon n} f\Big(\frac{x}{s^{\varepsilon n}}\Big)\Big\| \le \frac{1}{T}|s|^{\varepsilon n}\varphi\Big(\frac{1}{s^{\varepsilon n}}\overline{x}_{h,k}(x, \delta^1_{|r|}x)\Big), \quad x \in X.$$

Letting n tend to infinity and using the fact that the mapping $t \mapsto ut$ is continuous from Y into itself, we arrive at

$$L(ux) = uL(x), \quad x \in X, \quad u \in \mathcal{U}(A),$$

which achieves the proof. ∎

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Remark 3.3 (i) As for the preceding lemma, one can replace $D_u f(\overline{x})$ by $D'_u f(\overline{x})$ in (3.2), where

$$D'_u f(\overline{x}) := r f\Big(\frac{u(x_h + x_k) + \sum_{j\notin\{h,k\}} x_j}{r}\Big) + \sum_{\substack{i(j)\in\{0,1\}\\ \sum_{j=1}^{d} i(j)=\ell}} r f\Big(\frac{u((-1)^{i(h)}x_h + (-1)^{i(k)}x_k) + \sum_{j\notin\{h,k\}}(-1)^{i(j)}x_j}{r}\Big) - (C^{\ell}_{d-1} - C^{\ell-1}_{d-1} + 1)\Big(u(f(x_h) + f(x_k)) + \sum_{j\notin\{h,k\}} f(x_j)\Big).$$

(ii) If $X = A$ in Theorem 3.2, then L becomes a right multiplier.

The following result gives conditions under which an odd approximate solution f of (1.1) must be a C*-algebra homomorphism. It extends and improves [2, Theorem 4.1].

Theorem 3.4 *Assume that $\mathbb{K} = \mathbb{C}$, X and Y are C^*-algebras with Y a left Banach X-module, and $f: X \to Y$ is an odd mapping. Suppose that there exist $\varepsilon \in \{-1,1\}$, $1 \le h < k \le d$, and a function $\varphi: X_{h,k} \to [0,+\infty[$ satisfying (2.5), (2.6), and (3.2). If $\lim_{n\to\infty} s^{\varepsilon n} f(\frac{e_X}{s^{\varepsilon n}}) = e_Y$, $f(\frac{u}{s^{\varepsilon n}} y) = f(\frac{u}{s^{\varepsilon n}}) f(y)$ for all $u \in \mathcal{U}(X)$ and $y \in X$, and*

$$\Big\| f\Big(\frac{u^*}{s^{\varepsilon n}}\Big) - f\Big(\frac{u}{s^{\varepsilon n}}\Big)^*\Big\| \le \varphi\Big(\overline{x}_l\Big(\frac{u}{s^{\varepsilon n}}\Big)\Big), \quad \forall u \in \mathcal{U}(X), \quad n \in \mathbb{N}, \tag{3.4}$$

then f is a C^*-algebra homomorphism.

Proof By Lemma 3.1, $L(x) = \lim_{n\to\infty} s^{\varepsilon n} f(\frac{u}{s^{\varepsilon n}})$ defines a linear mapping $L: X \to Y$. Let us show that $f = L$ and that f is a C*-algebra homomorphism. From $f(\frac{u}{s^{\varepsilon n}} y) = f(\frac{u}{s^{\varepsilon n}}) f(y)$ for all $u \in \mathcal{U}(X)$ and all $y \in X$, we deduce $L(uy) = L(u)f(y)$. In particular, $L(y) = L(e_X)f(y) = f(y)$, $y \in X$, whereby $f = L$. Moreover, $L(uv) = L(u)L(v)f(e_X) = L(u)L(v)$. Since every element of X is a finite linear combination of unitary elements and L is linear, L is an algebra homomorphism. Finally, if we multiply (3.4) by $s^{\varepsilon n}$ then let n tend to infinity, we obtain $L(u^*) = L(u)^*$ for all $u \in \mathcal{U}(X)$. Again by the linearity of L, we obtain $L(x^*) = L(x)^*$ for all $x \in X$. Therefore f is a C*-algebra homomorphism. ∎

If in the preceding theorem, f is assumed to be one to one, then it is a C*-algebra isomorphism into. In particular, we get the following corollary.

Corollary 3.5 *Under the hypotheses of Theorem 3.4, if $f: X \to Y$ is bijective, then it is a C^*-algebra isomomorphism.*

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