MR. PETER GRAY'S DEMONSTRATION OF FORMULÆ.

To the Editor of the Assurance Magazine.

Sir,—The demonstration of the expressions for the values of single and annual premiums given by Mr. P. Gray (No. XLVIII., page 238, of the *Assurance Magazine*) is quite new and remarkable indeed, but yet it affords us a greater interest, if we suppose, for the sake of generalization, the consideration for forbearance *in infinitum*, we have

$$A_x=(1-v)a'_x(A_x+A_x^2+A_x^3\ldots\ldots) \quad . \quad . \quad . \quad (I.)$$

which, as is known, is equal to

$$(1-v)a'_x\,.\,\frac{A_x}{1-A_x},$$

and thus the expression for the present value will become

$$A_x=1-(1-v)a'_x\,.$$

Hence, dividing the foregoing formula (I.), by a'_x, we obtain immediately the annual premium

$$\pi_x=(1-v)(A_x+A_x^2+A_x^3+\ \ldots\ ),$$

which is equal to the expression

$$\pi_x=\frac{(1-v)A_x}{1-A_x}.$$

I have the honour to be,
Sir,
Your most obedient servant,

D. AUGUST WIEGARD,
Director of Life Assurance Society "Iduna."

Halle a/S. Prussia, Germany,
23rd August, 1862.

ON INCREASING AND DECREASING SCALES OF PREMIUM.

To the Editor of the Assurance Magazine.

Sir,—The following lines may have an interest for some of the junior members of the Institute, and that is the only reason for my venturing to address you upon a subject so simple as that of determining a premium, ascending or descending by a series of stages, for a whole-term life assurance.

In practice I have met with five varieties of this form of payment, viz.:—

1st. The premium to be p for the first stage, and to be increased or decreased so as to be $p\pm q$ (q being a quantity previously determined) for the second stage, $p\pm 2q$ for the third stage . . . and $p\pm(v-1)q$ for the vth stage, at which it is to remain constant for the remainder of life; to find the value of p.

2nd. The premium for the first stage to be p (p being determined), for the second stage $p\pm q$ (q being arbitrarily fixed), for the third stage $p\pm 2q$, &c., and for the vth stage $p\pm q'$, at which it is to remain constant; to find the value of q'.

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3rd. The premium for the first stage being determined, to find the increase or decrease to be paid at each of the subsequent stages.

4th. The premium for the first, second, third, &c., stages being fixed at p, p_1, p_2, &c., to find the premium to be paid for the vth stage, p_v, at which it is to remain constant.

5th. The premium for the first stage to be p, and to be increased or decreased by a proportion of p, so as to be $p+\frac{1}{q}p=p\left(1+\frac{1}{q}\right)$ for the second stage, $p\left(1+\frac{2}{q}\right)$ for the third stage . . . $p\left(1+\frac{v-1}{q}\right)$ for the vth stage, at which it is to remain constant.

These cases may be more shortly stated thus:—

1st. q being known, to find p.
2nd. p and q being known, to find q'.
3rd. p being known, to find q.
4th. p, p_1, p_2 . . . &c., being known, to find the increase to or decrease from p_{v-1} to form p_v.
5th. p to be increased or decreased at each series by $\frac{p}{q}$.

I. An example of the first of these forms will be found in *Jones on Annuities, &c.*, p. 194, viz.:—

"Suppose the annual premium to increase or decrease a *certain* sum every t years, and, at the end of v intervals of t years each, the premium to continue constant during the remainder of life; what annual premium should be required during the *first* t years?

"Let p=the annual premium required;
q=the increase or decrease *per* £1* every t years."

Departing for a moment from the text, the argument for this case may be stated as follows:—

Benefit term.

An assurance of £1 at death, the present value of which . . . . $=\mathrm{A}_x$.

Payment terms.

An annuity of £p for the whole term of life, the present value of which . . . . . . $=p(1+a_x)$.

± A deferred annuity of q (q being arbitrarily determined), first payment or deduction at end of t years . . . . $=q(a_{(x)\rceil t-1})$.

± A further deferred annuity of q; first payment or deduction at end of $2t$ years . . . . $=q(a_{(x)\rceil 2t-1})$.

. . . . .

± A further deferred annuity of q; first payment or deduction at end of vt years . . . . $=q(a_{(x)\rceil vt-1})$.

Equating, we have, as stated by Jones (*ibid.*):—

$$\text{“}\mathrm{A}_x=p(1+a_x)\pm q(a_{(x)\rceil t-1}+a_{(x)\rceil 2t-1}+a_{(x)\rceil 3t-1}+\ldots+a_{(x)\rceil vt-1});$$

* This, taken with the context, is evidently intended for the sum assured, not per £1 of the premium.

by transposition and division,

$$p=\frac{A_x \mp q(a_{(x)\overline{t-1}|}+a_{(x)\overline{2t-1}|}+a_{x\overline{3t-1}|}+\ldots+a_{x\overline{vt-1}|})}{1+a_x};$$

by substitution in the first equation,

$$\frac{M_x}{D_x}=p.\frac{N_{x-1}}{D_x}\pm q\left(\frac{N_{x+t-1}+N_{x+2t-1}+N_{x+3t-1}+\ldots+N_{x+vt-1}}{D_x}\right);$$

from which we obtain

$$p=\frac{M_x \mp q(N_{x+t-1}+N_{x+2t-1}+N_{x+3t-1}+\ldots+N_{x+vt-1}}{N_{x-1}}."$$

A numerical example follows at Carlisle 4 per cent., but for convenience I substitute one at Experience 3 per cent., viz.:—

Example.—" What annual premium should be required during the first 5 years to insure £100 on a life aged 31, the annual premium to increase 4*s.* every 5 years, and remain constant at the end of 20 years (Experience 3 per cent.)."

$$t=5, \qquad v=4, \qquad q=\frac{\cdot 2}{100}=\cdot 002.$$

$$\begin{aligned}
N_{x+t-1} &= N_{35} = 543540\cdot812\\
N_{x+2t-1} &= N_{40} = 412864\cdot847\\
N_{x+3t-1} &= N_{45} = 305877\cdot513\\
N_{x+4t-1} &= N_{50} = 219141\cdot537\\
&\qquad\quad 1481424\cdot709\\
&\qquad\quad 200\\
&\qquad\quad 2962\cdot849\\
M_x &= M_{31} = 13770\cdot543\\
&\qquad\quad 10807\cdot694
\end{aligned}$$

$$N_{x-1}=N_{30}=\frac{10807\cdot694}{702266\cdot223}=\cdot01539\times100=£1.\ 10s.\ 10d.;$$

or,

$$\begin{aligned}
a_{(x)\overline{t-1}|} &= a_{31\overline{4}|} = 15\cdot881\\
a_{(x)\overline{2t-1}|} &= a_{31\overline{9}|} = 12\cdot063\\
a_{(x)\overline{3t-1}|} &= a_{31\overline{14}|} = 8\cdot937\\
a_{(x)\overline{4t-1}|} &= a_{31\overline{19}|} = 6\cdot403\\
&\qquad\quad 43\cdot284\\
&\qquad\quad \cdot002\\
&\qquad\quad \cdot086568\\
A_x = A_{31} &= \quad \cdot40235\\
&\qquad\quad \cdot31579
\end{aligned}$$

$$1+a_x=1+a_{31}=\frac{\cdot 31579}{20\cdot 519}=\cdot 01539\times 100=£1.\ 10s.\ 10d.$$

On the above example I note that Mr. Chisholm, in his valuable work, has the following remarks:—

"In the fundamental equation given above, the right-hand side can only express the value of A_x when p multiplies the whole of that side. For the premium for the first t years being p, its value for the whole of life is $p(1+a_x)$; and the increase or decrease at the end of every t years being $p\times q$, its value is evidently

$$p\times q(a_{(x)\rceil t-1}+a_{(x)\rceil 2t-1}+a_{(x)\rceil 3t-1}+\ \ldots\ +a_{(x)\rceil vt-1}).$$

"The writer of the article has not adverted to this, and in the example which follows the rule there is a misconception as to the value of q, where it is made equal to ·002 in place of $=\cdot 2$, the increase per pound as assumed.

"The equation and solution should have stood thus:—

$$A_x=p\{(1+a_x)\pm q(a_{(x)\rceil t-1}+a_{(x)\rceil 2t-1}+a_{(x)\rceil 3t-1}+\ \ldots\ +a_{(x)\rceil vt-1})\};$$

by transposition and division,

$$p=\frac{A_x}{1+a_x\pm q(a_{(x)\rceil t-1}+{}_{(x)\rceil 2t-1}+a_{(x)\rceil 3t-1}+\ \ldots\ +a_{(x)\rceil vt-1})};$$

by substitution in the first equation,

$$\frac{M_x}{D_x}=p\frac{N_{x-1}\pm q(N_{x+t-1}+{}_{x+2t-1}+N_{x+3t-1}+\ \ldots\ +N_{x+vt-1})}{D_x};$$

from which we obtain

$$p=\frac{M_x}{N_{x-1}\pm q(N_{x+t-1}+N_{x+2t-1}+N_{x+3t-1}+\ \ldots\ +N_{x+vt-1})}."$$

On a careful review of the foregoing, I think it will be found that the misconception as to the value of q is not that of "the writer of the article" in *Jones on Annuities*, but may rather be imputed to Mr. Chisholm, who appears to have confounded the terms of the proposition—viz., the increase of the premium per £1 assured with what he terms "the increase per pound as assumed" (*scil.* in the premium). It is evident that Mr. Chisholm had in his mind the problem which follows (No. 5), the expression for which he has investigated in his work.

Example.—What annual premium should be required to insure £100 on a life aged 31, to be decreased 4*s.* every 5 years, and to remain constant at the end of 20 years?

As in last example	2962·849
$M_x=M_{31}$ =	13770·543
	16733·392

$$\text{and }\frac{16733\cdot 392}{N_{x-1}=N_{30}=702266\cdot 223}=\cdot 023827\times 100=£2.\ 7s.\ 8d.\text{ nearly.}$$

$$\begin{aligned}\text{Or, as above,}\quad & \cdot 08657\\ A_x = A_{31} = \quad & \cdot 40235\\ \hline & \cdot 48892\end{aligned}$$

$$\text{and } \frac{\cdot 48892}{1+a_x=1+a_{30}=20\cdot 519}=\cdot 023827 \text{ (as before).}$$

II. A second form of this mode of assurance, which sometimes occurs, is as follows:—

An annual premium of p is to be paid for first t years, p being an arbitrary sum, but one which, in the case of an increasing premium, should be greater than a temporary assurance for a like term; $p \pm q$ for the second term of t years, q also being an arbitrarily fixed sum; $p \pm 2q$ for the third term of t years . . . and $p \pm (v-2)q$ for the $(v-1)$th term, and $p \pm q'$ for the vth term, at which it is to continue constant during the remainder of life: what is the value of q'?

Proceeding as before:—

Benefit term.

An assurance of £1 at death, the present value of which . . . . $\Big\}=A_x$.

Payment terms.

An annuity of £p, previously determined, for the whole of life, the present value of which . . $\Big\}=p(1+a_x)$.

$\pm$ A deferred annuity of q (also first determined), first payment or deduction at end of t years . $\Big\}=q(a_{(x)\urcorner t-1})$.

$\pm$ A further deferred annuity of q (as before), first payment or deduction to be made at the end of $2t$ years $\Big\}=q(a_{(x)\urcorner 2t-1})$.

$\pm$ &c. &c.

$\pm$ A deferred annuity of q', first payment or deduction at the end of vt years . . . . $\Big\}=q'(a_{(x)\urcorner vt-1})$.

Equating, we obtain

$$A_x=p(1+a_x)\pm q(a_{(x)\urcorner t-1}+a_{(x)\urcorner 2t-1}+\ .\ .\ .\ .\ +a_{(x)\urcorner (v-1)t-1})\pm q'(a_{(x)\urcorner vt-1});$$

by transposition and division,

$$\pm q'=\frac{A_x-p(1+a_x)\mp q(a_{(x)\urcorner t-1}+a_{(x)\urcorner 2t-1}+\ .\ .\ .\ .\ +a_{(x)\urcorner (v-1)t-1})}{a_{(x)\urcorner vt-1}};$$

by substitution in the first equation,

$$\frac{M_x}{D_x}=p\cdot\frac{N_{x-1}}{D_x}\pm q\frac{(N_{x+t-1}+N_{x+2t-1}+\ .\ .\ .\ .\ +N_{x+(v-1)t-1})\pm q'.N_{x+vt-1}}{D_x};$$

from which we obtain

$$\pm q'=\frac{M_x-p.N_{x-1}\mp q.(N_{x+t-1}+N_{x+2t-1}+\ .\ .\ .\ .\ +N_{x+(v-1)t-1})}{N_{x+vt-1}}.$$

Example.—An annual premium of £1·539 is to be charged for the assurance of £100 on a life aged 31, for the first 5 years, to be increased to £1·739 for the second 5 years, £1·939 for the third 5 years, and £2·139 for the fourth 5 years: what further increase should be made for the fifth stage, which is to continue constant for the remainder of life?

$$t=5, \qquad v=4, \qquad q=\cdot 002 \text{ per £1 assured.}$$

$N_{x-1}=N_{30}=702266\cdot223$
93510

70226622
35113311
2106799
632040

10807·8772

$N_{x+t-1}=543540\cdot812$
$N_{x+2t-1}=412864\cdot847$
$N_{x+3t-1}=305877\cdot513$

1262283·172
·002

2524·566344
10807·8772

13332·4435
$M_x=M_{31}=13770\cdot543$

438·100

$$N_{vt-1}=N_{50}=\frac{438\cdot1}{219141\cdot5}=\cdot002\times100=\cdot2=4s.;$$

or,

$1+a_x=1+a_{31}=20\cdot519$
p inverted$=$ 93510

20519
10260
615
180

·31574

$a_{(x)\overline{t-1}|}=a_{31\overline{4}|}=15\cdot881$
$a_{(x)\overline{2t-1}|}=a_{31\overline{9}|}=12\cdot063$
$a_{(x)\overline{3t-1}|}=a_{31\overline{14}|}=\ 8\cdot937$

36·881
·002

·073762
·31574

·38950
$A_{31}=40235$

·01285

$$a_{31\overline{19}|}=\frac{\cdot01285}{6\cdot403}=\cdot002\times100=\cdot2, \text{ as before.}$$

The first premium, and the addition at the second, third, and fourth stages respectively, might have been differently chosen, but the values taken were selected because they afford a proof of the first case. This remark applies also to cases 3 and 4.

Example.—An annual premium of £2·38278 is to be charged for an assurance of £100 on a life aged 31, for the first 5 years, to be decreased to £2·18278 for the second 5 years, to £1·98278 for the third five years, and to £1·78278 for the fourth 5 years, what is the decrease which should be made for the fifth stage, and to remain constant for the rest of life?

$$N_{x-1}=N_{30}=702266{\cdot}223\times{\cdot}0238278=16733{\cdot}459$$
As in preceding example, $2524{\cdot}566$

$$14208{\cdot}893$$
$$M_x=M_{31} = 13770{\cdot}543$$
$$-438{\cdot}350$$

and $$\frac{-438{\cdot}350}{N_{vt-1}=N_{50}=219141{\cdot}5}=-{\cdot}002\times100=-{\cdot}2=4s.$$

or, $$1+a_x=1+a_{31}=20{\cdot}519\times{\cdot}0238278={\cdot}48893$$
As in preceding example, ${\cdot}07376$

$${\cdot}41517$$
$$A_x=A_{31} = {\cdot}40235$$
$$-{\cdot}01282$$

and $$\frac{-{\cdot}01282}{a_{31\rceil 19}=a\,50=6{\cdot}403}=-{\cdot}002 \text{ (as before).}$$

III. The third case proposed is one where, the first premium being fixed, it is required to find the rate of increase or decrease for each succeeding stage.

Suppose the premium for the first t years to be p, what premium must be paid for the second, third, &c., series of t years, the premium for the vth stage being constant throughout the rest of life?

This may be stated as follows:—

Benefit term.

An assurance of £1 payable at death, the present value of which . . . . . . } $=A_x$.

Payment terms.

An annuity of £p, already determined, the present value of which . . . . . . } $=p(1+a_x)$.

$\pm$ A deferred annuity of q, first payment or deduction at end of t years . . . . . . } $=q(a_{(x)\rceil t-1})$.

$\pm$ do. do. $2t$ years $=q(a_{(x)\rceil 2t-1})$.

$\pm$ do. do. $3t$ years $=q(a_{(x)\rceil 3t-1})$.

. . . .

$\pm$ do. do. vt years $=q(a_{(x)\rceil vt-1})$.

Equating,

$$A_x=p(1+a_x)\pm q(a_{(x)\rceil t-1}+a_{(x)\rceil 2t-1}+a_{(x)\rceil 3t-1}+\ldots\ldots+a_{(x)\rceil vt-1});$$

by transposition and division,

$$q=\frac{A_x\mp p(1+a_x)}{a_{(x)\rceil t-1}+a_{(x)\rceil 2t-1}+a_{(x)\rceil 3t-1}+\ldots\ldots+a_{(x)\rceil vt-1}};$$

by substitution in the first equation,

$$\frac{M_x}{D_x}=p\,.\,\frac{N_{x-1}}{D_x}\pm\frac{q(N_{x+t-1}+N_{x+2t-1}+N_{x+3t-1}+\ldots\ldots+N_{x+vt-1})}{D_x};$$

from which we obtain

$$\pm q = \frac{M_x \mp p \, . \, N_{x-1}}{N_{x+t-1} + N_{x+2t-1} + N_{x+3t-1} + \; \ldots \; + N_{x+vt-1}}.$$

Example.—An annual premium of £1·539, for an assurance of £100 on a life aged 31, is to be paid for the first 5 years; what must be the increase at the end of 5, 10, 15, and 20 years, the premium for the last stage to be constant during the rest of life?

$$t=5, \qquad v=4.$$

$$M_x = M_{31} = 13770{\cdot}543$$

$$N_{x-1} = N_{30} = 702266{\cdot}223 \times {\cdot}01539 = 10807{\cdot}877 \text{ (as before)}$$

$$2962{\cdot}666$$

$$\begin{aligned} N_{x+t-1} &= N_{35} = 543540{\cdot}812 \\ N_{x+2t-1} &= N_{40} = 412864{\cdot}847 \\ N_{x+3t-1} &= N_{45} = 305877{\cdot}513 \\ N_{x+4t-1} &= N_{50} = 219141{\cdot}537 \end{aligned}$$

$$1481424{\cdot}709$$

$$\text{and } \frac{2962{\cdot}666}{1481424{\cdot}709} = {\cdot}002 \times 100 = {\cdot}2.$$

Otherwise,

$$A_x = A_{31} = {\cdot}40235$$

$$1 + a_x = 1 + a_{31} = 20{\cdot}519 \times {\cdot}01539 = {\cdot}31574 \text{ (as before)}$$

$${\cdot}08661$$

$$\begin{aligned} a_{(x)\overline{t-1}|} &= a_{31\overline{4}|} = 15{\cdot}881 \\ a_{(x)\overline{2t-1}|} &= a_{31\overline{9}|} = 12{\cdot}063 \\ a_{(x)\overline{3t-1}|} &= a_{31\overline{14}|} = 8{\cdot}937 \\ a_{(x)\overline{4t-1}|} &= a_{31\overline{19}|} = 6{\cdot}403 \end{aligned}$$

$$43{\cdot}284$$

$$\text{and } \frac{{\cdot}08661}{43{\cdot}284} = {\cdot}002 \times 100 = {\cdot}2.$$

Example.—An annual premium of £2·38278 for an assurance of £100 on a life aged 31 is to be paid for the first 5 years, what must be the decrease at the end of 5, 10, 15, and 20 years, the premium for the last stage to be constant during the remainder of life?

As before, $N_{x-1} = N_{30} = 702266{\cdot}223 \times {\cdot}0238278 = 16733{\cdot}459$

$$M_x = M_{31} = 13770{\cdot}543$$

$$-2962{\cdot}916$$

and, as before, $\frac{-2962{\cdot}916}{1481424{\cdot}709} = -{\cdot}002 \times 100 = -{\cdot}2;$

or, as before, $1 + a_x = 1 + a_{31} = 20{\cdot}519 \times {\cdot}0238278 = {\cdot}48893$

$$A_x = A_{31} = {\cdot}40235$$

$$-{\cdot}08658$$

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and as before, $\frac{-\cdot08658}{43\cdot284}=-\cdot002$ (as before).

IV. The fourth form of this mode of assurance above referred to is one where the premium for the first stage is fixed at p, for the second stage at p_1, for the third stage at p_2 . . . . and for the $(v-1)$th stage at $p_{(v-1)}$; required to find the value of p_v for the vth stage, at which it will remain constant.

The benefit and payment terms in this case are as follows:—

Benefit term.

An assurance of £1 payable at death, the present value of which . . . $=A_x$.

Payment terms.

An annuity of £p for the whole of life, the present value of which . . . $=p(1+a_x)$.

$\pm$ A deferred annuity of $(p_1\sim p)$, first payment at end of t years . . . $=(p_1\sim p)(a_{(x)\rceil t-1})$.

$\pm$ A deferred annuity of $(p_2\sim p_1)$, first payment at end of $2t$ years . . . $=(p_2\sim p_1)(a_{(x)\rceil 2t-1})$.

$\pm$ A deferred annuity of $(p_{(v-1)}\sim p_{(v-2)})$, first payment at end of $v-1$ years . $=(p_{(v-1)}\sim p_{(v-2)})(a_{(x)\rceil (v-1)t-1})$.

$\pm$ And a deferred annuity of $p_v\sim p_{(v-1)}$, first payment at end of v years . . $=(p_v\sim p_{(v-1)})(a_{(x)\rceil vt-1})$.

Equating,

$$A_x=p(1+a_x)\pm(p_1\sim p)(a_{(x)\rceil t-1})\pm(p_2\sim p_1)(a_{(x)\rceil 2t-1})\pm\ldots\pm$$
$$\pm(p_{(v-1)}\sim p_{(v-2)})(a_{(x)\rceil (v-1)t-1})\pm(p_v\sim p_{(v-1)})(a_{(x)\rceil vt-1});$$

by transposition and division,

$$\pm(p_v\sim p_{(v-1)})=$$
$$=\frac{A_x-p(1+a_x)\pm(p_1\sim p)(a_{(x)\rceil t-1})\pm(p_2\sim p_1)(a_{(x)\rceil 2t-1})\pm\ldots\pm(p_{(v-1)}\sim p_{(v-2)})(a_{(x)\rceil (v-1)t-1})}{a_{(x)\rceil vt-1}};$$

by substitution in the first equation,

$$\frac{M_x}{D_x}=p\frac{N_{x-1}}{D_x}\pm(p_1\sim p)\frac{N_{x+t-1}}{D_x}\pm(p_2\sim p_1)\frac{N_{x+2t-1}}{D_x}\pm\ldots\pm$$
$$\pm(p_{(v-1)}\sim p_{(v-2)})\frac{N_{x+(v-1)t-1}}{D_x}\pm(p_v\sim p_{(v-1)})\frac{N_{x+vt-1}}{D_x};$$

whence we obtain

$$\pm(p_v\sim p_{(v-1)})=$$
$$=\frac{M_x\pm p.N_{x-1}\pm(p_1\sim p)N_{x+t-1}\pm(p_2\sim p_1)N_{x+2t-1}\pm\ldots\pm(p_{(v-1)}\sim p_{(v-2)})N_{x+(v-1)t-1}}{N_{x+vt-1}}.$$

Example.—For an assurance of £100 on a life aged 31, the premium for the first stage of 5 years is to be £1·539, for the second stage £1·739, for the third stage of 5 years £1·939, for the fourth stage of 5 years £2·139: what further increase should be made for the fifth stage, and to continue for the rest of life?

$$M_x = M_{31} = \quad 13770{\cdot}543$$

$$N_{x-1} = N_{30} = 702266{\cdot}223 \times {\cdot}01539 = \quad 10807{\cdot}877 \text{ (as before)}$$
$$(p_1 - p) = {\cdot}01739 - {\cdot}01539 = {\cdot}002$$
$$N_{x+t-1} = N_{35} = 543540{\cdot}812 \times {\cdot}002 = \quad 1087{\cdot}082$$
$$(p_2 - p_1) = {\cdot}01939 - {\cdot}01739 = {\cdot}002$$
$$N_{x+2t-1} = N_{40} = 412864{\cdot}847 \times {\cdot}002 = \quad 825{\cdot}730$$
$$(p_3 - p_2) = {\cdot}02139 - {\cdot}01939 = {\cdot}002$$
$$N_{x+3t-1} = N_{45} = 305877{\cdot}513 \times {\cdot}002 = \quad 611{\cdot}755$$

$$13332{\cdot}444$$

$$438{\cdot}099$$

and
$$N_{x+vt-1} = N_{50} = \frac{438{\cdot}099}{219141{\cdot}709} = {\cdot}002 \times 100 = {\cdot}2;$$

or,

$$A_x = A_{31} = \quad {\cdot}40235$$

$$1 + a_x = 1 + a_{31} = 20{\cdot}519 \times {\cdot}01539 = \quad {\cdot}31574$$
$$a_{(x)\,\rceil t-1} = a_{31\,\rceil 4} = 15{\cdot}881 \times {\cdot}002 = \quad {\cdot}03176$$
$$a_{(x)\,\rceil 2t-1} = a_{31\,\rceil 9} = 12{\cdot}063 \times {\cdot}002 = \quad {\cdot}02413$$
$$a_{(x)\,\rceil 3t-1} = a_{31\,\rceil 14} = 8{\cdot}937 \times {\cdot}002 = \quad {\cdot}01787$$

$${\cdot}38950$$

$${\cdot}01285$$

$$a_{(x)\,\rceil vt-1} = a_{31\,\rceil 19} = \frac{{\cdot}01285}{6{\cdot}403} = {\cdot}002 \times 100 = {\cdot}2.$$

Example.—For an assurance of £100 on a life aged 31, the premium for the first 5 years is to be £2, for the second 5 years £1, for the third 5 years £1. 10*s*., for the fourth 5 years £1. 5*s*.; what further increase or decrease should be made for the fifth stage, and to continue during the remainder of life?

$$N_{x-1} = N_{30} = 702266{\cdot}223 \times {\cdot}02 \quad . \quad . \quad . \quad . \quad . \quad = 14045{\cdot}324$$
$$(p_1 - p) = {\cdot}02 - {\cdot}01 = 01$$
$$N_{x+t-1} = N_{35} = 534540{\cdot}812 \times {\cdot}01 \quad = 5435{\cdot}408$$
$$(p_2 \sim p_1) = {\cdot}01 \sim {\cdot}015 = {\cdot}005$$
$$N_{x+2t-1} = N_{40} = 412864{\cdot}847 \times {\cdot}005 \quad . \quad . \quad . \quad = 2064{\cdot}324$$
$$(p_3 - p_2) = {\cdot}015 - {\cdot}0125 = {\cdot}0025$$
$$N_{x+3t-1} = N_{45} = 305877{\cdot}513 \times {\cdot}0025 = 764{\cdot}694$$

$$16109{\cdot}648$$
$$6200{\cdot}102$$

$$9909{\cdot}546$$
$$M_x = M_{31} = \quad 13770{\cdot}543$$

$$3860{\cdot}997$$

and
$$N_{x+4t-1} = N_{50} = \frac{3860{\cdot}997}{219141{\cdot}709} = {\cdot}01762 \times 100 = 1{\cdot}762;$$

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or,

$$
\begin{array}{llll}
1+a_x = 1+a_{31} & =20{\cdot}519\times{\cdot}02 & = \ . \ . & ={\cdot}41038 \\
-a_{(x)\rceil t-1} = a_{31\rceil 4} & =15{\cdot}881\times{\cdot}01 & =15881 & \\
+a_{(x)\rceil 2t-1} = a_{31\rceil 9} & =12{\cdot}063\times{\cdot}005 & \ . \ . \ . & ={\cdot}06031 \\
-a_{(x)\rceil 3t-1} = a_{31\rceil 14} & = \ 8{\cdot}937\times{\cdot}0025 & =\underline{02234} & \\
 & & & \overline{{\cdot}47069} \\
 & & & {\cdot}18115 \\
 & & & \overline{{\cdot}28954} \\
 & & A_x = A_{31} \ = & {\cdot}40235 \\
 & & & \overline{{\cdot}11281}
\end{array}
$$

$$\text{and } \frac{{\cdot}11281}{a_{(x)\rceil 4t-1}=a_{30\rceil 14}=6{\cdot}403} = {\cdot}01762 \text{ (as before).}$$

V. The fifth and last case proposed to be treated is one where the first premium is increased or decreased by a fixed proportion at each of the succeeding stages; thus:—

An annual premium of p is to be paid for the first t years, to be increased or decreased by $\frac{p}{q}$ for the second t years, by $2.\frac{p}{q}$ for the third t years . . . . and by $(v-1)\frac{p}{q}$ for the vth t years; to find the value of p.

The statement of this case is as follows:—

Benefit term.

An assurance of £1 for life, the present value of which . . . . $=A_x$.

Payment terms.

An annuity of £p for the whole of life, the present value of which . . . . $p(1+a_x)$.

$\pm$ A deferred annuity of $\frac{p}{q}$, first payment or deduction at the end of t years . . . . . $\frac{p}{q}\cdot a_{(x)\rceil t-1}$.

$\pm$ A deferred annuity of $\frac{p}{q}=\left(2\frac{p}{q}-\frac{p}{q}\right)$, first payment or deduction at the end of $2t$ years . $\frac{p}{q}\cdot a_{(x)\rceil 2t-1}$.

. . . . .

$\pm$ A deferred annuity of $\frac{p}{q}=\left((v-2).\frac{p}{q}-(v-1).\frac{p}{q}\right)$, first payment or deduction at the end of vt years $\frac{p}{q}\cdot a_{(x)\rceil vt-1}$.

Equating,

$$A_x=p\left\{(1+a_x)\pm\frac{1}{q}\left(a_{(x)\rceil t-1}+a_{(x)\rceil 2t-1}+\ .\ .\ .\ .\ +a_{(x)\rceil vt-1}\right)\right\};$$

by substitution and division,

$$p=\frac{A_x}{(1+a_x)\mp\frac{1}{q}a_{(x)\rceil t-1}+a_{(x)\rceil 2t-1}+\ .\ .\ .\ .\ +a_{(x)\rceil vt-1}};$$

by substitution in the first equation,

$$\frac{M_x}{D_x}=p.\left\{\frac{N_{x-1}\pm\frac{1}{q}(N_{x+t-1}+N_{x+2t-1}+\ldots\ldots+N_{x+vt-1})}{D_x}\right\};$$

from which we get

$$p=\frac{M_x}{N_{x-1}\mp\frac{1}{q}(N_{x+t-1}+N_{x+2t-1}+\ldots\ldots+N_{x+vt-1})}.$$

Example.—What annual premium should be required during the first five years to insure £100 on a life aged 31, the annual premium to increase 4*s*. per £1, or one-fifth, every 5 years, and to remain constant at the end of 20 years?

$$t=5,\qquad v=4,\qquad \frac{1}{q}=\cdot 2.$$

$$\begin{array}{lr} N_{x+t-1}=N_{35}= & 543540\cdot 812 \\ N_{x+2t-1}=N_{40}= & 412864\cdot 847 \\ N_{x+3t-1}=N_{45}= & 305877\cdot 513 \\ N_{x+4t-1}=N_{50}= & 219141\cdot 537 \\ \hline & 1481424\cdot 709 \\ \frac{1}{q}= & \cdot 2 \\ \hline & 296284\cdot 9418 \\ N_{x-1}=N_{30}= & 702266\cdot 223 \\ \hline & 998551\cdot 165 \end{array}$$

$$M_x=M_{31}=\frac{13770\cdot 543}{998551\cdot 165}=\cdot 01379\times 100=£1.\ 7s.\ 7d.,$$

$$\text{and }\frac{1}{q}=\cdot 2\ \therefore\ \cdot 2\times\cdot 01379.\ \times 100=5s.\ 4d.\text{ nearly;}$$

or,

$$\begin{array}{lr} a_{(x)\overline{t-1}|}=a_{31\overline{4}|}= & 15\cdot 881 \\ a_{(x)\overline{2t-1}|}=a_{31\overline{9}|}= & 12\cdot 063 \\ a_{(x)\overline{3t-1}|}=a_{31\overline{14}|}= & 8\cdot 937 \\ a_{(x)\overline{3t-1}|}=a_{31\overline{19}|}= & 6\cdot 403 \\ \hline & 43\cdot 284 \\ \frac{1}{q}= & \cdot 2 \\ \hline & 8\cdot 6568 \\ 1+a_x=1+a_{31}= & 20\cdot 519 \\ \hline & 29\cdot 176 \end{array}$$

$$A_x=A_{31}=\frac{\cdot 40235}{29\cdot 176}=\cdot 01379\times 100=£1.\ 7s.\ 7d.$$

Example.—What annual premium should be required during the first 5 years to insure £100 on a life aged 31, the annual premium to be decreased 4*s*. per £1, or one-fifth, every 5 years, and to remain constant at the end of 20 years?

$$N_{x-1}=N_{30}=702266{\cdot}223$$

—as in preceding example . . . 296284·942

$$405981{\cdot}281$$

$$\text{and } M_x=M_{31}=\frac{13770{\cdot}543}{405981{\cdot}281}={\cdot}033919\times100=\pounds3{\cdot}3919$$

$$\text{and } \frac{1}{q}={\cdot}2 \therefore {\cdot}2\times {\cdot}033919 \times100=13s.\ 9d. \text{ nearly.}$$

Or,

$$1+a_x=1+a_{31}=20{\cdot}519$$

—as in preceding example . 8·657

$$11{\cdot}862$$

$$\text{and } A_x=A_{31}=\frac{{\cdot}40235}{11{\cdot}862}={\cdot}033919 \text{ (as before).}$$

In conclusion, it will be obvious that the values or ratio assigned to p and q, as well as the terms t and v, may, within certain limits, be varied at pleasure.

Apologising for the length of this communication,

I am, Sir,

Your obedient servant,

SAMUEL L. LAUNDY.

Eagle Life Office,
24th March, 1862.