Note on M. Collignon's Paper on the Integration of $a^n\cos a\,da$ and $a^n\sin a\,da$.

By Professor GIBSON.

The properties of the functions P_n, Q_n may also be determined very briefly as follows:—

Let u be a function of x; then by integration by parts we have

$$\int e^{\frac{x}{a}}u\,dx = ae^{\frac{x}{a}}u - a\int e^{\frac{x}{a}}\frac{du}{dx}\,dx$$

$$= ae^{\frac{x}{a}}\left\{u - a\frac{du}{dx} + a^2\frac{d^2u}{dx^2} - \ldots + (-1)^r a^r\frac{d^ru}{dx^r} + \ldots\right\}.$$

Now let $u = x^n$ and let ${}_nP_r = n(n-1)(n-2)\ldots(n-r+1)$, where n is a positive integer. Then

$$\int e^{\frac{x}{a}}x^n dx = ae^{\frac{x}{a}}\left\{x^n - {}_nP_1 a x^{n-1} + {}_nP_2 a^2 x^{n-2} - \ldots + (-1)^r {}_nP_r a^r x^{n-r} + \ldots\right\}.$$

Put $a = i$ so that $e^{\frac{x}{a}} = e^{-xi} = \cos x - i\sin x$; then

$$\int(\cos x - i\sin x)x^n\,dx$$

$$= (\sin x + i\cos x)\{x^n - i\,{}_nP_1x^{n-1} - {}_nP_2x^{n-2} + i\,{}_nP_3x^{n-3} + {}_nP_4x^{n-4} - \ldots$$
$$+ (-1)^r{}_nP_{2r}x^{n-2r} + i(-1)^{r+1}{}_nP_{2r+1}x^{n-2r-1} + \ldots\}.$$

Equating real and imaginary parts we get

$$\int\cos x\, x^n dx = \sin x\{x^n - {}_nP_2x^{n-2} + {}_nP_4x^{n-4} - \ldots + (-1)^r{}_nP_{2r}x^{n-2r} + \ldots\}$$
$$+ \cos x\{{}_nP_1x^{n-1} - {}_nP_3x^{n-3} + {}_nP_5x^{n-5} - \ldots + (-1)^r{}_nP_{2r+1}x^{n-2r-1} + \ldots\}$$
$$= P_n\sin x + Q_n\cos x,$$

$$\int\sin x . x^n dx = -P_n\cos x + Q_n\sin x.$$

The method of deriving the series proves the relation

$$Q_n = \frac{dP_n}{dx};$$

the actual values verify the relation and give also the general term.

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