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A VORONOVSKAYA THEOREM FOR VARIATION-DIMINISHING SPLINE APPROXIMATION

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1. Preliminary. In [7] Schoenberg introduced the following variation-diminishing spline approximation methods.

Let $m > 1$ be an integer and let $\Delta = \{x_i\}$ be a biinfinite sequence of real numbers with $x_i \leqq x_{i+1} < x_{i+m}$. To a function f associate the spline function Vf of order m with knots Δ defined by

$$(1.1) \quad Vf(x) = \sum_j f(\xi_j) N_j(x)$$

where

$$\xi_j = (x_{j+1} + x_{j+2} + \ldots + x_{j+m-1})/(m - 1)$$

and the $N_j(x)$ are B-splines with support $x_j < x < x_{j+m}$ normalized so that $\sum_j N_j(x) = 1$. See, e.g., [2] for a precise definition of the $N_j(x)$ and a discussion of the properties of Vf.

We shall be concerned with only the special case

$$\begin{aligned} & x_i = 0 \quad \text{for } i = 1 - m, \ldots, -1, 0 \\ (1.2) \quad & x_i = i/n \text{ for } i = 1, 2, \ldots, n - 1 \\ & x_i = 1 \quad \text{for } i = n, \ldots, n + m - 1 \end{aligned}$$

where n is a positive integer. Thus, we suppose that f is defined on [0, 1] and restrict Vf to [0, 1].

Note that (1.2) implies that $Vf(0) = f(0)$ and $Vf(1) = f(1)$ and that (1.1) becomes the finite sum

$$(1.3) \quad Vf(x) = \sum_{-m<j<n} f(\xi_j) N_j(x).$$

We shall henceforth use $\sum_j$ to denote the range of this finite sum.

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THEOREM 1. (Schoenberg). *Let f be bounded in* [0, 1] *and let Vf be given by* (1.2) *and* (1.3) *with $m > 2$. If $x \in (0, 1)$ is such that $f''(x)$ exists, then*

$$\lim_{m/n \to 0} \frac{n^2}{m}[Vf(x) - f(x)] = \frac{f''(x)}{24}.$$

This theorem was extended to higher derivatives by Marsden and Riemenschneider [**6**]. Theorem 1 and its extension are analogues of Voronovskaya's theorem about Bernstein polynomial approximation and its extension by Bernstein (see e.g., [**4**]). Theorem 1 does not include Voronovskaya's theorem since Bernstein polynomial approximation is Vf for the special case $n = 1$. Indeed, m/n tends to infinity in Voronovskaya's theorem.

Since Vf converges to f at points of continuity of f if and only if $m + n$ tends to infinity, we are led to consider the following:

Question. Let f be bounded in [0, 1] and let Vf be given by (1.2) and (1.3). Let $x \in (0, 1)$ be such that $f''(x)$ exists. How does $Vf(x) - f(x)$ behave as $m + n \to \infty$ if m/n does not tend to zero?

Note that $m + n - 1$ is the number of data points $f(\xi_j)$ needed to specify Vf so that $m + n$ is a measure of the "complexity" of Vf.

2. The main result. The following theorem, which answers the above question, was stated in [**5**] as a conjecture.

THEOREM 2. *Let f be bounded in* [0, 1] *and let Vf be given by* (1.2) *and* (1.3). *Let $x \in [0, 1]$ be such that $f''(x)$ exists. If*

$$\lim_{m+n \to \infty} \frac{m-1}{n} = t \tag{2.1}$$

exists as a nonnegative extended real number, then

$$\lim_{m+n\to\infty} (m + n)[Vf(x) - f(x)] = f''(x)[e(x, t)/2]$$

where

$$e(x, t) = \begin{cases} \dfrac{1+t}{3t^2}[(2tx)^{3/2} - 3tx^2] & \left(0 \leqq x \leqq \dfrac{1}{2}, 2x \leqq t \leqq \dfrac{1}{2x}\right) \\ \dfrac{1}{12}t(t+1) & \left(\dfrac{t}{2} \leqq x \leqq 1 - \dfrac{t}{2}, 0 \leqq t \leqq 1\right) \\ \dfrac{1+t}{t}\left(x - x^2 - \dfrac{1}{6t}\right) & \\ \left(\dfrac{1}{2t} \leqq x \leqq 1 - \dfrac{1}{2t}, 1 \leqq t \leqq \infty\right) & \end{cases}$$

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and

$$e(x, t) = e(1 - x, t) \quad for \frac{1}{2} \leqq x \leqq 1.$$

3. The functions $E_r(x)$. If $m/n \to 0$, the conclusion of Theorem 2 follows from Theorem 1. Hence, we shall always assume that $m/n \to t > 0$. In particular, this forces $m \to \infty$.

For nonnegative integers r, set

$$E_r(x) = \sum_j (\xi_j - x)^r N_j(x).$$

Note that $E_0(x) = 1$ and $E_1(x) = 0$.

An important preliminary argument is the following: If f is bounded in $[0, 1]$ and $f^{(r)}(x)$ exists at a certain x in $(0, 1)$, Taylor series expansion at x implies that

$$Vf(x) = f(x) + \sum_{k=2}^{r} \frac{f^{(k)}(x)}{k!} E_k(x) + \sum_j \eta_r(x, \xi_j)(\xi_j - x)^r N_j(x)$$

where $\eta_r(x, \xi)$ is bounded and tends to zero as ξ tends to x. Now, we let $\delta > 0$, H_r be an upper bound on $|\eta_r(x, \xi)|$, $\omega(\delta, \eta_r)$ be the modulus of continuity function for η_r, $\sum'$ and $\sum''$ denote summation over those j for which, respectively,

$$|\xi_j - x| < \delta \quad \text{and} \quad |\xi_j - x| \geqq \delta,$$

and r be an even integer. Then

$$\left| \sum_j \eta_r(x, \xi_j)(\xi_j - x)^r N_j(x) \right|$$

$$\leqq \sum_j |\eta_r(x, \xi_j)| (\xi_j - x)^r N_j(x)$$

$$\leqq \omega(\delta, \eta_r) \sum{}' (\xi_j - x)^r N_j(x) + \frac{H_r}{\delta^2} \sum{}'' (\xi_j - x)^{r+2} N_j(x)$$

$$\leqq \omega(\delta, \eta_r) E_r(x) + \frac{H_r}{\delta^2} E_{r+2}(x).$$

If $h = h(m)$ is some parameter tending to zero as $m \to \infty$ and if it can be shown that

$$E_r(x) = O(h^r),\ E_{r+2}(x) = O(h^{r+2}) \text{ as } m \to \infty,$$

then, with the choice $\delta^2 = h$, we have

$$h^{-r} \left| \sum_j \eta_r(x, \xi_j)(\xi_j - x)^r N_j(x) \right|$$

$$\leqq \omega(\sqrt{h}, \eta_r) \cdot O(1) + H_r \cdot O(h)$$

and, hence,

$$\lim_{m\to\infty} h^{-r}\left[Vf(x) - f(x) - \sum_{k=2}^{r-1} \frac{f^{(k)}(x)}{k!}E_k(x)\right]$$

$$= \frac{f^{(r)}(x)}{r!} \lim_{m\to\infty} h^{-r}E_r(x).$$

This argument with $r = 2$ and $h = m^{-1/2}$ will complete the proof of Theorem 2 once we have established

LEMMA 1. *As* $m \to \infty$, $E_2(x) = O(m^{-1})$ *and* $E_4(x) = O(m^{-2})$. *Moreover, if* (2.1) *holds, then*

$$\lim_{m\to\infty} mE_2(x) = \frac{t}{1 + t}e(x, t). \tag{3.1}$$

The remainder of this paper will be devoted to a proof of this lemma.

4. Consequences of a *B*-spline identity. In [5] was proved the *B*-spline identity

$$x^k = \sum_j \xi_{j,k}N_j(x) \quad \text{for } k = 0, 1, \dots, m - 1$$

where $\xi_{j,0} = 1$ and, for $k > 0$,

$$\xi_{j,k} = \binom{m-1}{k}^{-1} \sum_{j<i_1<\dots<i_k<j+m} x_{i_1}x_{i_2}\dots x_{i_k}.$$

Of course, $\xi_{j,1} = \xi_j$.

After some manipulation, we obtain

$$E_r(x) = \sum_j \frac{f_r(x, \xi_j)}{(m - 2)}N_j(x)$$

where

$$\frac{f_r(x, \xi_j)}{(m - 2)} = \sum_{k=2}^{r} (-x)^{r-k}\binom{r}{k}(\xi_j^k - \xi_{j,k}).$$

Note that $f_2(x, \xi_j)$ is independent of x.

As a first step in the further analysis, we shall show that the $f_r(x, y)$ are well-defined bounded functions by exhibiting them explicitly. While we do this only for $r \leqq 4$, it is clear from an induction argument that the process can be continued.

A second step is to interpret $\sum_j f_r(x, \xi_j)N_j(y)$ as being almost $Vg_r(x, y)$

in the y variable for some $g_r(x, y)$ and then showing that $Vg_r(x, x)$ has the appropriate behavior. We do this only for $r = 2$. For $r = 4$ we are content to show only that

$$Vg_4(x, x) = O(m^{-1}).$$

See, however, the Remarks section.

5. Formulae for the $f_r(x, y)$. Set

$$y_i = nx_i = \begin{cases} 0 \text{ if } 2 - m \leqq i \leqq 0 \\ i \text{ if } 0 \leqq i \leqq n \\ n \text{ if } n \leqq i \leqq n + m - 2. \end{cases}$$

Set $A_{o,n,j,i} = 1$ and

$$A_{r,n,j,i} = \sum_{j<i_1<\ldots<i_r<i} y_{i_1}y_{i_2}\ldots y_{i_r}.$$

Note that

$$(5.1) \quad n^r\binom{m-1}{r}\xi_{j,r} = A_{r,n,j,j+m}.$$

For $r > 0$

$$A_{r,n,j,i} = \sum_{k=j+r}^{i-1} y_k A_{r-1,n,j,k}$$

$$= \sum_{k=\max(r,j+r)}^{i-1} \min(k, n)A_{r-1,n,j,k}$$

a recurrence which we will now solve, case by case.

Case 1. $j \leqq 0 \leqq r < i \leqq n + 1$. The recurrence becomes

$$A_{r,n,j,i} = \sum_{k=r}^{i-1} kA_{r-1,n,j,k}$$

which solves as

$$A_{r,n,j,i} = \sum_{k=0}^{r} p_{k,r}\binom{i}{2r-k}$$

with $p_{-1,r} = 0$, $p_{r,r} = \delta_{0,r}$, and, for $0 \leqq k \leqq r$,

$$p_{k,r+1} = (2r - k + 1)(p_{k,r} + p_{k-1,r}).$$

In particular,

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$$A_{1,n,j,i} = \binom{i}{2}, \; A_{2,n,j,i} = 3\binom{i}{4} + 2\binom{i}{3},$$

$$A_{3,n,j,i} = 15\binom{i}{6} + 20\binom{i}{5} + 6\binom{i}{4},$$

$$A_{4,n,j,i} = 105\binom{i}{8} + 210\binom{i}{7} + 130\binom{i}{6} + 24\binom{i}{5}.$$

With $A_1 = A_{1,n,j,i} = i(i - 1)/2$ and $R = 2i - 1 = (8A_1 + 1)^{1/2}$, we have, after a tedious argument,

$$2A_{2,n,j,i} = A_1^2 - A_1R/3,$$
$$6A_{3,n,j,i} = A_1^3 - A_1^2R + 2A_1^2,$$
$$24A_{4,n,j,i} = A_1^4 - 2A_1^3R + \frac{32}{3}A_1^3 - \frac{12}{5}A_1^2R + \frac{1}{3}A_1^2 + \frac{2}{5}A_1R.$$

With $i = j + m$, $\xi_j = \xi_{j,1}$, $Q_j = R/n$, and $\xi_{j,r}$ given by (5.1), we have

$$(m - 2)(\xi_j^2 - \xi_{j,2}) = -\xi_j^2 + \frac{1}{3}\xi_jQ_j,$$

$$(m - 2)(m - 3)(\xi_j^3 - \xi_{j,3}) = (-3m + 5)\xi_j^3$$
$$+ (m - 1)\xi_j^2Q_j - 2\frac{m - 1}{n}\xi_j^2,$$

$$(m - 2)(m - 3)(m - 4)(\xi_j^4 - \xi_{j,4}) = (-6m^2 + 23m - 23)\xi_j^4$$
$$+ 2(m - 1)^2\xi_j^3Q_j - \frac{32}{3}\frac{(m - 1)^2}{n}\xi_j^3$$
$$+ \frac{12}{5}\frac{(m - 1)}{n}\xi_j^2Q_j - \frac{1}{3}\frac{(m - 1)}{n^2}\xi_j^2 - \frac{2}{5}\frac{1}{n^2}\xi_jQ_j$$

whence

$$f_2(x, \xi_j) = -\xi_j^2 + \frac{1}{3}\xi_jQ_j,$$

$$f_3(x, \xi_j) = 3(\xi_j - x)f_2(x, \xi_j) + \frac{-4\xi_j^3 + 2\xi_j^2Q_j - 2\dfrac{m - 1}{n}\xi_j^2}{(m - 3)},$$

$$f_4(x, \xi_j) = 6(\xi_j - x)^2f_2(x, \xi_j)$$
$$+ 4(\xi_j - x)\frac{-4\xi_j^3 + 2\xi_j^2Q_j - 2\dfrac{m - 1}{n}\xi_j^2}{m - 3}$$

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$$+ \frac{a_j}{(m-3)(m-4)}$$

with

$$a_j = (-3m - 15)\xi_j^4 + (2m + 10)\xi_j^3 Q_j - \frac{(8m+64)(m-1)}{3n}\xi_j^3$$

$$+ \frac{12(m-1)}{5n}\xi_j^2 Q_j - \frac{m-1}{3n^2}\xi_j^2 - \frac{2}{5n^2}\xi_j Q_j.$$

Case 2. $j \leqq 0 < n \leqq i$. The recurrence becomes, if $n < i$,

$$A_{r,n,j,i} = A_{r,n,j,n+1} + n \sum_{k=n+1}^{i-1} A_{r-1,n,j,k}$$

which solves as

$$(5.2) \quad A_{r,n,j,i} = \sum_{k=0}^{r} q_{k,r,n} \binom{i - \frac{n+1}{2}}{k}$$

with $q_{0,0,n} = 1$,

$$q_{0,r,n} = \sum_{k=0}^{r-1} p_{k,r} \binom{n+1}{2r-k} - \sum_{k=1}^{r} n q_{k-1,r-1,n} \binom{\frac{n+1}{2}}{k}$$

and $q_{k,r,n} = nq_{k-1,r-1,n}$ for $0 < k \leqq r$. The $p_{k,r}$ are as defined in Case 1. One easily checks that (5.2) is valid for $i = n$. In particular, with $I = i - (n+1)/2$ and $A_1 = A_{1,n,j,i} = nI$,

$$\begin{aligned}
2A_{2,n,j,i} &= n^2 I(I-1) + (n^3 - n)/6 \\
&= A_1^2 - nA_1 + (n^3 - n)/6, \\
6A_{3,n,j,i} &= n^3 I(I-1)(I-2) + (n^4 - n^2)(I-1)/2 \\
&= A_1^3 - 3nA_1^2 + (n^3 + 4n^2 - n)A_1/2 - (n^4 - n^2)/2, \\
24A_{4,n,j,i} &= n^4 I(I-1)(I-2)(I-3) + (n^5 - n^3)I(I-1) \\
&\quad - 2(n^5 - n^3)I + (n^3 - n)(9n^2 - 1)/5 + (n^3 - n)^2/12 \\
&= A_1^4 - 6nA_1^3 + (n^3 + 11n^2 - n)A_1^2 \\
&\quad - (3n^4 + 6n^3 - 3n^2)A_1 \\
&\quad + (n^3 - \mathrm{n})(9n^2 - 1)/5 + (n^3 - \mathrm{n})^2/12
\end{aligned}$$

with $i = j + m$, $\xi_j = \xi_{j,1}$, and $\xi_{j,r}$ given by (5.1), we have

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$$(m - 2)(\xi_j^2 - \xi_{j,2}) = -\xi_j^2 + \xi_j - (n^2 - 1)/6n(m - 1),$$
$$(m - 2)(m - 3)(\xi_j^3 - \xi_{j,3}) = (-3m + 5)\xi_j^3 + (3m - 3)\xi_j^2$$
$$- (n^2 + 4n - 1)\xi_j/2n + (n^2 - 1)/2n(m - 1),$$
$$(m - 2)(m - 3)(m - 4)(\xi_j^4 - \xi_{j,4}) = (-6m^2 + 23m - 23)\xi_j^4$$
$$+ 6(m - 1)^2\xi_j^3 - (n^2 + 11n - 1)(m - 1)\xi_j^2/n$$
$$+ (3n^2 + 6n - 3)\xi_j/n - (n^2 - 1)^2/12(m - 1)n^2$$
$$- (n^2 - 1)(9n^2 - 1)/5(m - 1)n^3$$

whence

$$f_2(x, \xi_j) = -\xi_j^2 + \xi_j - (n^2 - 1)/6n(m - 1),$$
$$f_3(x, \xi_j) = 3(\xi_j - x)f_2(x, \xi_j) + b_j/(m - 3),$$
$$f_4(x, \xi_j) = 6(\xi_j - x)^2 f_2(x, \xi_j) + 4(\xi_j - x)b_j/(m - 3)$$
$$+ c_j/(m - 3)(m - 4)$$

with

$$b_j = -4\xi_j^3 + 6\xi_j^2 - (n^2 + 2nm - 2n - 1)\xi_j/n(m - 1)$$
$$+ (n^2 - 1)/2n(m - 1),$$
$$c_j = (-3m - 15)\xi_j^4 + (6m + 30)\xi_j^3$$
$$- \left[\frac{(n^2 - 1)(m + 5)}{n(m - 1)} + 3m + 21\right]\xi_j^2$$
$$+ \left[\frac{(n^2 - 1)(m + 5)}{n(m - 1)} + 6\right]\xi_j$$
$$- \left[\frac{n^2 - 1}{12n} + \frac{9n^2 - 1}{5n^2}\right]\frac{n^2 - 1}{n(m - 1)}.$$

Case 3. $-1 \leqq j \leqq j + r < i \leqq n + 1$. This case, which does not involve the coalesced endpoint knots, was completely discussed in [**6**]. Here we make the results "fit in" with the other cases.

The recurrence becomes

$$A_{r,n,j,i} = \sum_{j+r}^{i-1} kA_{r-1,n,j,k}$$

which solves as

$$A_{r,n,j,i} = \sum_{k=0}^{r} c_{k,r,j}\binom{i - j - 1}{2r - k}$$

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with

$$c_{0,0,j} = 1,\ c_{-1,r-1,j} = c_{r,r-1,j} = 0, \quad \text{and}$$
$$c_{k,r,j} = (2r - k + j)c_{k-1,r-1,j} + (2r - k - 1)c_{k,r-1,j} \qquad \text{for } 0 \leqq k \leqq r.$$

Proceeding as in Cases 1 and 2, we have

$$f_2(x, \xi_j) = m(m - 2)/12n^2,$$
$$f_3(x, \xi_j) = 3(\xi_j - x)f_2(x, \xi_j),$$
$$f_4(x, \xi_j) = 6(\xi_j - x)^2 f_2(x, \xi_j) - (5m^2 + 2m)(m - 2)/240n^4.$$

There is a Case 4 which is symmetric with Case 1.

Reflection on the intervals in which ξ_j must lie in each case permits us to summarize thusly:

$$f_2(x, y) = -y^2 + \frac{1}{3}y\sqrt{8\frac{m-1}{n}y + \frac{1}{n^2}}$$

for $0 \leqq y \leqq \min\left(\frac{m-1}{2n}, \frac{n}{2(m-1)}\right)$,

$$= y - y^2 - \frac{n^2 - 1}{6n(m-1)}$$

for $\frac{n}{2(m-1)} \leqq y \leqq \frac{1}{2}$,

$$= \frac{m(m-2)}{12n^2}$$

for $\frac{m-1}{2n} \leqq y \leqq \frac{1}{2}$,

$$= f_2(x, 1 - y)$$

for $\frac{1}{2} \leqq y \leqq 1$. Similar statements hold for $f_3(x, y)$ and $f_4(x, y)$.

6. Proof of lemma 1. One can show easily that, for $0 \leqq y \leqq 1/2$, $f_2(x, y) \leqq y$. Hence, using symmetry,

$$f_4(x, y) = 6(y - x)^2 f_2(x, y) + O\left(\frac{1}{m}\right)$$
$$\leqq 3(y - x)^2 + C_1/m$$

with C_1 a constant. Thus,

$$E_2(x) \leqq \min(x, 1 - x)/(m - 2) \leqq \frac{2}{m}$$

and

$$E_4(x) \leqq 3E_2(x)/(m - 2) + C_1/m(m - 2)$$
$$\leqq (6 + C_1)/m(m - 2) \leqq (24 + 4C_1)/m^2.$$

Since both $E_2(x)$ and $E_4(x)$ are positive, the first assertion of Lemma 1 is proved.

Now let $g_2(y) = g_2(y, (m - 1)/n)$ be defined by

$$g_2(y) = -y^2 + \frac{2y}{3}\sqrt{2\frac{m-1}{n}y}$$

for $0 \leqq y \leqq \min\left(\frac{m-1}{2n}, \frac{n}{2(m-1)}\right)$,

$$= y - y^2 - \frac{n}{6(m-1)}$$

for $\frac{n}{2(m-1)} \leqq y \leqq \frac{1}{2}$,

$$= \frac{(m-1)^2}{12n^2}$$

for $\frac{m-1}{2n} \leqq y \leqq \frac{1}{2}$,

$$= g_2(1 - y)$$

for $\frac{1}{2} \leqq y \leqq 1$. Then,

$$f_2(x, y) = g_2(y) + O\left(\frac{1}{m}\right)$$

and, hence,

$$(m - 2)E_2(x) = Vg_2(x) + O\left(\frac{1}{m}\right) \quad \text{as } m \to \infty.$$

LEMMA 2. *As* $m \to \infty$, $Vg_2(x)$ *tends to* $g_2(x)$. *Hence, if* (2.1) *holds, then*

$$\lim_{m\to\infty} mE_2(x) = \lim_{m\to\infty} g_2(x) = \frac{t}{1+t}e(x, t).$$

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The following "false proof" of Lemma 2 is instructive:

The fact (see [5]) that

$$\xi_j^2 - \xi_{j,2} = \frac{1}{(m-1)^2(m-2)} \sum_{j<r<s<j+m} (x_s - x_r)^2$$
$$\leqq \frac{1}{2(m-1)}$$

and standard arguments about positive linear approximation (see, e.g. [3]) yield

A. If $f \in C[0, 1]$, then

$$\|Vf - f\|_\infty \leqq \frac{3}{2}\omega(1/\sqrt{m-1}, f).$$

B. If $f \in C'[0, 1]$, then

$$\|Vf - f\|_\infty \leqq (2/\sqrt{m-1})\omega(1/\sqrt{m-1}, f').$$

C. If $f \in C''[0, 1]$, then

$$\|Vf - f\|_\infty \leqq \|f''\|/4(m-1).$$

The last fact would prove Lemma 2 if g_2 had a bounded second derivative. Unfortunately, it does not. Indeed, in the interiors of its respective domains of definition,

$$g_2''(x) = -2 + \sqrt{\frac{m-1}{2nx}}, -2, 0, -2 + \sqrt{\frac{m-1}{2n(1-x)}}.$$

A correct proof of Lemma 2 follows from the integrability of g_2''. Suppose first that $(m-1)/n$ does not tend to infinity. If $0 \leqq x \leqq 1/2$, Taylor's series with integral remainder gives

$$|Vg_2(x) - g_2(x)| \leqq \sum_j \left| \int_x^{\xi_j} g_2''(s)(\xi_j - s)ds \right| N_j(x)$$
$$\leqq \sum_j |\xi_j - x| \cdot \left| \int_x^{\xi_j} \left| g_2''(s) \right| ds \right| N_j(x)$$
$$\leqq \sum_j |\xi_j - x| \cdot \left| \int_x^{\xi_j} \max\left(\sqrt{\frac{m-1}{2ns}}, 2, \sqrt{\frac{m-1}{2n(1-s)}} \right) ds \right| N_j(x)$$
$$\leqq \sum_j |\xi_j - x| (2 + \sqrt{2})\left(1 + \frac{m-1)}{4n}\right) \times |\sqrt{\xi_j} - \sqrt{x}| N_j(x)$$

$$\leqq (2 + \sqrt{2})\left(1 + \frac{m-1}{4n}\right)E_2(x)/\sqrt{x}$$

$$\leqq (2 + \sqrt{2})\left(1 + \frac{m-1}{4n}\right)\sqrt{x}/(m-2)$$

$$= O\left(\frac{1}{m}\right).$$

The "worst case" in the integral estimation occurs with $x = 1/2$, $\xi_j = 1$, and $m - 1 < 4n$.

If $(m - 1)/n$ tends to infinity, we have

$$0 \leqq x - x^2 - g_2(x) \leqq n/6(m - 1)$$

so that

$$|Vg_2(x) - g_2(x) + E_2(x)| \leqq n/6(m - 1)$$

and, hence,

$$|Vg_2(x) - g_2(x)| = O(n/(m - 1)).$$

Lemma 2 is proved and, hence, also Lemma 1 and Theorem 2.

7. Remarks. From Theorem 2 one can produce other results, for example, pointwise versions of the facts A, B, C stated in Section 6. One interesting result is the following analogue of the Bajsanski-Bojanic theorem. See their paper [**1**] for the proof.

THEOREM 3. *Let f be continuous in* [0, 1] *and let Vf be given by* (1.2) *and* (1.3). *If*

$$Vf(x) - f(x) = O((m + n)^{-1}) \quad \text{as } m/n \to t > 0$$

holds for each x in (a, b) with $0 \leqq a < b \leqq 1$, then f is a linear function on $[a, b]$.

The requirement that $t > 0$ is necessary since $e(x, t) > 0$ is needed in the proof.

The obvious open problem is concerned with the behaviour of $Vf(x) - f(x)$ when $f^{(r)}(x)$ exists with $r > 2$. Presumably, one could extend the arguments in Sections 5 and 6 above to produce theorems for the case of even r, but the prospect is not appealing. A comparison of the approach here with that in [**6**] and that used to extend Voronovskaya's theorem (see [**4**]) show that three distinctly different methods have been used. One would like to see a recurrence formula involving the $E_r(x)$ rather than the $A_{r,n,j,i}$.

Application of Theorem 2 to $f_4(x, y)$ yields

$$\lim_{m+n\to\infty} (m + n)^2 E_4(x) = 3(e(x, t))^2.$$

This supports the conjecture that

$$\lim_{m+n\to\infty} (m + n)^r E_{2r}(x) \overset{?}{=} \frac{(2r)!}{r!}\left(\frac{e(x, t)}{2}\right)^r.$$

Given these, one could use the argument in Section 3 above to extend Theorem 2. This conjectured extension has been stated (with a slight misprint) in [**5**].

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