ON EDGE-DISJOINT CYCLES IN A GRAPH

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1. **Introduction.** Let $g(k)$ denote the least integer such that every graph $G(n, n+g(k))$, with n vertices and $n+g(k)$ edges, contains at least k edge-disjoint cycles; let $h(k)$ be similarly defined for planar graphs. Loops and multiple edges (i.e., cycles of length one and two) are permitted in both cases.

A trivial result is that $g(1) = h(1) = 0$. Erdős and Pósa [2] showed that $g(2) = 4$; Dirac and Erdős [1] showed that $h(2) = 3$. In the present paper their arguments will be extended to show that

$$g(3) = 10 ,$$

$$h(3) = 7 ,$$

and

$$h(4) = 11 .$$

Further exact results would appear to require a stronger method of attack or the examination of a considerable number of special cases.

2. **Proof that** $g(3) = 10$. We first show that every graph $G(n, n+10)$ contains at least three edge-disjoint cycles. This assertion is clearly true for $n = 1$; suppose that it has been established for all $n < m$, where m is some integer greater than one. Consider any graph $G = G(m, m+10)$. We may assume that G has no cycle of length less than seven, for if it did the graph obtained by removing the edges of this cycle would still have at least $m+4$ edges and hence would contain

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at least two edge-disjoint cycles since $g(2) = 4$.   So G would contain at least three edge-disjoint cycles and there would be nothing to prove. If x is a vertex of degree less than two, then the induction hypothesis may be applied to the graph obtained from G by removing x; if x is of degree two then the induction hypothesis may be applied to the graph obtained from G by removing x and adding an edge joining the two vertices originally joined to x.   From this it follows that we may assume that the degree of each vertex of G is at least three.   This implies that $m+10 \geq \frac{3}{2}m$, or $m \leq 20$.   But it is easy to show (see the proof of Lemma 3 in [2]) that any graph each vertex of which has degree at least three and which contains no cycles of length less than seven must have at least 22 vertices.   This contradiction suffices to establish the above assertion by induction.   Therefore $g(3) \leq 10$.

The graph in Figure 1 has 18 vertices, is regular of degree three, and its shortest cycles are of length six. Hence, if it had three edge-disjoint cycles they each would be of length six and each vertex would belong to one and only one of them.
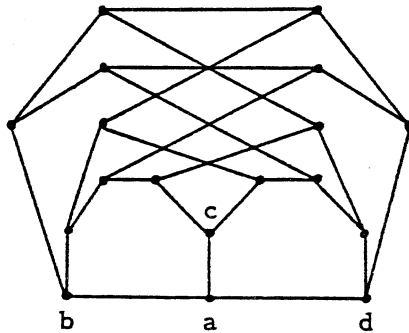

Figure 1

Consider the vertex a ; it is easily verified that there are no cycles of length six passing through a via the edges (b, a) and (a, d).   Hence, by symmetry, we may assume that if this graph contains three edge-disjoint cycles then the cycle passing through a contains the edges (c, a) and (a, d).   There are only two cycles of length six with this property.   But when the vertices of either of these cycles are removed, along with their incident edges, the remaining graph consists of a hexagon

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with its three diagonals joining opposite vertices, some edges
of which have been subdivided by the insertion of additional
vertices. It is clear that such a graph does not contain two
edge-disjoint cycles. From this it follows that the graph in
Figure 1 does not contain three edge-disjoint cycles. Since
it has 27 edges it must be that $g(3) > 9$, and it is trivial to
use this example to construct graphs $G(m, m+9)$ not containing
three edge-disjoint cycles for any $m \geq 18$. Combining this
inequality with the earlier one, we have that $g(3) = 10$.

3. **A lemma.** We sketch the proof of the following lemma,
although it employs a rather familiar type of argument. (See
Steinitz [4], pp. 7-8.)

LEMMA. If $G = G(n, e)$ is a planar graph such that the
length of each cycle is at least five and the degree of each vertex
is at least three, then $3e \leq 5n - 10$.

Proof. We may assume that G is a planar map to which
Euler's polyhedral formula may be applied; for, if it was not it
could be made into one by adding appropriate edges, and it is
not difficult to see that this could be done without forming any
cycles of length less than five. Suppose that G has a total of
f faces of which f_i have i sides. Then

$$2e = 5f_5 + 6f_6 + \cdots \geq 5 (f_5 + f_6 + \cdots) = 5f,$$

appealing to the hypothesis on the lengths of the cycles.
Substituting this inequality in Euler's formula, $n - e + f = 2$,
yields the required result.

4. **Proof that** $h(3) = 7$. We first show that every planar
graph $G(n, n+7)$ contains at least three edge-disjoint cycles.
This assertion is certainly true for small values of n. Let
m be the smallest integer for which there exists a planar graph
$G = G(m, m+7)$ for which it is not true. Using the fact that
$h(2) = 3$, the argument in §2 can be applied to show that the
length of each cycle of G is at least five and the degree of
each vertex is at least three. Hence, $m+7 \geq \frac{3}{2}m$ or $m \leq 14$.
But by the lemma, $3(m+7) \leq 5m - 10$ or $m \geq 15\frac{1}{2}$. This
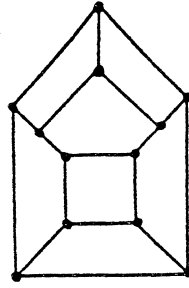contradiction establishes the assertion and shows that $h(3) \leq 7$.

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Figure 2

Consider the planar graph in Figure 2. If it contained three edge-disjoint cycles they would each have to be of length four. But it is clearly impossible to select three of the four cycles of length four without some edge being used more than once. This example, with 12 vertices and 18 edges, shows that h(3) > 6 and such examples with n vertices can be constructed for any n ≥ 12. Combining the two inequalities, we have that h(3) = 7.

5. <u>Proof that</u> h(4) = 11. It is easily verified that the graph of a regular dodecahedron, having 20 vertices and 30 edges, does not contain four edge-disjoint cycles. The examples obtained from this by subdividing its edges by inserting more vertices all show that h(4) > 10.

We now show that every planar graph G (n, n+11) contains at least four edge-disjoint cycles. Let m be the smallest integer for which there exists a planar graph $G = G$(m, m+11) without four such cycles. The same argument as used before shows that the length of each cycle of G is at least five and the degree of each vertex is at least three. Hence, $m+11 \geqq \frac{3}{2}m$ or $m \leqq 22$, and $3(m+11) \leqq 5m - 10$ or $m \geqq 21\frac{1}{2}$, by the lemma. As m is an integer it must be that m = 22 and that the degree of each vertex of G is exactly three.

A more careful analysis of the argument used in proving the lemma shows that if the planar graph G has 22 vertices, is regular of degree three, and has no cycles of length less than or equal to four, then G must be a planar map with 12

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pentagonal faces and one hexagonal face. But, as pointed out by Grünbaum and Motzkin [3], no such map exists.

This contradiction establishes the above assertion which when combined with the earlier inequality completes the proof that h(4) = 11.

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