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Comparison of K-Theory Galois Module Structure Invariants

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Abstract. We prove that two, apparently different, class-group valued Galois module structure invariants associated to the algebraic K-groups of rings of algebraic integers coincide. This comparison result is particularly important in making explicit calculations.

1 Introduction

Let G be a finite group and let M, N be finitely generated $\mathbf{Z}[G]$-modules. If $\alpha \in \mathrm{Ext}^2_{\mathbf{Z}[G]}(M, N)$ is a 2-extension whose cup-product induces an isomorphism, $(\alpha \cup -)$: $\hat{H}^i(G\ ;\ M) \to \hat{H}^{i+2}(G\ ;\ N)$, in Tate cohomology for all i then α may be realised by a 2-extension of the form $N \to A \to B \to M$ in which A and B are finitely generated, cohomologically trivial $\mathbf{Z}[G]$-modules. Since A and B have finite projective dimension, they define classes, $[A]$ and $[B]$, in the class-group of finitely generated, projective $\mathbf{Z}[G]$-modules, $\mathcal{CL}(\mathbf{Z}[G])$, and we may form the Euler characteristic, $[A] - [B] \in \mathcal{CL}(\mathbf{Z}[G])$. The Euler characteristic depends only on the isomorphism class of α. In number theory examples of Euler characteristics are given by the Chinburg invariants, $\Omega(L/K, i)$ ($i = 1, 2, 3$), of [4].

Let L be a number field and let G be a finite group of automorphisms of L. Independently, Snaith [19, Chapter 7] and Pappas (unpublished) used work of Kahn [11] to define an invariant associated to $K_3(\mathcal{O}_L)$ and $K_2(\mathcal{O}_L)$ in $Cl(\mathbf{Z}[G])$ for arbitrary G in the absence of ramification at infinity. In [5], [6] we gave an unconditional construction of invariants in $\mathcal{CL}(\mathbf{Z}[G])$ related to $K_{2r-1}(\mathcal{O}_L)$ and $K_{2r-2}(\mathcal{O}_L)$ for all L, G and $r > 1$. The purpose of this paper is to show that the invariants of [5], [6] coincide with those defined earlier in [19, Chapter 7] when $r = 2$. This comparison is necessitated by the fact that the quaternion examples of [7], [8] are evaluated in terms of the original construction of [19, Chapter 7]. Related constructions appear in [1] and [2], using work of Bloch-Kato [3] and Kato [12] to define invariants in $\mathcal{CL}(\mathbf{Z}[G])$ associated to motives.

The invariants of [5], [6] are denoted by $\Omega_n(L/K)$ and $\Omega_1(L/K)$ is the class associated to K_2 and K_3. The numbering of the invariants is meant to indicate the belief that $\Omega_n(L/K)$ is related to the values of the associated Artin L-functions at $s = -n$. The calculations of [7], [8] and the conjecture of [6] (see also [1], [2]) make this belief more explicit.

The invariant, $\Omega_1(L/K)$, will be defined in Section 4.1. The invariant of [19, Chapter 7] is simpler to describe.

Take $G = G(L/K)$, the Galois group of L/K. Let S be a finite $G(L/K)$-invariant set of places of L contain the infinite places, $S_\infty(L)$, and all finite places which ramify over K.

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Let $K_2'(\mathcal{O}_{L,S})$ denote the kernel of the natural map, $K_2(\mathcal{O}_{L,S}) \to \bigoplus_{w \in S_\infty(L)} K_2(L_w)$. In [19, Chapter 7] a 2-extension was obtained from a K-theory sequence of [11, Section 5] of the form

$$K_3^{\mathrm{ind}}(\mathcal{O}_{L,S}) \longrightarrow \bigoplus_{w \in S_\infty(L)} K_3^{\mathrm{ind}}(L_w) \longrightarrow C \longrightarrow K_2'(\mathcal{O}_{L,S})$$

which, when there is no ramification at infinity, defines an Euler characteristic, $\Omega_1(L/K,3) \in \mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big)$. In Section 2 we shall describe an elementary modification of this sequence to produce, for all L/K, a 2-extension of the form

$$K_3^{\mathrm{ind}}(\mathcal{O}_{L,S}) \longrightarrow \bigoplus_{w \in S_\infty(L)} \tilde{K}_3^{\mathrm{ind}}(L_w) \longrightarrow \tilde{C} \longrightarrow \tilde{K}_2'(\mathcal{O}_{L,S})$$

in which $\tilde{K}_2'(\mathcal{O}_{L,S})$ is defined by an extension of $\mathbf{Z}[G(L/K)]$-modules of the form

$$0 \longrightarrow K_2'(\mathcal{O}_{L,S}) \longrightarrow \tilde{K}_2'(\mathcal{O}_{L,S}) \longrightarrow \bigoplus_{v \in S'_\infty(K)} \mathrm{Ind}_{G_{w(v)}}^{G(L/K)}(\mathbf{Z}_-) \oplus T \longrightarrow 0$$

where $S'_\infty(K)$ consists of the infinite places of K which ramify in L/K and $G_w \cong G(L_w/K_v) \cong \mathbf{Z}/2$ is a decomposition group for v, $w = w(v)$ being a place of L above v. The $\mathbf{Z}[G_w]$-module, $\mathbf{Z}_-$ denotes the integers on which the generator acts by multiplication by minus one and T is isomorphic to the free module $\bigoplus_{w \in S_\infty(K), w \,\mathrm{complex}} \mathrm{Ind}_{\{1\}}^{G(L/K)}(\mathbf{Z})$. Cup-product with the resulting class in $\mathrm{Ext}^2_{\mathbf{Z}[G(L/K)]}\big(\tilde{K}_2'(\mathcal{O}_{L,S}), K_3^{\mathrm{ind}}(\mathcal{O}_{L,S})\big)$ induces an isomorphism in Tate cohomology in all dimensions and the resulting Euler characteristic defines $\Omega_1(L/K,3) \in \mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big)$. This Euler characteristic is independent of S.

The following is our main result:

Theorem 1.1 *Let L/K be a Galois extension of number fields with group $G(L/K)$. Then*

$$\Omega_1(L/K) = \Omega_1(L/K,3) \in \mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big).$$

The most delicate part of the proof of Theorem 1.1 concerns the 2-adic behaviour. As explained in Section 5, the key step here is to evaluate, in the class-group, the Euler characteristic of an element in $\mathrm{Ext}^3_{\mathbf{Z}[G(L/\mathbf{Q})]}(E_-, E_+)$. We know of two ways to do this. In the context of this paper it will be quicker, and more convenient for the reader, to produce a litany of explicit elements, listed with their salient properties in Section 7, to evaluate the Euler characteristic as in Section 5. In [23] another method, using modular Hecke algebras, is given by which to evaluate such Euler characteristics. The method we have used here was chosen for two reasons. Firstly, [23] only addresses the 2-adic part of Theorem 1.1 and, secondly, the inclusion of the necessary prerequisites in order to use the alternative method would have made the paper even longer.

The paper is organised in the following manner. In Section 2 the exact sequence of [11, Section 5] is described together with its modification to produce the Euler characteristic, $\Omega_1(L/K,3)$. In Section 3 we establish the existence of a commutative diagram of 2-extensions (Theorem 3.2) which will be the basis of our proof of Theorem 1.1. In Section 4 we define the invariant $\Omega_1(L/K)$ and establish a commutative diagram (Theorem 4.22) which is sufficient to yield Theorem 1.1 in the totally real case (Corollary 4.3). In Section 5 we conclude the proof of Theorem 1.1 in the presence of ramification at infinity, using the results of Appendix Section 7. In Section 6, for completeness, we give a second definition of $\Omega_1(L/K,3)$, leaving to the reader the proof that it agrees with the first definition.

Finally, we are extremely grateful to the referee for many suggestions which have helped to improve our presentation of this material.

2 The Invariant, $\Omega_1(L/K,3)$

2.1 Double Cosets and Archimedean Places

Let L/K be a Galois extension of number fields and let $E/\mathbf{Q}$ be a large Galois extension of number fields such that $L \subset E$ and E is totally complex. Let c denote complex conjugation. Let Ω_L denote the absolute Galois group, $\Omega_L = G(\mathbf{Q}^{\mathrm{sep}}/L)$, where $\mathbf{Q}^{\mathrm{sep}}$ is a separable closure of $\mathbf{Q}$, the rationals.

Let $w_\infty \colon L \to E \to \mathbf{Q}^{\mathrm{sep}}$ be a fixed embedding which restricts to a real embedding, $v_\infty \colon K \to E^{\langle c\rangle} \to (\mathbf{Q}^{\mathrm{sep}})^{\langle c\rangle}$.

Proposition 2.2 *In the notation of Section 2.1, there is an isomorphism of $\mathbf{Z}[G(L/K)]$-modules of the form*

$$\Big(\mathrm{Ind}_{\langle c\rangle}^{\Omega_{\mathbf{Q}}}\big(K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big)\Big)^{\Omega_L} \cong \bigoplus_{w\in S_\infty(L)} K_3^{\mathrm{ind}}(L_w)$$

where L_w is the completion of L at the Archimedean place, w.

Lemma 2.3 *In the notation of Section 2.1:*

(i) There is a bijection between the set of double cosets, $\Omega_L \setminus \Omega_{\mathbf{Q}}/\langle c\rangle$, and the set of Archimedean places of L, $S_\infty(L)$.

(ii) The intersection, $\Omega_L \cap \langle gcg^{-1}\rangle$, is trivial if $(w_\infty)g$ is a complex place and has order two if $(w_\infty)g$ is real.

(iii) In the first case of part (ii), if $(v_\infty)g$ is a real place of K then $gcg^{-1} \in \Omega_K$ and its image in $G(L/K) \cong \Omega_K/\Omega_L$ is the decomposition group, $H_g = G(L_{(w_\infty)g}/K_{(v_\infty)g})$.

Proof For part (i), assigning to $g \in \Omega_{\mathbf{Q}}$ the embedding $(w_\infty)g \colon L \xrightarrow{v_\infty} \mathbf{Q}^{\mathrm{sep}} \xrightarrow{g} \mathbf{Q}^{\mathrm{sep}}$ defines a bijection between embeddings of L and $\Omega_L\backslash\Omega_{\mathbf{Q}}$. The set of embeddings, $\{(w_\infty)g, (w_\infty)gc\}$, corresponds to an Archimedean place of L, since the completions of $(w_\infty)g$ and $(w_\infty)gc$ coincide. Hence assigning the double coset $\Omega_L g\langle c\rangle$ to this Archimedean place defines a bijection between $\Omega_L \setminus \Omega_{\mathbf{Q}}/\langle c\rangle$ and $S_\infty(L)$.

For part (ii), if $(w_\infty)g$ is a complex place then gcg^{-1} does not belong to Ω_L and $\Omega_L \cap \langle gcg^{-1}\rangle = \{1\}$. If $(w_\infty)g$ is real then $\Omega_L \cap \langle gcg^{-1}\rangle = \langle gcg^{-1}\rangle$ is of order two.

Part (iii) is clear.

Lemma 2.4 *In the notation of Section 2.1, suppose that M is a $\mathbf{Z}[G(E/\mathbf{Q})]$-module.*

(i) There is a double coset isomorphism of $\mathbf{Z}[G(E/L)]$-modules of the form

$$\bigoplus_{g\in G(E/L)\backslash G(E/\mathbf{Q})/\langle c\rangle} \operatorname{Ind}_{G(E/L)\cap\langle gcg^{-1}\rangle}^{G(E/L)}\big((g^{-1})^*(M)\big) \xrightarrow{\cong} \operatorname{Res}^{G(E/\mathbf{Q})}_{G(E/L)}\big(\operatorname{Ind}_{\langle c\rangle}^{G(E/\mathbf{Q})}(M)\big).$$

(ii) Also, taking $G(E/L)$-fixed points, there is an isomorphism of $\mathbf{Z}[G(L/K)]$-modules

$$\bigoplus_{g\in G(E/L)\backslash G(E/\mathbf{Q})/\langle c\rangle} \big((g^{-1})^*(M)\big)^{G(E/L)\cap\langle gcg^{-1}\rangle} \xrightarrow{\cong} \big(\operatorname{Ind}_{\langle c\rangle}^{G(E/\mathbf{Q})}(M)\big)^{G(E/L)}.$$

Here, if $G(E/L)\cap\langle gcg^{-1}\rangle = \langle gcg^{-1}\rangle$, $(g^{-1})^*(M)$ denotes M with a new action whereby gcg^{-1} acts on m to send it to $c(m)$. If $G(E/L)\cap\langle gcg^{-1}\rangle$ is trivial then $(g^{-1})^*(M)$ is M and the action is, of course, trivial.

Proof In part (i) the isomorphism is given by the well-known Double Coset isomorphism (see [20, Theorem 1.2.40], for example) which sends

$$u \bigotimes_{G(E/L)\cap\langle gcg^{-1}\rangle} m \in \operatorname{Ind}_{G(E/L)\cap\langle gcg^{-1}\rangle}^{G(E/L)}\big((g^{-1})^*(M)\big)$$

to $ug\bigotimes_{\langle c\rangle} m \in \operatorname{Ind}_{\langle c\rangle}^{G(E/\mathbf{Q})}(M)$. Hence generators of the $G(E/L)g\langle c\rangle$-component of the $G(E/L)$-fixed points are given by

$$\sum_{u\in G(E/L)/(G(E/L)\cap\langle gcg^{-1}\rangle)} u \bigotimes_{G(E/L)\cap\langle gcg^{-1}\rangle} m$$

where $m \in (g^{-1})^*(M))^{G(E/L)\cap\langle gcg^{-1}\rangle}$. In $\operatorname{Ind}_{\langle c\rangle}^{G(E/\mathbf{Q})}(M)$ such a generator corresponds to

$$\sum_{u\in G(E/L)/(G(E/L)\cap\langle gcg^{-1}\rangle)} ug \bigotimes_{\langle c\rangle} m.$$

It remains to verify that this yields an isomorphism of $\mathbf{Z}[G(L/K)]$-modules as claimed in part (ii). We may lift $y \in G(L/K) \cong G(E/K)/G(E/L)$ to $y \in G(E/K)$ and then the action of y on $\sum_{u\in G(E/L)/(G(E/L)\cap\langle gcg^{-1}\rangle)} ug\bigotimes_{\langle c\rangle} m$ sends it to

$$\sum_{u\in G(E/L)/(G(E/L)\cap\langle gcg^{-1}\rangle)} yug \bigotimes_{\langle c\rangle} m = \sum_{v\in G(E/L)/(G(E/L)\cap\langle ygcg^{-1}y^{-1}\rangle)} uyg \bigotimes_{\langle c\rangle} m,$$

since $G(E/L) \triangleleft G(E/K)$. Therefore the element $m \in i\big((g^{-1})^*(M)\big)^{G(E/L)\cap\langle gcg^{-1}\rangle}$ is mapped by y to $m \in \big((g^{-1})^*(M)\big)^{G(E/L)\cap\langle ygcg^{-1}y^{-1}\rangle}$, as required.

2.5 Proof of Proposition 2.2

Setting $M = K_3^{\mathrm{ind}}(E)$ in Lemma 2.4 and passing to the limit over $E/\mathbf{Q}$ we obtain an isomorphism of $\mathbf{Z}[G(L/K)]$-modules of the form

$$(\mathrm{Ind}_{\langle c\rangle}^{\Omega_{\mathbf{Q}}}\big(K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big)^{\Omega_L} \cong \bigoplus_{w\in S_\infty(L)} K_3^{\mathrm{ind}}(L_w)$$

where L_w is the completion of L at the Archimedean place, w. Here we have used the isomorphism, $\big(K_3^{\mathrm{ind}}(E)\big)^{G(E/L)\cap\langle gcg^{-1}\rangle} \cong K_3^{\mathrm{ind}}(E^{G(E/L)\cap\langle gcg^{-1}\rangle})$, together with Lemma 2.3(iii).

2.6 Towards the 2-Extension

Let M be a $\mathbf{Z}[G(E/\mathbf{Q})]$-module. Define an injective $\mathbf{Z}[G(E/\mathbf{Q})]$-homomorphism

$$\phi\colon M \longrightarrow \mathrm{Ind}_{\langle c\rangle}^{G(E/\mathbf{Q})}(M) \cong \mathbf{Z}[G(E/\mathbf{Q})] \bigotimes_{\mathbf{Z}[\langle c\rangle]} M$$

by $\phi(m) = \sum_{h\in G(E/\mathbf{Q})/\langle c\rangle} h \bigotimes_{\langle c\rangle} h^{-1}(m)$. Set $M_+ = \mathrm{coker}(\phi)$ so that there is a short exact sequence of $\mathbf{Z}[G(E/\mathbf{Q})]$-modules of the form

$$0 \longrightarrow M \longrightarrow \mathrm{Ind}_{\langle c\rangle}^{\Omega_{\mathbf{Q}}}(M) \longrightarrow M_+ \longrightarrow 0.$$

At several points in this paper we shall make crucial use of the deep results of [14], [15], [17], [18] concerning the connection between algebraic K-groups of fields and cohomology as well as their consequences derived in [11]. The reader is referred to [9, Section 18] for a description and comparison of the different approaches to these results. In particular we shall need the following canonical isomorphisms:

$$K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})^{\Omega_L} \cong K_3^{\mathrm{ind}}(L) \quad [14], [18],$$
$$H^1\big(L\,;K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big) \cong K_2(L), \quad H^1\big(L_w\,;K_3^{\mathrm{ind}}(L_w^{\mathrm{sep}})\big) \cong K_2(L_w). \quad [11]$$

Proposition 2.7 *In the notation of Section 2.6, there is a 2-extension of $\mathbf{Z}[G(L/K)]$-modules of the form*

$$K_3^{\mathrm{ind}}(L) \longrightarrow \bigoplus_{w\in S_\infty(L)} K_3^{\mathrm{ind}}(L_w) \longrightarrow \big(K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})_+\big)^{\Omega_L} \longrightarrow K_2'(L).$$

Here, as in Section 1, $K_2'(L) = \ker\big(K_2(L) \to \bigoplus_{w\in S_\infty(L)} K_2(L_w)\big)$.

Proof Setting $M = K_3^{\mathrm{ind}}(E)$ and taking the limit over $E/\mathbf{Q}$ we obtain a short exact sequence of continuous $\Omega_{\mathbf{Q}}$-modules of the form

$$0 \longrightarrow K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}}) \longrightarrow \mathrm{Ind}_{\langle c\rangle}^{\Omega_{\mathbf{Q}}}\big(K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big) \longrightarrow K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})_+ \longrightarrow 0.$$

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Applying $H^*(L\,;-)$ to this short exact sequence yields, in dimensions 0 and 1, the following 2-extension

$$K_3^{\mathrm{ind}}(L) \longrightarrow \bigoplus_{w\in S_\infty(L)} K_3^{\mathrm{ind}}(L_w) \longrightarrow \big(K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})_+\big)^{\Omega_L} \longrightarrow K_2'(L).$$

Here we have used Proposition 2.2 to identify the second module and the cohomological isomorphisms of Section 2.6 to complete the proof.

Remark 2.8 (i) As explained in [11], in Proposition 2.7 $(K_3^{\mathrm{ind}}\big(\mathbf{Q}^{\mathrm{sep}})_+\big)^{\Omega_L}$ is a cohomologically trivial $\mathbf{Z}[G(L/K)]$-module. However the module, $\bigoplus_{w\in S_\infty(L)} K_3^{\mathrm{ind}}(L_w)$, is cohomologically trivial if and only if L/K is unramified at infinity. This is one of the reasons for all the modifications which are to follow. A further reason for needing to modify the 2-extension of Proposition 2.7 is that the $K_2'(L)$ is not finitely generated and so we shall replace it below by $K_2'(\mathcal{O}_{L,S})$.

(ii) Notice that, by Section 2.5, as Ω_L-modules,

$$\mathrm{Ind}_{\langle c\rangle}^{\Omega_{\mathbf{Q}}}\big(K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big) \cong \bigoplus_{w\in S_\infty(L)} i_{w,*}i_w^*K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}}) = \big(K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big)_\infty$$

in the notation of Section 4.

Proposition 2.9 *Let S be a finite $G(L/K)$-stable set of places of L containing $S_\infty(L)$ and all places which ramify over K such as, for example, the set S of Section 1. Then the natural homomorphisms*

$$K_2'(\mathcal{O}_{L,S}) \longrightarrow K_2'(L) \quad \text{and} \quad K_3^{\mathrm{ind}}(\mathcal{O}_{L,S}) \xrightarrow{\cong} K_3^{\mathrm{ind}}(L)$$

induce an isomorphism of the form

$$\mathrm{Ext}^2_{\mathbf{Z}[G(L/K)]}\big(K_2'(L), K_3^{\mathrm{ind}}(L)\big) \xrightarrow{\cong} \mathrm{Ext}^2_{\mathbf{Z}[G(L/K)]}\big(K_2'(\mathcal{O}_{L,S}), K_3^{\mathrm{ind}}(\mathcal{O}_{L,S})\big).$$

Proof The homomorphism $K_2(\mathcal{O}_{L,S}) \to \bigoplus_{w\in S_\infty(L)} K_2(L_w)$, is surjective onto the torsion (with uniquely divisible cokernel) so that, from the long exact localisation sequence for K-groups, we obtain a short exact localisation sequence of the form

$$0 \longrightarrow K_2'(\mathcal{O}_{L,S}) \longrightarrow K_2'(L) \longrightarrow \bigoplus_{P\notin S, P \text{ finite}} (\mathcal{O}_L/P)^* \longrightarrow 0.$$

However, if $R \lhd \mathcal{O}_K$ is unramified in L/K then

$$\bigoplus_{P|R} (\mathcal{O}_L/P)^* \cong \mathrm{Ind}_{G(L_{P_0}/K_R)}^{G(L/K)}\big((\mathcal{O}_L/P_0)^*\big)$$

for some $P_0 \lhd \mathcal{O}_L$ lying over R. In addition, since L_{P_0}/K_R is unramified, there is a $\mathbf{Z}[G(L_{P_0}/K_R)]$-resolution [20, Section 7.3.39]

$$0 \longrightarrow \mathbf{Z}[G(L_{P_0}/K_R)] \longrightarrow \mathbf{Z}[G(L_{P_0}/K_R)] \longrightarrow (\mathcal{O}_L/P_0)^* \longrightarrow 0.$$

Therefore $\mathrm{Ext}^i_{\mathbf{Z}[G(L/K)]}(\bigoplus_{P\notin S, P \text{ finite}} (\mathcal{O}_L/P)^*, M) = 0$ for all M and all $i \geq 1$. The result follows easily from the long exact sequence of Ext-groups.

Corollary 2.10 *Pulling back the 2-extension of Proposition 2.7 via the homomorphism, $K_2'(\mathcal{O}_{L,S}) \to K_2'(L)$, yields a 2-extension of $\mathbf{Z}[G(L/K)]$-modules of the form*

$$K_3^{\mathrm{ind}}(\mathcal{O}_{L,S}) \longrightarrow \bigoplus_{v \in S_\infty(L)} K_3^{\mathrm{ind}}(L_v) \longrightarrow C \longrightarrow K_2'(\mathcal{O}_{L,S})$$

in which C is cohomologically trivial.

Proof It suffices to notice that, by Remark 2.8, C is the pullback of a cohomologically trivial module via a homomorphism which induces isomorphisms on all Tate cohomology groups.

2.11 Modification For Ramification at Infinity

When L/K is unramified at infinity then $\bigoplus_{w \in S_\infty(L)} K_3^{\mathrm{ind}}(L_w)$ is also cohomologically trivial and in this case $\Omega_1(L/K, 3) \in \mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big)$ was defined in [19, Chapter 7] to be equal to the Euler characteristic, in the sense of Section 1, of a representative of this 2-extension in which the central modules are finitely generated and cohomologically trivial. From the localisation sequence it is straightforward to show that the resulting Euler characteristic is independent of S [20, Section 7.1.3]. We shall now describe how to modify this 2-extension in the case of ramification at infinity.

As in Section 1, let $S_\infty'(K)$ denote the set of real places of K which become complex in L. For each $v \in S_\infty'(K)$, choose $w = w(v)$ to be a place of L over v with decomposition group, $G_w = G(L_{w(v)}/K_v) = \{1, \tau_w\}$ of order two. The short exact sequence

$$0 \longrightarrow (\mathbf{Q}/\mathbf{Z})(2) \longrightarrow K_3^{\mathrm{ind}}(L_w) \longrightarrow K_3^{\mathrm{ind}}(L_w) \otimes \mathbf{Q} \longrightarrow 0$$

shows that there exists a unique element of order two, $(-1)_w \in K_3^{\mathrm{ind}}(L_w)$, which is fixed by τ_w. Define a $\mathbf{Z}[G_w]$-module, $\tilde{K}_3^{\mathrm{ind}}(L_w)$, in the following manner. The underlying abelian group of $\tilde{K}_3^{\mathrm{ind}}(L_w)$ is $K_3(L_w) \oplus \mathbf{Z}$. If we denote by $a \tilde{\oplus} b$ the element of $\tilde{K}_3^{\mathrm{ind}}(L_w)$ corresponding to $a \oplus b \in K_3^{\mathrm{ind}}(L_w) \oplus \mathbf{Z}$ then the action of τ_w on $\tilde{K}_3^{\mathrm{ind}}(L_w)$ is given by $\tau_w(a \tilde{\oplus} b) = \big(\tau_w(a) \cdot (-1)_w^b\big) \tilde{\oplus} b$.

The following result is straightforward.

Lemma 2.12 *In the notation of Section 2.11, $\tilde{K}_3^{\mathrm{ind}}(L_w)$ is a cohomologically trivial G_w-module.*

The lower row of the following commutative diagram is the 2-extension whose Euler characteristic will define $\Omega_1(L/K, 3) \in \mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big)$ in general.

Proposition 2.13 *With the notation of Corollary 2.10 and Lemma 2.12, there exists a canonical commutative diagram of 2-extensions of $\mathbf{Z}[G(L/K)]$-modules in which $\bigoplus_w \tilde{K}_3^{\mathrm{ind}}(L_w)$ and $C \oplus T_2$ are cohomologically trivial and $\tilde{K}_2'(O_{L,S})$ is a finitely generated extension of $K_2'(O_{L,S})$.*

$$\begin{array}{ccccccc}
K_3^{\mathrm{ind}}(L) & \longrightarrow & \bigoplus_w K_3^{\mathrm{ind}}(L_w) & \xrightarrow{\delta} & C & \longrightarrow & K_2'(O_{L,S}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_3^{\mathrm{ind}}(L) & \longrightarrow & \bigoplus_w \tilde{K}_3^{\mathrm{ind}}(L_w) & \xrightarrow{\tilde{\delta}_w} & C \oplus T_2 & \longrightarrow & \tilde{K}_2'(O_{L,S})
\end{array}$$

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Proof The proof of the proposition will consist of the construction of a sequence of diagrams (1)–(5). The required 2-extension will be found as the middle row of diagram (5).

Consider once more the 2-extension of $\mathbf{Z}[G(L/K)]$-modules

$$K_3^{\text{ind}}(\mathcal{O}_{L,S}) \longrightarrow \bigoplus_{w \in S_\infty(L)} K_3^{\text{ind}}(L_w) \xrightarrow{\delta} C \longrightarrow K_2'(\mathcal{O}_{L,S}).$$

Suppose that $v \in S'_\infty(K)$ and $w = w(v)$, as in Section 2.6. Since $(1+\tau_w)\big((-1)_w\big) = 0$, $(1+\tau_w)\Big(\delta\big((-1)_w\big)\Big) = 0$ in C. Therefore, because C is cohomologically trivial, there is an $\alpha_w \in C$ such that $(1-\tau_w)(\alpha_w) = \delta\big((-1)_w\big)$.

We now construct a commutative diagram of $\mathbf{Z}[G_w]$-modules of the form

$$(1) \qquad \begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
K_3(L_w) & \xrightarrow{\delta} & C & \longrightarrow & K_2'(O_{L,S}) \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{K}_3^{\text{ind}}(L_w) & \xrightarrow{\tilde{\delta}_w} & C \oplus \mathbf{Z}[G_w] & \longrightarrow & \tilde{K}_2'(O_{L,S})_w \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{Z} & \longrightarrow & \mathbf{Z}[G_w] & \longrightarrow & \mathbf{Z}_{w,-} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

with the following properties. The vertical columns are exact and the terms of the second column map surjectively onto those of the third column. The morphisms $C \to C \oplus \mathbf{Z}[G_w]$ and $C \oplus \mathbf{Z}[G_w] \to \mathbf{Z}[G_w]$ in the middle column are defined by $(c \mapsto c \oplus 0)$ and $(c \oplus a \mapsto a)$, respectively. The morphism $\tilde{\delta}_w\colon \tilde{K}_3^{\text{ind}}(L_w) = K_3^{\text{ind}}(L_w) \tilde{\oplus} \mathbf{Z} \to C \oplus \mathbf{Z}[G_w]$ is defined by $\tilde{\delta}(a \tilde{\oplus} 0) = \delta(a) \oplus 0$ and $\tilde{\delta}(0 \tilde{\oplus} 1) = \alpha_w \oplus (1+\tau_w)$. The module $\tilde{K}_2'(O_{L,S})_w$ is defined to be the cokernel of $\tilde{\delta}_w$, while $\mathbf{Z}_{w,-}$ is the G_w-module with underlying abelian group $\mathbf{Z}$ on which τ_w acts by multiplication by -1. To check that one has a commutative diagram in (1) it suffices to check that

$$(2) \qquad \begin{array}{ccc}
K_3^{\text{ind}}(L_w) & \xrightarrow{\delta} & C \\
\downarrow & & \downarrow \\
\tilde{K}_3^{\text{ind}}(L_w) & \xrightarrow{\tilde{\delta}_w} & C \oplus \mathbf{Z}[G_w]
\end{array}$$

is a commutative square of $\mathbf{Z}[G_w]$-module homomorphisms. This follows from the formulae

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$$(1-\tau_w)(0\,\tilde{\oplus}\,1) = (-1)_w\,\tilde{\oplus}\,0 \in \tilde{K}_3^{\mathrm{ind}}(L_w)$$

and

$$(1-\tau_w)\tilde{\delta}(0\,\tilde{\oplus}\,1) = (1-\tau_w)\big(\alpha_w \oplus (1+\tau_w)\big) = \delta(-1)_w \oplus 0 \in C \oplus \mathbf{Z}[G_w].$$

It will be convenient to define versions of the diagram (1) when $w = w(v)$ and v is an arbitrary infinite place of F for which purpose we must pause in order to introduce some temporary notation.

Definition 2.14 Let $S_1(K)$ denote the set of complex places of K and let $S_2(K)$ denote the set of real places of K which do not ramify in L. Thus $S_\infty(K)$ is the disjoint union, $S_1(K) \cup S_2(K) \cup S'_\infty(K)$.

Suppose $v \in S_\infty(K)$ and $w = w(v)$, as in Section 2.6. Set $\tilde{K}_3^{\mathrm{ind}}(L_w) = K_3^{\mathrm{ind}}(L_w)$ if $v \in S_1(K) \cup S_2(K)$. Let $T_{1,w} = \mathbf{Z}$ if $v \in S'_\infty(K)$ and let $T_{1,w} = 0$ otherwise. Define $T_{2,w} = \mathbf{Z}[G_w]$ if $v \in S'_\infty(K) \cup S_1(K)$ and $T_{2,w} = 0$ otherwise. Finally, let $T_{3,w} = \mathbf{Z}_{w,-}$ if $v \in S'_\infty(K)$, $T_{3,w} = 0$ if $v \in S_2(K)$ and $T_{3,w} = \mathbf{Z}[G_w]$ if $v \in S_1(K)$.

2.15 The Proof of Proposition 2.13 Continued

For all infinite places v of K and $w = w(v)$, we now have a commutative diagram of $\mathbf{Z}[G_w]$-modules

$$\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
K_3^{\mathrm{ind}}(L_w) & \xrightarrow{\delta} & C & \longrightarrow & K'_2(O_{L,S}) \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{K}_3^{\mathrm{ind}}(L_w) & \xrightarrow{\tilde{\delta}_w} & C \oplus T_{2,w} & \longrightarrow & \tilde{K}'_2(O_{L,S})_w \\
\downarrow & & \downarrow & & \downarrow \\
T_{1,w} & \longrightarrow & T_{2,w} & \longrightarrow & T_{3,w} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array} \tag{3}$$

of the following kind. If $v \in S'_\infty(K)$ then (3) is diagram (1). If $v \in S_1(K) \cup S_2(K)$ then G_w and $T_{1,w}$ are trivial and (3) is obtained from the 2-extension of Section 2.15 by push-out via the inclusion morphism, $C \to \tilde{C} = C \oplus T_{2,w}$, where $T_{2,w} = T_{3,w}$ in this case. Once again the maps from the middle to the right-hand column are all surjective.

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Inducing up from G_w to $G(L/K)$ and using the fact that C and $K_2'(O_{L,S})$ are $\mathbf{Z}[G(L/K)]$-modules, we may use (3) to construct a diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Ind}_{G_w}^{G(L/K)} K_3^{\mathrm{ind}}(L_w) & \xrightarrow{\delta} & C & \longrightarrow & K_2'(O_{L,S}) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Ind}_{G_w}^{G(L/K)} \tilde{K}_3^{\mathrm{ind}}(L_w) & \xrightarrow{\tilde{\delta}_w} & C \oplus \operatorname{Ind}_{G_w}^{G(L/K)} T_{2,w} & \longrightarrow & \tilde{K}_2'(O_{L,S})_{w,0} \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Ind}_{G_w}^{G(L/K)} T_{1,w} & \longrightarrow & \operatorname{Ind}_{G_w}^{G(L/K)} T_{2,w} & \longrightarrow & \operatorname{Ind}_{G_w}^{G(L/K)} T_{3,w} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array} \tag{4}$$

whose columns are exact and in which the terms of the second column map surjectively onto those of the third.

We now identify $\operatorname{Ind}_{G_w}^{G(L/K)} K_3^{\mathrm{ind}}(L_w)$ with $\bigoplus_{w'|v} K_3^{\mathrm{ind}}(L_{w'})$, and define $\bigoplus_{w'|v} \tilde{K}_3^{\mathrm{ind}}(L_w)$ to be $\operatorname{Ind}_{G_w}^{G(L/K)} \tilde{K}_3^{\mathrm{ind}}(L_w)$. Using (4) to push-out the 2-extension of Corollary 2.10 yields a diagram of the following form in which the columns are short exact and the rows are exact 2-extensions.

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_3^{\mathrm{ind}}(L) & \longrightarrow & \bigoplus_w K_3^{\mathrm{ind}}(L_w) & \xrightarrow{\delta} & C & \longrightarrow & K_2'(O_{L,S}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_3^{\mathrm{ind}}(L) & \longrightarrow & \bigoplus_w \tilde{K}_3^{\mathrm{ind}}(L_w) & \xrightarrow{\tilde{\delta}_w} & C \oplus T_2 & \longrightarrow & \tilde{K}_2'(O_{L,S}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & T_1 & \longrightarrow & T_2 & \longrightarrow & T_3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array} \tag{5}$$

For $i = 1, 2, 3$ one has

$$T_i = \bigoplus_{v \in S_\infty(K)} \operatorname{Ind}_{G_{w(v)}}^{G(L/K)} T_{i,w(v)}$$

where the sum is over the infinite places v of K. The module $\tilde{K}_2'(O_{L,S})$ is defined to be the cokernel of $\tilde{\delta}$.

Since $\tilde{K}_3^{\text{ind}}(L_{w(v)})$ and $T_{2,w}$ are cohomologically trivial $\mathbf{Z}[G_{w(v)}]$-modules, the modules $\bigoplus_w \tilde{K}_3^{\text{ind}}(L_w)$ and $C \oplus T_2$ in the middle row of (5) are cohomologically trivial $\mathbf{Z}[G(L/K)]$-modules, which completes the proof of Proposition 2.13.

2.16

Since $K_3^{\text{ind}}(L)$ and $\tilde{K}_2'(O_{L,S})$ are finitely generated $\mathbf{Z}[G(L/K)]$-modules, we may construct a commutative diagram representing an equivalence of 2-extensions

$$\begin{array}{ccccccc} K_3^{\text{ind}}(L) & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & \tilde{K}_2'(O_{L,S}) \\ \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\ K_3^{\text{ind}}(L) & \longrightarrow & \bigoplus_w \tilde{K}_3^{\text{ind}}(L_w) & \xrightarrow{\tilde{\delta}} & C \oplus T_2 & \longrightarrow & \tilde{K}_2'(O_{L,S}) \end{array} \tag{6}$$

in which A_1 and B_1 are finitely generated, cohomologically trivial $\mathbf{Z}[G(L/K)]$-modules.

Theorem 2.17 *(i) For A_1 and B_1 as in diagram (6) of Section 2.16, the Euler characteristic*

$$\Omega_1(L/K,3) = [A_1] - [B_1] \in \mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big).$$

(ii) Also $\Omega_1(L/K,3)$ depends only on the extension L/K. In particular, $\Omega_1(L/K,3)$ does not depend on the choice of places S in Section 2.6, on the $w(v)$ chosen for $v \in S_\infty(K)$, on the $\alpha_{w(v)} \in C$ for $v \in S'_\infty(K)$ used to construct the diagrams (2) and (5) or on the choice of a top row of diagram (6).

Proof For part (i), it is clear that $[A_1] - [B_1] \in K_0\big(\mathbf{Z}[G(L/K)]\big)$ and, by definition, $\mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big)$ is equal to the kernel of the rank homomorphism, $K_0\big(\mathbf{Z}[G(L/K)]\big) \to \mathbf{Z}$. Firstly we remark that the rank of $T_{3,w(v)}$ over $\mathbf{Z}$ is 1 if $w(v)$ is complex and is 0 if $w(v)$ is real. Let $r_2(L)$ be the number of complex places of L. Since $K_2'(O_{L,S})$ is finite, T_3 and $\tilde{K}_2'(O_{L,S})$ both have rank $r_2(L)$ over $\mathbf{Z}$. By a result of Borel, $K_3^{\text{ind}}(L)$ also has rank $r_2(L)$ and the exactness of the first row of (6) shows $\Omega_1(L/K,3)$ has rank 0 and therefore lies in $\mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big) \subset K_0\big(\mathbf{Z}[G(L/K)]\big)$.

For part (ii), we now consider, in reverse order, the dependence of $\Omega_1(L/K,3)$ upon the choices listed in the statement of Theorem 2.17.

It is well-known that if the bottom row of (6) is fixed and the top row is changed, the class $[A_1] - [B_1]$ in $K_0\big(\mathbf{Z}[G(L/K)]\big)$ is unaltered (see [19, Proposition 2.2.2 p. 47], for example).

Suppose now that we fix S and the choice of the $w(v)$ for v an infinite place of K. We must show that $\Omega_1(L/K,3)$ does not depend on the choice of the elements $\alpha_{w(v)} \in C$ for $v \in S'_\infty(K)$ which were used in the construction of (2) and (5). Fix $v \in S'_\infty(K)$ and let $w = w(v)$. Suppose α'_w is another element of C which satisfies the defining condition for α_w, namely that

$$(1-\tau_w)\alpha_w = (1-\tau_w)\alpha'_w = \delta(-1)_w$$

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where, as in Section 2.11, $\tau_w \in G_w$ is complex conjugation at w and $\delta\colon K_3^{\mathrm{ind}}(L_w) \to C$ is the homomorphism in the 2-extension of Section 2.15. Then $(1-\tau_w)(\alpha_w - \alpha'_w) = 0$ and, since C is cohomologically trivial, there is an element, $\gamma_w \in C$, such that

$$(1+\tau_w)\gamma_w = \alpha_w - \alpha'_w.$$

Define a $\mathbf{Z}[G_w]$-automorphism $\pi_w\colon C \oplus \mathbf{Z}[G_w] \to C \oplus \mathbf{Z}[G_w]$ by $\pi_w(c \oplus 0) = c \oplus 0$ and $\pi_w(0 \oplus 1) = \gamma_w \oplus 1$. We claim that the diagram

$$\begin{array}{ccc} \tilde{K}_3^{\mathrm{ind}}(L_w) = K_3^{\mathrm{ind}}(L_w) \tilde{\oplus} \mathbf{Z} & \xrightarrow{\tilde{\delta}'_w} & C \oplus \mathbf{Z}[G_w] \\ \downarrow & & \downarrow{\scriptstyle \pi_w} \\ \tilde{K}_3^{\mathrm{ind}}(L_w) = K_3^{\mathrm{ind}}(L_w) \tilde{\oplus} \mathbf{Z} & \xrightarrow{\tilde{\delta}_w} & C \oplus \mathbf{Z}[G_w] \end{array} \tag{7}$$

commutes, where the morphisms are described in the following manner. The left vertical morphism is the identity. The bottom row is the morphism defined in the middle row of (1). Thus $\tilde{\delta}_w(a \tilde{\oplus} 0) = \delta(a) \oplus 0$ and $\tilde{\delta}_w(0 \tilde{\oplus} 1) = \alpha_w \oplus (1+\tau_w)$. The morphism $\tilde{\delta}'_w$ in the top row of (7) is defined in the same way as $\tilde{\delta}_w$ except that α_w is replaced by α'_w in the definition. The commutativity of (7) follows from

$$\begin{aligned} \pi_w\big(\tilde{\delta}'_w(0 \tilde{\oplus} 1)\big) &= \pi_w\big(\alpha'_w \oplus (1+\tau_w)\big) \\ &= (\alpha'_w \oplus 0) + (1+\tau_w)\pi_w(0 \oplus 1) \\ &= (\alpha'_w \oplus 0) + (1+\tau_w)(\gamma_w \oplus 1) \\ &= (\alpha'_w \oplus 0) + \big((\alpha_w - \alpha'_w) \oplus (1+\tau_w)\big) \\ &= \alpha_w \oplus (1+\tau_w) \\ &= \tilde{\delta}_w(0 \tilde{\oplus} 1). \end{aligned}$$

For all infinite places v of K we now have a diagram of $\mathbf{Z}[G_{w(v)}]$-modules

$$\begin{array}{ccc} \tilde{K}_3^{\mathrm{ind}}(L_{w(v)}) & \xrightarrow{\tilde{\delta}'_{w(v)}} & C \oplus T_{2,w(v)} \\ \downarrow & & \downarrow{\scriptstyle \pi_{w(v)}} \\ \tilde{K}_3^{\mathrm{ind}}(L_{w(v)}) & \xrightarrow{\tilde{\delta}_{w(v)}} & C \oplus T_{2,w(v)} \end{array} \tag{8}$$

which specializes to (7) if $v \in S'_\infty(K)$ and in which the vertical homomorphisms are the identity otherwise. Fitting together the inductions of these diagrams from $G_{w(v)}$ to $G(L/K)$ yields a commutative diagram

$$\begin{array}{ccccccc} K_3^{\mathrm{ind}}(L) & \longrightarrow & \bigoplus_w \tilde{K}_3^{\mathrm{ind}}(L_w) & \xrightarrow{\tilde{\delta}'} & C \oplus T_2 & \longrightarrow & \tilde{K}'_2(O_{L,S}) \\ \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle 1} \\ K_3^{\mathrm{ind}}(L) & \longrightarrow & \bigoplus_w \tilde{K}_3^{\mathrm{ind}}(L_w) & \xrightarrow{\tilde{\delta}} & C \oplus T_2 & \longrightarrow & \tilde{K}'_2(O_{L,S}) \end{array} \tag{9}$$

in which the top row is the sequence resulting from using $\alpha'_{w(v)}$ instead of $\alpha_{w(v)}$ for $v \in S'_\infty(K)$. All of the vertical morphisms in (9) are $\mathbf{Z}[G(L/K)]$-isomorphisms.

Suppose now that we use the top row of (9) to compute $\Omega_1(L/K,3)$. Thus we choose a diagram

$$\begin{array}{ccccccc} K_3^{\mathrm{ind}}(L) & \longrightarrow & A_1' & \longrightarrow & B_1' & \longrightarrow & \tilde{K}_2'(O_{L,S}) \\ \downarrow 1 & & \downarrow & & \downarrow & & \downarrow 1 \\ K_3^{\mathrm{ind}}(L) & \longrightarrow & \bigoplus_w \tilde{K}_3^{\mathrm{ind}}(L_w) & \xrightarrow{\tilde{\delta}'} & C \oplus T_2 & \longrightarrow & \tilde{K}_2'(O_{L,S}) \end{array} \tag{10}$$

in which A_1' and B_1' are finitely generated and cohomologically trivial $\mathbf{Z}[G(L/K)]$-modules.

Composing the vertical morphisms in (10) with the vertical isomorphisms in (9) gives a diagram of the form (6). This shows that we can take the top rows of (6) and (10) to be the same sequence. We conclude that $\Omega_1(E/F,3)$ does not depend on the choice of the $\alpha_{w(v)}$ for $v \in S'_\infty(K)$.

We must now show that $\Omega_1(L/K,3)$ does not depend on the choice of the $w(v)$ for v an infinite place of F. Suppose v is fixed, $\sigma \in G$ and that $\sigma w(v) = w(v)'$ is another place of L over v. If $v \in S'_\infty(K)$, let $\alpha_{w(v)'} = \sigma\alpha_{w(v)}$ in C. Via the identification of $\mathrm{Ind}_{G_{w(v)}}^{G(L/K)} K_3^{\mathrm{ind}}(L_{w(v)})$ with $\bigoplus_{w'|v} K_3^{\mathrm{ind}}(L_{w'})$, the choice of $w(v)$ (together with $\alpha_{w(v)}$ if $v \in S'_\infty(K)$) leads to a diagram (5) which is isomorphic to the one resulting from the choice of $w(v)'$ (together with $\alpha'_{w(v)}$ if $v \in S'_\infty(K)$). Thus the invariant, $\Omega_1(L/K,3)$, is independent of the choice of the $w(v)$, since we have already shown it is independent of the choice of the $\alpha_{w(v)}$ for $v \in S'_\infty(K)$.

The last point is to check that $\Omega_1(L/K,3)$ is independent of the choice of the $G(L/K)$-stable set of places S of L which contains the ramified primes of L. The argument for this uses the localisation sequence (*cf.* Proposition 2.9(proof)) and is given in [19, Section 7.1], so the proof of Theorem 2.17 is complete.

Corollary 2.18 *In Theorem 2.17, if L/K is unramified at infinity then $\Omega_1(L/K,3)$ is equal to the Euler characteristic of the 2-extension of Proposition 2.13*

$$K_3^{\mathrm{ind}}(\mathcal{O}_{L,S}) \longrightarrow \bigoplus_{w \in S_\infty(L)} K_3^{\mathrm{ind}}(L_w) \xrightarrow{\delta} C \longrightarrow K_2'(\mathcal{O}_{L,S}).$$

Proof By construction, we may choose a commutative diagram of 2-extensions of the form

$$\begin{array}{ccccccc} K_3^{\mathrm{ind}}(L) & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & K_2'(O_{L,S}) \\ \downarrow 1 & & \downarrow & & \downarrow & & \downarrow i \\ K_3^{\mathrm{ind}}(L) & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & \tilde{K}_2'(O_{L,S}) \end{array}$$

in which A_1, B_1, A_2, B_2 are finitely generated, cohomologically trivial $\mathbf{Z}[G(L/K)]$-modules with A_1, B_1 as in diagram (6). The homomorphism, i, is the injection of diagram (5). Hence the cokernel of i is equal to T_3, which is free in the absence of ramification at infinity. Hence there is an exact sequence of the form

$$0 \longrightarrow A_2 \longrightarrow A_1 \oplus B_2 \longrightarrow B_1 \longrightarrow T_3 \longrightarrow 0.$$

Therefore, in $\mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big)$, we have

$$\begin{aligned}\Omega_1(L/K,3) &= [A_1]-[B_1]\\ &= [A_2]-[B_2]+[T_3]\\ &= [A_2]-[B_2],\end{aligned}$$

as required.

3 An Important Diagram

3.1

The purpose of this section is to establish a commutative diagram in Theorem 3.2 which will be the basis of a series of commutative diagrams and 2-equivalences which are used to establish Theorem 1.1. The main feature of this diagram is that the lower row involves the bar resolution, which is related to the definition of $\Omega_1(L/K)$, while the upper row is the sequence from which $\Omega_1(L/K,3)$ is defined when $M = K_3^{\mathrm{ind}}(E)$.

Let L/K be a Galois extension of number fields with group, $G(L/K)$. As in Section 2.1, let $E/\mathbf{Q}$ be a large Galois extension of number fields such that $L \subset E$ and E is totally complex. Let c denote complex conjugation. Let M be a $\mathbf{Z}[G(E/\mathbf{Q})]$-module and consider the short exact sequence

$$0 \longrightarrow M \xrightarrow{\phi} \mathrm{Ind}_{\langle c\rangle}^{G(E/\mathbf{Q})}(M) \longrightarrow M_+ \longrightarrow 0$$

where ϕ is as in Section 2.6. The resulting $G(E/L)$-cohomology sequence yields a 2-extension of the form

$$M^{G(E/L)} \longrightarrow \mathrm{Ind}_{\langle c\rangle}^{G(E/\mathbf{Q})}(M)^{G(E/L)} \longrightarrow M_+^{G(E/L)} \longrightarrow H^1\big(G(E/L)\,;M\big)'$$

where $H^1\big(G(E/L)\,;M\big)' = \ker\Big(H^1\big(G(E/L)\,;M\big) \xrightarrow{\phi_*} H^1\big(G(E/L)\,;\mathrm{Ind}_{\langle c\rangle}^{G(E/\mathbf{Q})}(M)\big)\Big)$.

Let $\{B_iG(E/\mathbf{Q}), d_i\}$ denote the bar resolution of $G(E/\mathbf{Q})$ [10, p. 216]; [19, p. 1]) so that $B_iG(E/\mathbf{Q})$ is the free $\mathbf{Z}[G(E/\mathbf{Q})]$-module on i-tuples of elements of $G(E/\mathbf{Q})$, written $[g_1|g_2|\cdots|g_i]$ (and [ ] if $i=0$).

Theorem 3.2 *(i) In the notation of Section 3, there exists a commutative diagram of 2-extensions of $\mathbf{Z}[G(L/K)]$-modules of the form*

$$\begin{array}{ccccccc}
M^{G(E/L)} & \longrightarrow & \big(\mathrm{Ind}_{\langle c\rangle}^{G(E/\mathbf{Q})}(M)\big)^{G(E/L)} & \xrightarrow{\pi} & M_+^{G(E/L)} & \longrightarrow & H^1\big(G(E/L)\,;M\big)' \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle (-1)} \\
M^{G(E/L)} & \longrightarrow & \mathrm{Hom}_{G(E/L)}\big(B_0G(E/\mathbf{Q}),M\big) & \xrightarrow{d_0^*} & \mathrm{Ker}(d_1^*) & \longrightarrow & H^1\big(G(E/L)\,;M\big)'
\end{array}$$

where $H^1\big(G(E/L)\,;M\big)' = \ker\Big(H^1\big(G(E/L)\,;M\big) \to H^1\big(G(E/L)\,;\mathrm{Ind}_{\langle c\rangle}^{G(E/\mathbf{Q})}(M)\big)\Big)$.

(ii) Let M be a finitely generated $\mathbf{Z}[G(L/K)]$-module. Then, in Section 3, the $\mathbf{Z}[G(L/K)]$-module, $\mathrm{Hom}_{G(E/L)}\big(B_sG(E/\mathbf{Q}),M\big)$*, is cohomologically trivial for all s.*

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Proof Part (ii) is proved by a straightforward spectral sequence argument.

For part (i), define a homomorphism

$$\lambda_0 \colon \operatorname{Ind}_{\langle c\rangle}^{G(E/\mathbf{Q})}(M) \longrightarrow \operatorname{Hom}\big(B_0 G(E/\mathbf{Q}), M\big)$$

by the formula

$$\lambda_0\Big(\sum_{h\in G(E/\mathbf{Q})/\langle c\rangle} h\otimes m_h\Big)(h') = h'(m_{h'})$$

for $h' \in G(E/\mathbf{Q})$. This is well-defined since the formula selects either $h = h'$ or $h = h'c$ and $h'\otimes m_{h'} = h'c\otimes m_{h'c}$ if and only if $m_{h'c} = cm_{h'}$ so that $h'c(m_{h'c}) = h'c(cm_{h'}) = h'(m_{h'})$.

If $v \in G(E/K)$,

$$\lambda_0\bigg(v\Big(\sum_{h\in G(E/\mathbf{Q})/\langle c\rangle} h\otimes m_h\Big)\bigg)(h') = h'(m_{v^{-1}h'})$$

while

$$v\bigg(\lambda_0\Big(\sum_{h\in G(E/\mathbf{Q})/\langle c\rangle} h\otimes m_h\Big)\bigg)(h') \\ = v\bigg(\lambda_0\Big(\sum_{h\in G(E/\mathbf{Q})/\langle c\rangle} h\otimes m_h\Big)(v^{-1}h')\bigg) v\big(v^{-1}h'(m_{v^{-1}h'})\big),$$

so that λ_0 is a $\mathbf{Z}[G(E/K)]$-homomorphism.

Define

$$\lambda_1 \colon M_+ \longrightarrow \operatorname{Hom}\big(B_1 G(E/\mathbf{Q}), M\big)$$

by, $g, h' \in G(E/\mathbf{Q})$,

$$\lambda_1\Big(\Big[\sum_{h\in G(E/\mathbf{Q})/\langle c\rangle} h\otimes m_h\Big]\Big)(g[h']) = gh'(m_{gh'}) - g(m_g).$$

This is well-defined since

$$\begin{aligned}\lambda_1\Big(\Big[\sum_{h\in G(E/\mathbf{Q})/\langle c\rangle} h\otimes m_h + \phi(a)\Big]\Big)&(g[h']) \\ &= gh'(m_{gh'}) - g(m_g) + gh'\big(h'^{-1}g^{-1}(a)\big) - g\big(g^{-1}(a)\big) \\ &= gh'(m_{gh'}) - g(m_g) \\ &= \lambda_1\Big(\Big[\sum_{h\in G(E/\mathbf{Q})/\langle c\rangle} h\otimes m_h\Big]\Big)(g[h']).\end{aligned}$$

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It is a $G(E/K)$-map because, if $g_1 \in G(E_+/K)$, then

$$\begin{aligned}
\lambda_1\Bigl(\sum g_1 h \otimes m_h\Bigr)(g[h']) &= gh'\bigl(m_{g_1^{-1}gh'} - g(m_{g_1^{-1}g})\bigr) \\
&= g_1 g_1^{-1} gh'(m_{g_1^{-1}gh'} - g_1 g_1^{-1} g\bigl(m_{g_1^{-1}g}\bigr)) \\
&= g_1\Bigl(\lambda_1\Bigl(\sum h \otimes m_h\Bigr)(g_1^{-1}g[h'])\Bigr) \\
&= g_1\Bigl(\lambda_1\Bigl(\sum h \otimes m_h\Bigr)\Bigr)(g[h']).
\end{aligned}$$

We have a commutative diagram of the following form.

$$\begin{array}{ccc}
\bigl(\mathrm{Ind}_{\langle c\rangle}^{G(E/\mathbf{Q})}(M)\bigr)^{G(E/L)} & \xrightarrow{\pi} & M_+^{G(E/L)} \\
\downarrow{\scriptstyle \lambda_0} & & {\scriptstyle \lambda_1}\downarrow \\
\mathrm{Hom}_{G(E/L)}\bigl(B_0 G(E/\mathbf{Q}), M\bigr) & \xrightarrow{d_0^*} & \mathrm{Hom}_{G(E/L)}\bigl(B_1 G(E/\mathbf{Q}), M\bigr)
\end{array}$$

in which π is the canonical surjection and d_0^* is induced by the bar-resolution differential. The commutativity of this diagram follows from the formulae

$$\begin{aligned}
\lambda_1\Bigl(\pi\Bigl(\sum h \otimes m_h\Bigr)\Bigr)(g[h']) &= \lambda_1\Bigl(\Bigl[\sum h \otimes m_h\Bigr]\Bigr)(g[h']) \\
&= gh'(m_{gh'}) - g(m_g) \\
&= \lambda_0\Bigl(\sum h \otimes m_h\Bigr)(gh' - g) \\
&= \lambda_0\Bigl(\sum h \otimes m_h)(d_0(g[h'])\Bigr) \\
&= d_0^*\Bigl(\lambda_0\Bigl(\sum h \otimes m_h\Bigr)\Bigr)(g[h']).
\end{aligned}$$

The $G(E/L)$-cohomology of M may be computed from the complex

$$\mathrm{Hom}_{G(E/L)}\bigl(B_0 G(E/\mathbf{Q}), M\bigr) \xrightarrow{d_0^*} \mathrm{Hom}_{G(E/L)}\bigl(B_1 G(E/\mathbf{Q}), M\bigr) \xrightarrow{d_1^*} \cdots$$

and the resulting isomorphism, $\rho\colon M^{G(E/L)} \xrightarrow{\cong} \mathrm{Ker}(d_0^*)$, is given by $\bigl(m \mapsto (g \mapsto m)\bigr)$ for $g \in G(E/\mathbf{Q})$, if we identify $B_0 G(E/\mathbf{Q})$ with $\mathbf{Z}[G(E/\mathbf{Q})]$. Furthermore, if $m \in M^{G(E/L)}$, the formulae

$$\begin{aligned}
\lambda_0\bigl(\phi(m)\bigr)[h'] &= \lambda_0\Bigl(\sum_{h \in G(E/\mathbf{Q})/\langle c\rangle} h \bigotimes_{\langle c\rangle} h^{-1}(m)\Bigr)[h'] \\
&= h'((h')^{-1}(m) \\
&= m \\
&= \rho(m)[h']
\end{aligned}$$

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show that λ_0 induces the identity map between $M^{G(E/L)}$ and $\mathrm{Ker}(d_0^*) \cong M^{G(E/L)}$.

Next we consider the map induced by λ_1 on the cokernel of π, which is isomorphic to

$$\mathrm{Ker}(\phi_*) = \ker\Big(H^1\big(G(E/L)\,;M\big) \longrightarrow H^1\big(G(E/L)\,;\mathrm{Ind}_{\langle c\rangle}^{G(E/\mathbf{Q})}(M)\big)\Big).$$

The homomorphism, d_1, induces

$$d_1^*\colon \mathrm{Hom}_{G(E/L)}\big(B_1G(E/\mathbf{Q}),M\big) \longrightarrow \mathrm{Hom}_{G(E/L)}\big(B_2G(E/\mathbf{Q}),M\big)$$

and we observe that

$$\begin{aligned} d_1^*\Big(\lambda_1\Big(\sum_{h\in G(E/\mathbf{Q})/\langle c\rangle}\Big)\Big)(g[h_1|h_2]) &= gh_1h_2(m_{h_1h_2}) - gh_1(m_{h_1}) - gh_1h_2(m_{h_1h_2}) \\ &\qquad + g(m_g) + gh_1(m_{h_1}) - g(m_g) \\ &= 0 \end{aligned}$$

so that the cokernel of π, maps to

$$\mathrm{Ker}(d_1^*)/d_0^*\Big(\mathrm{Hom}_{G(E/L)}(B_0G(E/\mathbf{Q}),M)\Big) \cong H^1\big(G(E/L)\,;M\big).$$

Let $z \in (M_+)^{G(E/L)}$ be represented by $\sum_{h\in G(E/\mathbf{Q})/\langle c\rangle} h \bigotimes_{\langle c\rangle} m_h$. Therefore, if $g \in G(E/L)$, there exists $f(g) \in M$ such that $g(z) - z = \phi\big(f(g)\big)$. This means that

$$\sum_{h\in G(E/\mathbf{Q})/\langle c\rangle} gh \bigotimes_{\langle c\rangle} m_h - h \bigotimes_{\langle c\rangle} m_h = \sum_h h \bigotimes_{\langle c\rangle} h^{-1}\big(f(g)\big)$$

so that $f(g) = h(m_{g^{-1}h}) - h(m_h)$ for all h.

The coboundary map

$$\delta\colon (M_+)^{G(E/L)} \longrightarrow H^1\big(G(E/L)\,;M\big)$$

is given in terms of the bar resolution for $G(E/L)$ by $\delta(z) = [f]$, the class of the 1-cocycle, f. On the other hand, we have a composition

$$(M_+)^{G(E/L)} \xrightarrow{\lambda_1} \mathrm{Hom}_{G(E/L)}\big(B_1G(E/\mathbf{Q}),M\big) \longrightarrow \mathrm{Hom}_{G(E/L)}\big(B_1G(E/L),M\big)$$

in which the second map is induced by the inclusion, $G(E/L) \subset G(E/\mathbf{Q})$. The image of z under this composite is

$$([g] \mapsto \lambda_1(z)(1[g]) = g(m_g) - m_1).$$

Since $f(g) = g(m_1) - g(m_g)$ the sum of $\delta(z)$ and $\lambda_1(z)$ is $([g] \mapsto g(m_1) - m_1)$ which is the coboundary of $m_1 \in M$. Hence

$$[\delta(z)] = -[\lambda_1(z)] \in H^1\big(G(E/L)\,;M\big)$$

which completes the proof.

3.3

We shall need the case in which $M = K_3^{\text{ind}}(E)$ in Theorem 3.2. More precisely, let $R\Gamma_s\big(G(E/L), K_3^{\text{ind}}(E)\big) = \operatorname{Hom}_{G(E/L)}\big(B_sG(E/\mathbf{Q}), K_3^{\text{ind}}(E)\big)$ and let $R\Gamma_*\big(L, K_3^{\text{ind}}(\mathbf{Q}^{\text{sep}})\big)$ denote the limit of $R\Gamma_*\big(G(E/L), K_3^{\text{ind}}(E)\big)$ taken over Galois extensions, $\mathbf{Q} \subset E \subset \mathbf{Q}^{\text{sep}}$. We have a chain complex

$$R\Gamma_0\big(L, K_3^{\text{ind}}(\mathbf{Q}^{\text{sep}})\big) \xrightarrow{d_0^*} R\Gamma_1\big(L, K_3^{\text{ind}}(\mathbf{Q}^{\text{sep}})\big) \xrightarrow{d_0^*} \cdots .$$

From Theorem 3.2 we obtain a homomorphism between the 2-extension of Section 2.6

$$K_3^{\text{ind}}(L) \longrightarrow \bigoplus_{w \in S_\infty(L)} K_3^{\text{ind}}(L_w) \longrightarrow \big(K_3^{\text{ind}}(\mathbf{Q}^{\text{sep}})_+\big)^{\Omega_L} \longrightarrow K_2'(L)$$

and

$$K_3^{\text{ind}}(L) \longrightarrow R\Gamma_0\big(L, K_3^{\text{ind}}(\mathbf{Q}^{\text{sep}})\big) \xrightarrow{d_0^*} \operatorname{Ker}(d_1^*)' \longrightarrow K_2'(L)$$

(where $\operatorname{Ker}(d_1^*)'$ is the inverse image of $K_2'(L)$) using the natural identification of $K_2'(L)$ with

$$\operatorname{Ker}(H^1\Big(L\,;K_3^{\text{ind}}\big(\mathbf{Q}^{\text{sep}})\big)\Big) \longrightarrow H^1\Big(L\,;\operatorname{Ind}_{\langle c\rangle}^{\Omega_{\mathbf{Q}}}\big(K_3^{\text{ind}}(\mathbf{Q}^{\text{sep}})\big)\Big)$$

of [11, Section 5]. In fact, this homomorphism is almost an equivalence of 2-extensions, being the identity on $K_3^{\text{ind}}(L)$ and minus the identity on $K_2'(L)$.

4 The Invariant, $\Omega_1(L/K)$

Definition 4.1 Let L/K be a Galois extension of number fields with group $G(L/K)$. Denote by $S_\infty(L)$ the set of Archimedean places of L. Let S be a finite $G(L/K)$-stable set of places of L containing $S_\infty(L)$, all places above 2 and all places which ramify over K. We also assume that S is large enough so that, for each intermediate field $K \subset F \subset L$, one can find a Galois extension, M/F, with $G(M/F)$ an elementary abelian 2-group, unramified outside S and having prescribed ramification at the real places of F. Let $\mathcal{O}_{L,S}$ denote the S-integers of L and, for each rational prime, l, set $X_l = \operatorname{Spec}(\mathcal{O}_{L,S}[1/l])$. If F_l is an étale $\mathbf{Z}_l$-sheaf on X_l, set $(F_l)_\infty = \bigoplus_{w \in S_\infty(L)} i_{w,*}i_w^*F_l$ where $i_w\colon \operatorname{Spec}(L_w) \to X_l$ is the canonical morphism and L_w is the completion of L at w. For example, we may take $F_l = \mathbf{Z}_l(2)$. For each prime, l, let $C_l(2)$ denote the cone of the natural injection of sheaves, $\mathbf{Z}_l(2) \to \big(\mathbf{Z}_l(2)\big)_\infty$, which is an object in the derived category of étale $\mathbf{Z}_l$-sheaves. Let $\mathbf{H}^*\big(X_l\,; C_l(2)\big)$ denote the hypercohomology of $C_l(2)$ in the derived category of the homotopy category of $\mathbf{Z}_l[G(L/K)]$-modules. From Artin-Verdier duality and the long exact sequence one finds that these cohomology groups vanish except in dimensions 0 and 1. One shows ([6] *cf.* [12, Section 4.17]) that there exists a two-term complex, $M^* = (M_l \to M_l')$, of $\mathbf{Z}_l[G(L/K)]$-modules of finite projective dimension concentrated in dimensions 0 and 1 such that there is an isomorphism, $\psi\colon M^* \to \mathbf{H}^*\big(X_l\,; C_l(2)\big)$, in the derived category of $\mathbf{Z}_l[G(L/K)]$-modules. This gives rise to a 2-extension

$$\mathbf{H}^0\big(X_l\,;C_l(2)\big) \longrightarrow M_l \longrightarrow M_l' \longrightarrow \mathbf{H}^1\big(X_l\,;C_l(2)\big)$$

whose class

$$E_l \in \operatorname{Ext}^2_{\mathbf{Z}_l[G(L/K)]}\Big(\mathbf{H}^1\big(X_l\,;C_l(2)\big),\mathbf{H}^0\big(X_l\,;C_l(2)\big)\Big)$$

is independent of the choice of ψ. Since M_l and M_l' are cohomologically trivial, cup-product with E_l induces an isomorphism in Tate cohomology in all dimensions. Now let $K_2'(\mathcal{O}_{L,S})$ denote the kernel of the natural map, $K_2(\mathcal{O}_{L,S}) \to \bigoplus_{w\in S_\infty(L)} K_2(L_w)$ and let $K_3^{\mathrm{ind}}(L) = K_3^{\mathrm{ind}}(\mathcal{O}_{L,S})$ denote the indecomposable K_3 (*i.e.* the quotient of $K_3(L)$ by the image of the Milnor K-group, $K_3^M(L)$) as in [19, Chapter 7]. There are isomorphisms of the form [11], [15], [17], [18],

$$K_2'(\mathcal{O}_{L,S}) \otimes \mathbf{Z}_l \cong \mathbf{H}^1\big(X_l\,;C_l(2)\big)$$

and exact sequences

$$0 \longrightarrow \bigoplus_{w\in S_\infty(L)} \mathbf{Z}_l(2) \longrightarrow \mathbf{H}^0\big(X_l\,;C_l(2)\big) \longrightarrow K_3^{\mathrm{ind}}(\mathcal{O}_{L,S}) \otimes \mathbf{Z}_l \longrightarrow 0$$

in which $G(L/K)$ acts on the left-hand group via its permutation action on $S_\infty(L)$. Hence there is a finitely generated $\mathbf{Z}[G(L/K)]$-submodule, $K_3^{\mathrm{ind}}(\mathcal{O}_{L,S})'$, of $\prod_l \mathbf{H}^0\big(X_l\,;C_l(2)\big)$ and an exact sequence of the form

$$0 \longrightarrow \bigoplus_{w\in S_\infty(L)} \mathbf{Z} \longrightarrow K_3^{\mathrm{ind}}(\mathcal{O}_{L,S})' \xrightarrow{\pi} K_3^{\mathrm{ind}}(\mathcal{O}_{L,S}) \longrightarrow 0$$

which is constructed by choosing isomorphisms $\hat{\mathbf{Z}}(2) \cong \prod_l \mathbf{Z}_l(2)$ and $\hat{\mathbf{Z}} \cong \prod_l \mathbf{Z}_l$. Since $K_2'(\mathcal{O}_{L,S})$ is a finite group there is an isomorphism of the form

$$\operatorname{Ext}^2_{\mathbf{Z}[G(L/K)]}\big(K_2'(\mathcal{O}_{L,S}),K_3^{\mathrm{ind}}(\mathcal{O}_{L,S})'\big) \cong \bigoplus_l \operatorname{Ext}^2_{\mathbf{Z}_l[G(L/K)]}\Big(\mathbf{H}^1\big(X_l\,;C_l(2)\big),\mathbf{H}^0\big(X_l\,;C_l(2)\big)\Big)$$

induced by $(M \mapsto \{M \otimes \mathbf{Z}_l\}_l)$ which sends E to $\bigoplus_l E_l$. The cup-product with E induces isomorphisms in Tate cohomology in all dimensions and the Euler characteristic associated to E defines $\Omega_1(L/K) \in \mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big)$.

The following is the main result of this section.

Theorem 4.2 *Let L/K be a Galois extension of number fields. Then the set, S, of places of L may be chosen so that there is a commutative diagram of 2-extensions of $\mathbf{Z}[G(L/K)]$-modules of the form*

$$\begin{array}{ccccccc}
K_3^{\mathrm{ind}}(\mathcal{O}_{L,S})' & \longrightarrow & A & \xrightarrow{a} & B & \longrightarrow & K_2'(\mathcal{O}_{L,S}) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & {\scriptstyle \pi_3}\downarrow{\scriptstyle \cong} \\
K_3^{\mathrm{ind}}(\mathcal{O}_{L,S}) & \longrightarrow & \bigoplus_{w\in S_\infty(L)} K_3^{\mathrm{ind}}(L_w) & \xrightarrow{a'} & C & \longrightarrow & K_2'(\mathcal{O}_{L,S})
\end{array}$$

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such that

(i) π *is the surjection of Definition 4.1, with kernel* $\bigoplus_{w\in S_\infty(L)} \mathbf{Z}$,

(ii) π_1 *and* π_2 *are surjective and the isomorphism,* π_3*, is given by* ± 1 *on each Sylow* p*-subgroup of the finite group,* $K_2'(\mathcal{O}_{L,S})$,

(iii) the upper 2-extension defines the class

$$E \in \operatorname{Ext}^2_{\mathbf{Z}[G(L/K)]}\big(K_2'(\mathcal{O}_{L,S}), K_3^{\mathrm{ind}}(\mathcal{O}_{L,S})'\big)$$

whose Euler characteristic is equal to

$$\Omega_1(L/K) \in \mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big),$$

(iv) the lower 2-extension is that of Corollary 2.10, as used in the construction of $\Omega_1(L/K,3)$.

The proof of Theorem 4.2 will take the form of a discussion occupying the remainder of this section and culminating in Section 4.7. In order to assist the reader, the strategy of the proof is outlined in Section 4.5 (see also Remark 4.4). In Theorem 4.2(ii) the ambiguity of signs in π_3 is irrelevant for our purposes, all we shall need is that π_3 is an isomorphism. We have left the signs unresolved in order to simplify the proof of Theorem 4.2 in Section 4.7 for the reader's convenience. It would be very strange if the signs actually varied with l. Before proceeding to the proof we record the following corollary, which is the totally real case of Theorem 1.1.

Corollary 4.3 *If* L/K *is totally real then*

$$\Omega_1(L/K) = \Omega_1(L/K,3) \in \mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big).$$

Proof From the commutative diagram of Theorem 4.2 we may obtain a commutative diagram of equivalent 2-extensions in which the modules—A, B, $\bigoplus_{w\in S_\infty(L)} K_3^{\mathrm{ind}}(L_w)$ and C—are replaced by finitely generated, cohomologically trivial modules—A_1, B_2, A_2 and B_2—respectively. This modified diagram gives rise to an exact sequence of the form

$$0 \longrightarrow \bigoplus_{w\in S_\infty(L)} \mathbf{Z} \longrightarrow A_1 \longrightarrow A_2 \oplus B_1 \longrightarrow B_2 \longrightarrow 0.$$

Since $\bigoplus_{w\in S_\infty(L)} \mathbf{Z}$ is a free module in the totally real case, in $\mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big)$ we obtain the equation

$$\Omega_1(L/K) = [A_1] - [B_1] = [A_2] - [B_2]$$

and the result follows from Corollary 2.18.

Remark 4.4 In preparation for the proof of Theorem 4.2 we make some simplifying remarks.

Firstly, if we have constructed a commutative diagram of 2-extensions of the required form but in which π_1 and π_2 are not surjective then we may easily remedy this lack of surjectivity in the following manner. Choose a free $\mathbf{Z}[G(L/K)]$-module, F, together with a surjection, $\phi\colon F \to \ker\bigl(C \to K_2'(\mathcal{O}_{L,S})\bigr)$. Lift ϕ to a homomorphism, $\psi\colon F \to \bigoplus_{w\in S_\infty(L)} K_3^{\mathrm{ind}}(L_w)$, such that $a'\psi = \phi$. Now replace $a\colon A \to B$, π_1 and π_2 by $a \oplus 1\colon A \oplus F \to B \oplus F$, $\pi_1 + \psi$ and $\pi_2 + \phi$, respectively.

Secondly, we may establish the existence of the diagram one prime at a time, using the isomorphisms

$$\mathrm{Ext}^2_{\mathbf{Z}[G(L/K)]}\bigl(K_2'(\mathcal{O}_{L,S}), K_3^{\mathrm{ind}}(\mathcal{O}_{L,S})'\bigr) \cong \bigoplus_{l \text{ prime}} \mathrm{Ext}^2_{\mathbf{Z}_l[G(L/K)]}\bigl(K_2'(\mathcal{O}_{L,S}) \otimes \mathbf{Z}_l, K_3^{\mathrm{ind}}(\mathcal{O}_{L,S})' \otimes \mathbf{Z}_l\bigr)$$

and

$$\mathrm{Ext}^2_{\mathbf{Z}[G(L/K)]}\bigl(K_2'(\mathcal{O}_{L,S}), K_3^{\mathrm{ind}}(\mathcal{O}_{L,S})\bigr) \cong \bigoplus_{l \text{ prime}} \mathrm{Ext}^2_{\mathbf{Z}_l[G(L/K)]}\bigl(K_2'(\mathcal{O}_{L,S}) \otimes \mathbf{Z}_l, K_3^{\mathrm{ind}}(\mathcal{O}_{L,S}) \otimes \mathbf{Z}_l\bigr),$$

which follow from the fact that $K_2'(\mathcal{O}_{L,S})$ is finite.

4.5 The Strategy For Proving Theorem 4.2

We are now ready to relate the $\mathbf{Z}_l[G(L/K)]$-modules $K_3^{\mathrm{ind}}(\mathcal{O}_{L,S}) \otimes \mathbf{Z}_l$, $K_3^{\mathrm{ind}}(\mathcal{O}_{L,S})' \otimes \mathbf{Z}_l$ and $K_2'(\mathcal{O}_{L,S}) \otimes \mathbf{Z}_l$ to the hypercohomology of the mapping cone, $C_l(2)$ of Definition 4.1.

In the notation of Definition 4.1 let l be a prime and let $X_l = \mathrm{Spec}(\mathcal{O}_{L,S}[1/l])$.

We are going to work towards the construction of the following rather daunting commutative diagram in which the rows are 2-extensions of $\mathbf{Z}_l[G(L/K)]$-modules, in which Ind denotes $\mathrm{Ind}_{\langle c\rangle}^{\Omega_{\mathbf{Q}}}\bigl(K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\bigr)^{\Omega_L}$ and the bottom row is equivalent to the 2-extension, E_l, of Definition 4.1:

4.6

$$\begin{array}{ccccccc}
K_3^{\mathrm{ind}}(X_l) \otimes \mathbf{Z}_l & \longrightarrow & \mathrm{Ind} & \longrightarrow & C & \longrightarrow & K_2'(X_l) \otimes \mathbf{Z}_l \\
\downarrow & & \downarrow & & \downarrow & & {\scriptstyle -1}\downarrow \\
K_3^{\mathrm{ind}}(X_l) \otimes \mathbf{Z}_l & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & K_2'(X_l) \otimes \mathbf{Z}_l \\
\uparrow{\scriptstyle \pm 1} & & \uparrow & & \uparrow & & {\scriptstyle \pm 1}\uparrow \\
K_3^{\mathrm{ind}}(X_l) \otimes \mathbf{Z}_l & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & K_2'(X_l) \otimes \mathbf{Z}_l \\
\downarrow{\scriptstyle 1} & & \downarrow & & \downarrow & & {\scriptstyle 1}\downarrow \\
K_3^{\mathrm{ind}}(X_l) \otimes \mathbf{Z}_l & \longrightarrow & A_3 & \longrightarrow & B_3 & \longrightarrow & K_2'(X_l) \otimes \mathbf{Z}_l \\
\uparrow & & \uparrow & & \uparrow & & {\scriptstyle \pm 1}\uparrow \\
\mathbf{H}^0\bigl(X_l\,; C_l(2)\bigr) & \longrightarrow & A_4 & \longrightarrow & B_4 & \longrightarrow & K_2'(X_l) \otimes \mathbf{Z}_l
\end{array}$$

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As explained in Remark 4.4, it is sufficient to construct the commutative diagram of 2-extensions after tensoring with $\mathbf{Z}_l$, for each prime, l. Also, from the localisation sequences

$$0 \longrightarrow K_n(\mathcal{O}_{L,S}) \longrightarrow K_n(X_l) \longrightarrow \bigoplus_{P \text{ prime over } l} K_{n-1}(\mathcal{O}_L/P) \longrightarrow 0$$

one sees that $\mathcal{O}_{L,S}$ may be replaced by X_l, since $K_n(\mathcal{O}_L/P) \otimes \mathbf{Z}_l = 0$ for $n > 0$. Also we construct the required commutative diagrams of 2-extensions involving X_l by pulling back corresponding diagrams of 2-extensions in which L replaces X_l.

We shall use Theorem 3.2 to obtain the diagram relating the first and second rows. The all-important third row is obtained in Section 4.12 by truncating the complex $R\Gamma_*\big(G(E/L), D_{l,m,*}\big)$ where $D_{l,m,*}$ is a mapping cone complex associated to the Kummer sequence of multiplication by l^m on $K_3^{\mathrm{ind}}(E)$. There are two canonical maps from the chain complex of a mapping cone and "projection on the second factor" yields the map connecting the second and third rows. The second map goes from the third row to the fourth row with $\mathbf{Z}/l^m$, rather than $\mathbf{Z}_l$, coefficients. On the other hand, reduction modulo l^m yields a diagram, given in Section 4.19, which relates the fifth row to the fourth row modulo l^m. This yields the diagram of Section 4.20. Then Section 4.21 is devoted to the verification that the rows remain 2-extensions as we take the limit over m to obtain the desired diagram described above.

Assuming the existence of the diagram of Section 4.6, we can prove Theorem 4.2.

4.7 Proof of Theorem 4.2

It is sufficient to construct the commutative diagram of 2-extensions after tensoring with $\mathbf{Z}_l$, for each prime, l, and with $\mathcal{O}_{L,S}$ replaced by X_l.

Let us temporarily refer to an isomorphism of 2-extensions in which the maps on the ends are ± 1 as an "equivalence up to sign". In the diagram of Section 4.6 the bottom row is equivalent to E_l of Section 4.11, by the remark following the diagram of Section 4.19. The other four rows are equivalent up to sign to the top row, which is the lower 2-extension in the statement of Theorem 4.2, after tensoring with $\mathbf{Z}_l$. Hence we may produce a diagram of l-adic 2-extensions of the required type by forming the pull-back 2-extension form the bottom two rows of the diagram of Section 4.6. The result is a 2-extension which maps to the second row of the diagram of Section 4.6 and we may form the pull-back of this 2-extension with the top row over the second row. This yields a commutative diagram of 2-extensions, equivalent up to sign to the bottom row and mapping into the top row, as required.

Finally, an equivalence up to sign in which the sign is the same at the ends can be transformed into an equivalence by reversing the signs of all the vertical maps, if necessary. If the signs are different we may reverse the signs, if necessary, to make them +1 on the left and -1 on $K_2'(X_l) \otimes \mathbf{Z}_l$, which is just the Sylow l-subgroup of $K_2'(\mathcal{O}_{L,S})$.

The rest of the proof follows from Remark 4.4 together with the fact that the map, $\mathbf{H}^0\big(X_l\,;C_l(2)\big) \to K_3^{\mathrm{ind}}(X_l) \otimes \mathbf{Z}_l$, is equal to $\pi \otimes \mathbf{Z}_l$.

For each positive integer, m, we may form the mapping cone, $C_{l,m}(2)$ on X_l, defined by

$$\mathbf{Z}/l^m(2) \longrightarrow \bigoplus_{w \in S_\infty(L)} i_{w,*} i_w^*\big(\mathbf{Z}/l^m(2)\big) \longrightarrow C_{l,m}(2).$$

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Let $\mathbf{H}^i\big(X_l\,;C_{l,m}(2)\big)$ denote the i-th étale hypercohomology group of $C_{l,m}(2)$ and let $H^i(X_l\,;-)$ denote ordinary étale cohomology. Write A/n for $A\otimes\mathbf{Z}/n$.

The next result follows from [6, Proposition 2.4], together with the results of [11], [14], [15], [17], [18].

Lemma 4.8 *The set of places, S, may be chosen so that:*

(i) For $i\neq 0,1$, $\mathbf{H}^i\big(X_l\,;C_{l,m}(2)\big)=0$.

(ii) The hypercohomology of the cone sequence yields an exact sequence of the form

$$0\longrightarrow \big(\mathbf{Z}/l^m(2)\big)^{\Omega_L}\longrightarrow \bigoplus_{w\in S_\infty(L)}\mathbf{Z}/l^m(2)\longrightarrow \mathbf{H}^0\big(X_l\,;C_{l,m}(2)\big)\xrightarrow{\ \delta\ } K_3^{\mathrm{ind}}(X_l)/l^m\longrightarrow 0$$

and an isomorphism

$$\delta\colon \mathbf{H}^1\big(X_l\,;C_{l,m}(2)\big)\xrightarrow{\ \cong\ }\big(K_2(X_l)/l^m\big)'$$

where $\big(K_2(X_l)/l^m\big)'=\mathrm{Ker}\big(K_2(X_l)/l^m\to\bigoplus_{w\in S_\infty(L)}K_2(L_w)/l^m\big)$.

4.9

We may define $C_l(2)$ to be the mapping cone on X_l defined by

$$\mathbf{Z}_l(2)\longrightarrow\bigoplus_{w\in S_\infty(L)} i_{w,*}i_w^*\big(\mathbf{Z}_l(2)\big)\longrightarrow C_l(2).$$

The l-adic étale hypercohomology of $C_l(2)$ is defined, as usual, to be

$$\mathbf{H}^2\big(X_l\,;C_l(2)\big)=\lim_{\overleftarrow{m}}\mathbf{H}^2\big(X_l\,;C_{l,m}(2)\big).$$

Corollary 4.9 *The set of places, S, may be chosen so that:*

(i) For $i\neq 0,1$, $\mathbf{H}^i\big(X_l\,;C_l(2)\big)=0$.

(ii) The exact hypercohomology sequence associated to the cone sequence yields an isomorphism

$$\delta\colon\mathbf{H}^1\big(X_l\,;C_l(2)\big)\xrightarrow{\ \cong\ }H^2\big(X_l\,;\mathbf{Z}_l(2)\big)\cong K_2'(X_l)\otimes\mathbf{Z}_l$$

and a short exact sequence of the form

$$0\longrightarrow\bigoplus_{w\in S_\infty(L)}\mathbf{Z}_l(2)\longrightarrow\mathbf{H}^0\big(X_l\,;C_l(2)\big)\xrightarrow{\ \delta\ }H^1\big(X_l\,;\mathbf{Z}_l(2)\big)\cong K_3^{\mathrm{ind}}(X_l)\otimes\mathbf{Z}_l\longrightarrow 0$$

where $K_2'(X_l)=\ker\big(K_2(X_l)\to\bigoplus_{w\in S_\infty(L)}K_2(L_w)\big)$, as in Proposition 2.9.

Proof The groups in Lemma 4.8(ii) are finite and therefore the sequences remain exact upon taking inverse limits. However, as explained in [24, Section III], $\big(\mathbf{Z}_l(2)\big)^{\Omega_L}=0$, which completes the proof.

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4.11

Let $\mathcal{D}_m$ denote the derived category of the homotopy category of $\mathbf{Z}/l^m[G(L/K)]$-modules. It is shown in [6] that there exists a chain complex of finitely generated, projective $\mathbf{Z}/l^m[G(L/K)]$-modules of the form

$$Q_{-1} \xrightarrow{d_{-1}} Q_0 \xrightarrow{d_0} Q_1$$

which is quasi-isomorphic in $\mathcal{D}_m$ to $\mathbf{H}^*\big(X_l\,;C_{l,m}(2)\big)$, for a suitable choice of S.

We may form a 2-extension of $\mathbf{Z}/l^m[G(L/K)]$-modules of the form

$$\mathbf{H}^0\big(X_l\,;C_{l,m}(2)\big) \longrightarrow Q_0/\big(d_{-1}(Q_{-1})\big) \longrightarrow Q_1 \longrightarrow \mathbf{H}^1\big(X_l\,;C_{l,m}(2)\big).$$

Since Q_1 and $Q_0/\big(d_{-1}(Q_{-1})\big)$ are cohomologically trivial, cup-product with the resulting 2-extension

$$E_{l,m} \in \mathrm{Ext}^2_{\mathbf{Z}_l[G(L/K)]}\Big(\mathbf{H}^1(X_l\,;C_{l,m}(2)\big), \mathbf{H}^0\big(X_l\,;C_{l,m}(2)\big)\Big)$$

induces an isomorphism in Tate cohomology

$$\hat{H}^i\Big(G(L/K)\,;\mathbf{H}^1\big(X_l\,;C_{l,m}(2)\big)\Big) \xrightarrow{\cong} \hat{H}^{i+2}\Big(G(L/K)\,;\mathbf{H}^0\big(X_l\,;C_{l,m}(2)\big)\Big)$$

for all i.

Since $\mathbf{H}^1\big(X_l\,;C_{l,m}(2)\big) \cong \big(K_2(X_l)/l^m\big)'$, this group is finite and the natural map

$$\mathbf{H}^1\big(X_l\,;C_{l,m+1}(2)\big) \longrightarrow \mathbf{H}^1\big(X_l\,;C_{l,m}(2)\big)$$

is an isomorphism for $m \geq m_0$, where m_0 depends on L, S and l. Furthermore the chain complexes, $\{Q_*, d_*\}$, may be constructed to form an inverse system as m varies which is compatible with the inverse system of hypercohomology groups. Therefore there is a natural isomorphism of the form

$$\begin{aligned}&\mathrm{Ext}^2_{\mathbf{Z}_l[G(L/K)]}\Big(\mathbf{H}^1\big(X_l\,;C_l(2)\big), \mathbf{H}^0\big(X_l\,;C_l(2)\big)\Big)\\ &\qquad \overset{\rightarrow}{\cong} \varprojlim_m \mathrm{Ext}^2_{\mathbf{Z}_l[G(L/K)]}\Big(\mathbf{H}^1\big(X_l\,;C_{l,m}(2)\big), \mathbf{H}^0\big(X_l\,;C_{l,m}(2)\big)\Big)\end{aligned}$$

under which the 2-extension, E_l of Section 1 maps to $\{E_{l,m}\}$. To see that this surjection is an isomorphism observe that the kernel is equal to

$$\varprojlim_m{}^1 \mathrm{Ext}^1_{\mathbf{Z}_l[G(L/K)]}\Big(\mathbf{H}^1\big(X_l\,;C_{l,m}(2)\big), \mathbf{H}^0\big(X_l\,;C_{l,m}(2)\big)\Big),$$

which vanishes since the Ext^1-groups are finite.

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Consider the following commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(L_w) & \xrightarrow{j_w} & \operatorname{Spec}(L) \\ & {\scriptstyle i_w}\searrow \quad \swarrow{\scriptstyle k_l} & \\ & X_l & \end{array}$$

for X_l as in Section 4.5.

The maps, k_l, induce homomorphisms which fit into the localisation exact sequence [24, Section III]

$$\cdots \longrightarrow H^i\big(X_l\,;\mathbf{Z}/l^m(2)\big) \xrightarrow{k_l^*} H^i\big(L\,;\mathbf{Z}/l^m(2)\big) \longrightarrow \bigoplus_{P\notin S} H^{i-1}\big(\mathcal{O}_L/P\,;\mathbf{Z}/l^m(1)\big) \longrightarrow H^{i+1}\big(X_l\,;\mathbf{Z}/l^m(2)\big) \longrightarrow \cdots .$$

In [24] (with a correction for the case $l = 2$ which is given in [25]) it is shown that this sequence breaks into short exact sequences. In particular, there are isomorphisms

$$k_l^*\colon H^i\big(X_l\,;\mathbf{Z}/l^m(2)\big) \longrightarrow H^i\big(L\,;\mathbf{Z}/l^m(2)\big)$$

when $i = 0, 1$ and a short exact sequence of the form

$$0 \longrightarrow H^2\big(X_l\,;\mathbf{Z}/l^m(2)\big) \xrightarrow{k_l^*} H^2\big(L\,;\mathbf{Z}/l^m(2)\big) \longrightarrow \bigoplus_{P\notin S}(\mathcal{O}_L/P)^*/l^m \longrightarrow 0.$$

Taking inverse limits over m, Corollary 4.9 yields an isomorphism of $\mathbf{Z}_l[G(L/K)]$-modules

$$k_l^*\colon \mathbf{H}^0\big(X_l\,;C_l(2)\big) \xrightarrow{\cong} \mathbf{H}^0\big(L\,;C_l(2)\big)$$

and a short exact sequence

$$0 \longrightarrow \mathbf{H}^1\big(X_l\,;C_l(2)\big) \xrightarrow{k_l^*} \mathbf{H}^1\big(L\,;C_l(2)\big) \longrightarrow \bigoplus_{P\notin S}(\mathcal{O}_L/P)^* \otimes \mathbf{Z}_l \longrightarrow 0.$$

For each prime, $R \triangleleft \mathcal{O}_K$, not lying below an element of S the $\mathbf{Z}_l[G(L/K)]$-module

$$\bigoplus_{P\mid R}(\mathcal{O}_L/P)^* \otimes \mathbf{Z}_l \cong \operatorname{Ind}_{G(L_{P_0}/K_R)}^{G(L/K)}\big((\mathcal{O}_L/P_0)^* \otimes \mathbf{Z}_l\big)$$

is cohomologically trivial. Here P_0 is a chosen prime above R. In fact, since $P_0 \notin S$, $(\mathcal{O}_L/P_0)^* \otimes \mathbf{Z}_l$ has a resolution [20, p. 352] of the form

$$0 \longrightarrow \mathbf{Z}_l[G(L_{P_0}/K_R)] \longrightarrow \mathbf{Z}_l[G(L_{P_0}/K_R)] \longrightarrow (\mathcal{O}_L/P_0)^* \otimes \mathbf{Z}_l \longrightarrow 0$$

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and therefore

$$\operatorname{Ext}^i_{\mathbf{Z}_l[G(L/K)]}\Big(\bigoplus_{P\notin S}(\mathcal{O}_L/P)^*\otimes\mathbf{Z}_l, M\Big)=0$$

for all $i>0$. Hence k_l induces an isomorphism of the form

$$\operatorname{Ext}^2_{\mathbf{Z}_l[G(L/K)]}\Big(\mathbf{H}^1\big(X_l\,;C_l(2)\big),\mathbf{H}^0\big(X_l\,;C_l(2)\big)\Big)\\ \cong \operatorname{Ext}^2_{\mathbf{Z}_l[G(L/K)]}\Big(\mathbf{H}^1\big(L\,;C_l(2)\big),\mathbf{H}^0\big(L\,;C_l(2)\big)\Big).$$

4.12

Now let us consider the situation of Section 3.3 in which E contains all the l^m-th roots of unity. We have an exact sequence of the form

$$0\longrightarrow \mathbf{Z}/l^m(2)\xrightarrow{d} K_3^{\mathrm{ind}}(E)\xrightarrow{l^m} l^mK_3^{\mathrm{ind}}(E)\longrightarrow 0.$$

Define a complex, $D_{l,m}$, of $\mathbf{Z}[G(E/\mathbf{Q})]$-modules of the form

$$0\longrightarrow D_{l,m,1}=\mathbf{Z}/l^m(2)\xrightarrow{d} D_{l,m,0}=K_3^{\mathrm{ind}}(E)\longrightarrow 0.$$

Let ΣA denote a copy of the module, A, in dimension one. We have chain maps of complexes of the form

$$\begin{array}{ccc} D_{l,m,1} & \xrightarrow{d} & D_{l,m,0}\\ \downarrow & & \downarrow\\ \Sigma\mathbf{Z}/l^m(2) & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccc} D_{l,m,1} & \xrightarrow{d} & D_{l,m,0}\\ \downarrow & & \downarrow\\ 0 & \longrightarrow & l^mK_3^{\mathrm{ind}}(E) \end{array}$$

Define a chain complex, $\{R\Gamma_*\big(G(E/L),D_{l,m,*}\big),d\}$, by the formulae

$$R\Gamma_t\big(G(E/L),D_{l,m,*}\big)=\operatorname{Hom}_{G(E/L)}\big(B_tG(E/\mathbf{Q}),D_{l,m,0}\big)\oplus\operatorname{Hom}_{G(E/L)}\big(B_{t+1}G(E/\mathbf{Q}),D_{l,m,1}\big).$$

If $h_i\in\operatorname{Hom}_{G(E/L)}\big(B_{t+i}G(E/\mathbf{Q}),D_{l,m,i}\big)$ the differential is given by

$$d(h_0,h_1)=\big(d^*h_0+(-1)^td_*h_1,d^*h_1\big)$$

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where d^* is induced by the differential in the bar resolution and d_* is induced by the differential in $D_{l,m,*}$.

Since $\lim_{\vec{E}} K_3^{\mathrm{ind}}(E) = K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})$ is divisible

$$\lim_{\vec{E}} \mathrm{Hom}_{G(E/L)}\big(B_t G(E/\mathbf{Q}), K_3^{\mathrm{ind}}(E)/l^m\big) = 0$$

and

$$\begin{aligned}\lim_{\vec{E}} \mathrm{Hom}_{G(E/L)}\big(B_t G(E/\mathbf{Q}), l^m K_3^{\mathrm{ind}}(E)\big) &\cong \lim_{\vec{E}} \mathrm{Hom}_{G(E/L)}\big(B_t G(E/\mathbf{Q}), K_3^{\mathrm{ind}}(E)\big)\\ &= R\Gamma_t\big(L, K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big)\end{aligned}$$

in the notation of Section 3.3.

Hence we obtain a complex, $R\Gamma_*(L, D_{l,m})$, with

$$R\Gamma_t(L, D_{l,m}) = R\Gamma_t\big(L, K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big) \oplus R\Gamma_{t+1}\big(L, \mathbf{Z}/l^m(2)\big).$$

In addition, we have a commutative diagram of maps of chain complexes of the following form:

$$\begin{array}{ccccc} 0 & \longrightarrow & R\Gamma_1\big(L, K_3^{\mathrm{ind}}(L)\big) & \xrightarrow{d} & R\Gamma_0\big(L, K_3^{\mathrm{ind}}(L)\big)\cdots \\ \uparrow & & \uparrow & & \uparrow \\ R\Gamma_{-1}(L, D_{l,m}) & \xrightarrow{d} & R\Gamma_0(L, D_{l,m}) & \xrightarrow{d} & R\Gamma_1(L, D_{l,m})\cdots \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma_2\big(L, \mathbf{Z}/l^m(2)\big) & \xrightarrow{d} & R\Gamma_1\big(L, \mathbf{Z}/l^m(2)\big) & \xrightarrow{d} & R\Gamma_0\big(L, \mathbf{Z}/l^m(2)\big)\cdots \end{array}$$

In this diagram of complexes of $\mathbf{Z}[G(L/K)]$-modules the upward vertical chain map is a quasi-isomorphism. In fact, it induces a map from the long exact cohomology sequence

$$\cdots \longrightarrow H^t\big(L\,;K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big) \longrightarrow \mathbf{H}^t(L\,;D_{l,m}) \xrightarrow{\beta} H^{t+1}\big(L\,;\mathbf{Z}/l^m(2)\big) \longrightarrow \cdots$$

to the long exact cohomology sequence associated with

$$0 \longrightarrow \mathbf{Z}/l^m(2) \longrightarrow K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}}) \xrightarrow{l^m} K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}}) \longrightarrow 0$$

resulting in a diagram which is commutative up to sign and is the identity on the cohomology of $\mathbf{Z}/l^m(2)$ and the middle $K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})$. Hence, by the Five Lemma, the map induces a cohomology isomorphism of the form

$$\mathbf{H}^i(L\,;D_{l,m}) \xrightarrow{\cong} H^i\big(L\,;K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big)$$

for all i.

The downward vertical map induces the homomorphism, β, on cohomology. This may be identified, via the above isomorphism, with

$$(-1)^{i-1}\delta\colon H^i\big(L\,;K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big)\longrightarrow H^{i+1}\big(L\,;\mathbf{Z}/l^m(2)\big)$$

where δ is the coboundary associated to multiplication by l^m on $K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})$.

From [11, Section 4], [14], [15], [17], [18] one finds that these maps induce isomorphisms when $i=0,1$ of the form

$$H^0\big(L\,;K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big)\otimes\mathbf{Z}_l\cong K_3^{\mathrm{ind}}(L)\otimes\mathbf{Z}_l\xrightarrow{\cong}H^1\big(L\,;\mathbf{Z}_l(2)\big)\cong\varprojlim_m H^1\big(L\,;\mathbf{Z}/l^m(2)\big)$$

and

$$H^1\big(L\,;K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big)\otimes\mathbf{Z}_l\cong K_2(L)\otimes\mathbf{Z}_l\xrightarrow{\cong}H^2\big(L\,;\mathbf{Z}_l(2)\big)\cong\varprojlim_m H^2\big(L\,;\mathbf{Z}/l^m(2)\big).$$

Any complex of the form $0\longrightarrow R_{-1}\xrightarrow{d_{-1}}R_0\xrightarrow{d_0}R_1\cdots$ may be truncated to yield a 2-extension of the form

$$H^0\longrightarrow R_0/d_{-1}(R_{-1})\longrightarrow\mathrm{Ker}(d_1)\longrightarrow H^1$$

in which H^i denotes the i-th homology group of the complex. Applying this truncation to the diagram of the $R\Gamma_*(L,-)$'s and identifying the cohomology groups as K-groups, as in [11, p. 71], yields a commutative diagram of 2-extensions in which δ is induced by the coboundary defined above. A similar remark applies to the diagrams of Section 4.13 and Section 4.14.

$$\begin{array}{ccccccc}
K_3^{\mathrm{ind}}(L) & \longrightarrow & R\Gamma_0K & \longrightarrow & \mathrm{Ker}(d_K) & \longrightarrow & K_2(L)\\
\uparrow{\scriptstyle\pm1} & & \uparrow & & \uparrow & & {\scriptstyle\pm1}\uparrow\\
K_3^{\mathrm{ind}}(L)/l^m & \longrightarrow & R\Gamma_0D/\big(d(R\Gamma_{-1}D)\big) & \longrightarrow & \mathrm{Ker}(d_D) & \longrightarrow & K_2(L)\\
\downarrow{\scriptstyle\delta} & & \downarrow & & \downarrow & & {\scriptstyle\delta}\downarrow\\
K_3^{\mathrm{ind}}(L) & \longrightarrow & R\Gamma_1\mathbf{Z}_l^E/d(R\Gamma_0\mathbf{Z}_l) & \longrightarrow & \mathrm{Ker}(d_{\mathbf{Z}_l}) & \longrightarrow & H^2\big(L\,;\mathbf{Z}/l^m(2)\big)
\end{array}$$

Here the subscripts K, D and $\mathbf{Z}_l$ refer to the modules $K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})$, $D_{l,m}$ and $\mathbf{Z}/l^m(2)$, respectively. Pulling this diagram back to the submodules $K_2'(X_l)$ and $H^2\big(X_l\,;\mathbf{Z}/l^m(2)\big)'\cong K_2'(X_l)/l^m$ (the latter being independent of m when m is large enough) we obtain the following diagram of 2-extensions:

4.13

$$\begin{array}{ccccccc}
K_3^{\mathrm{ind}}(X_l) & \longrightarrow & R\Gamma_0K & \longrightarrow & \mathrm{Ker}(d_K)' & \longrightarrow & K_2'(X_l)\\
\uparrow{\scriptstyle\pm1} & & \uparrow & & \uparrow & & {\scriptstyle\pm1}\uparrow\\
K_3^{\mathrm{ind}}(X_l)/l^m & \longrightarrow & R\Gamma_0D/\big(d(R\Gamma_{-1}D)\big) & \longrightarrow & \mathrm{Ker}(d_D)' & \longrightarrow & K_2'(X_l)\\
\downarrow{\scriptstyle\delta} & & \downarrow & & \downarrow & & {\scriptstyle\delta}\downarrow\\
K_3^{\mathrm{ind}}(X_l) & \longrightarrow & R\Gamma_1\mathbf{Z}_l^E/d(R\Gamma_0\mathbf{Z}_l) & \longrightarrow & \mathrm{Ker}(d_{\mathbf{Z}_l})' & \longrightarrow & K_2'(X_l)/l^m
\end{array}$$

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Taking the limit over m the δ's become equal to ± 1, after tensoring with $\mathbf{Z}_l$. Hence we obtain a commutative diagram of l-adic 2-extensions in which the subscript, l, denotes the tensor product with $\mathbf{Z}_l$—a (rather poor) temporary notation to avoid cluttering up the diagram.

4.14

$$\begin{array}{ccccccc}
K_3^{\mathrm{ind}}(X_l)_l & \longrightarrow & R\Gamma_0 K_l & \longrightarrow & \mathrm{Ker}(d_K)_l' & \longrightarrow & K_2'(X_l)_l \\
\uparrow \pm 1 & & \uparrow & & \uparrow & & \pm 1 \uparrow \\
K_3^{\mathrm{ind}}(X_l)_l & \longrightarrow & R\Gamma_0 D/\big(d(R\Gamma_{-1}D)\big)_l & \longrightarrow & \mathrm{Ker}(d_D)_l' & \longrightarrow & K_2'(X_l)_l \\
\downarrow \delta & & \downarrow & & \downarrow & & \delta \downarrow \\
K_3^{\mathrm{ind}}(X_l)_l & \longrightarrow & R\Gamma_1 \mathbf{Z}_l^E/d(R\Gamma_0 \mathbf{Z}_l)_l & \longrightarrow & \mathrm{Ker}(d_{\mathbf{Z}_l})_l' & \longrightarrow & K_2'(X_l)_l
\end{array}$$

4.15

Next we must compare the 2-extensions of Section 4.11

$$\mathbf{H}^0\big(X_l\,;C_{l,m}(2)\big) \longrightarrow Q_0/\big(d_{-1}(Q_{-1})\big) \longrightarrow \mathrm{Ker}(d_1) \longrightarrow \mathbf{H}^1\big(X_l\,;C_{l,m}(2)\big)$$

with the bottom row of the diagram of Section 4.13. To this end, let $\{R\Gamma_i\big(L, C_{l,m}(2)\big), d_i\}$ denote the chain complex of $\mathbf{Z}/l^m[G(L/K)]$-modules defined by replacing $D_{l,m}$ by $C_{l,m}$ in the above construction.

We have a commutative diagram in which the vertical maps are induced by projection onto the second summand.

4.16

$$\begin{array}{ccccc}
R\Gamma_{-1}\big(L, C_{l,m}(2)\big) & \xrightarrow{d_{-1}} & R\Gamma_0\big(L, C_{l,m}(2)\big) & \xrightarrow{d_0} & R\Gamma_1\big(L, C_{l,m}(2)\big)\cdots \\
\downarrow & & \downarrow & & \downarrow \\
R\Gamma_0\big(L, \mathbf{Z}/l^m(2)\big) & \xrightarrow{d_0} & R\Gamma_1\big(L, \mathbf{Z}/l^m(2)\big) & \xrightarrow{d_1} & R\Gamma_2\big(L, \mathbf{Z}/l^m(2)\big)\cdots
\end{array}$$

Now let $L_{S,l}/L$ denote the maximal extension which is unramified outside $S \cup \{\text{primes over } l\}$. Let $G_l = G(L_{S,l}/L)$ denote the Galois group, which is a quotient of Ω_L. Let $Y_l \triangleleft \Omega_{\mathbf{Q}}$ denote the normal subgroup generated by G_l. Hence the action of $\Omega_{\mathbf{Q}}$ on $C_{l,m}(2)$ factorises through $\Omega_{\mathbf{Q}}/Y_l = H_l$ and we may form a complex, $\{R\Gamma_i\big(X_l, C_{l,m}(2)\big), d_i\}$, by replacing $\Omega_L = \lim_{\overleftarrow{E/L}} G(E/L)$ and $\Omega_{\mathbf{Q}} = \lim_{\overleftarrow{E/\mathbf{Q}}} G(E/\mathbf{Q})$ by G_l and H_l, respectively. We obtain a chain complex of cohomologically trivial $\mathbf{Z}_l[G(L/K)]$-modules, by Theorem 3.2(ii), of the form

$$R\Gamma_{-1}\big(X_l, C_{l,m}(2)\big) \xrightarrow{d_{-1}} R\Gamma_0\big(X_l, C_{l,m}(2)\big) \xrightarrow{d_0} \cdots$$

whose cohomology is $\mathbf{H}^*\big(X_l\,;C_{l,m}(2)\big)$. We obtain a commutative diagram analogous to the diagram of Section 4.16.

4.17

$$\begin{array}{ccccc} R\Gamma_{-1}\big(X_l, C_{l,m}(2)\big) & \xrightarrow{d_{-1}} & R\Gamma_0\big(X_l, C_{l,m}(2)\big) & \xrightarrow{d_0} & R\Gamma_1\big(X_l, C_{l,m}(2)\big)\cdots \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma_0\big(X_l, \mathbf{Z}/l^m(2)\big) & \xrightarrow{d_0} & R\Gamma_1\big(X_l, \mathbf{Z}/l^m(2)\big) & \xrightarrow{d_1} & R\Gamma_2\big(X_l, \mathbf{Z}/l^m(2)\big)\cdots \end{array}$$

The truncation of the diagram of Section 4.16 yields the following commutative diagram.

4.18

$$\begin{array}{ccccccc} \mathbf{H}^0\big(L\,;C_{l,m}(2)\big) & \longrightarrow & R\Gamma_0 C/d_{-1}(R\Gamma_{-1}C) & \longrightarrow & \mathrm{Ker}(d_C) & \longrightarrow & \mathbf{H}^1\big(L\,;C_{l,m}(2)\big) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1\big(L\,;\mathbf{Z}/l^m(2)\big) & \longrightarrow & R\Gamma_1\mathbf{Z}_l/d_0(R\Gamma_1\mathbf{Z}_l) & \longrightarrow & \mathrm{Ker}(d_{\mathbf{Z}_l}) & \longrightarrow & H^2\big(L\,;\mathbf{Z}/l^m(2)\big) \end{array}$$

Similarly, the truncation of the diagram of Section 4.17 yields the following commutative diagram.

4.19

$$\begin{array}{ccccccc} \mathbf{H}^0\big(X_l\,;C_{l,m}(2)\big) & \longrightarrow & R\Gamma_0 C/d_{-1}(R\Gamma_{-1}C) & \longrightarrow & \mathrm{Ker}(d_{XC}) & \longrightarrow & \mathbf{H}^1\big(X_l\,;C_{l,m}(2)\big) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1\big(X_l\,;\mathbf{Z}/l^m(2)\big) & \longrightarrow & R\Gamma_1\mathbf{Z}_l/d_0(R\Gamma_1\mathbf{Z}_l) & \longrightarrow & \mathrm{Ker}(d_{X\mathbf{Z}_l}) & \longrightarrow & H^2\big(X_l\,;\mathbf{Z}/l^m(2)\big) \end{array}$$

The quotient maps $\Omega_L \to G_l$ and $\Omega_{\mathbf{Q}} \to H_l$ induce a map from the diagram of Section 4.19 to that of Section 4.18 which is a pull-back diagram.

Finally, since the $\mathbf{Z}_l[G(L/K)]$-module, $R\Gamma_i\big(X_l, C_{l,m}(2)\big)$, is cohomologically trivial (*cf.* Theorem 3.2(ii)) the argument of [6], which constructs the resolution

$$Q_{-1} \xrightarrow{d_{-1}} Q_0 \xrightarrow{d_0} Q_1 \longrightarrow \mathbf{H}^1\big(X_l\,;C_{l,m}(2)\big) \longrightarrow 0,$$

also shows that the complex $\{Q_i, d_i\}$ is isomorphic in $\mathcal{D}_m$ to $\{R\Gamma_i\big(X_l, C_{l,m}(2)\big), d_i\}$. Therefore the upper 2-extension in the diagram of Section 4.19 represents

$$E_{l,m} \in \mathrm{Ext}^2_{\mathbf{Z}_l[G(L/K)]}\Big(\mathbf{H}^1\big(X_l\,;C_{l,m}(2)\big), \mathbf{H}^0\big(X_l\,;C_{l,m}(2)\big)\Big).$$

From the preceding discussion, pulling back to $K_2'(X_l) \otimes \mathbf{Z}_l$ and $\big(K_2(X_l)/l^m\big)'$, we have the following diagram in which Ind denotes $\mathrm{Ind}_{\langle c\rangle}^{\Omega_{\mathbf{Q}}}\big(K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big)^{\Omega_L}$.

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4.20

$$
\begin{array}{ccccccc}
K_3^{\mathrm{ind}}(X_l)\otimes\mathbf{Z}_l & \longrightarrow & \mathrm{Ind} & \longrightarrow & C & \longrightarrow & K_2'(X_l)\otimes\mathbf{Z}_l \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & {\scriptstyle -1}\big\downarrow \\
K_3^{\mathrm{ind}}(X_l)\otimes\mathbf{Z}_l & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & K_2'(X_l)\otimes\mathbf{Z}_l \\
\big\uparrow{\scriptstyle \pm1} & & \big\uparrow & & \big\uparrow & & {\scriptstyle \pm1}\big\uparrow \\
K_3^{\mathrm{ind}}(X_l)\otimes\mathbf{Z}_l & \longrightarrow & A_2(m) & \longrightarrow & B_2(m) & \longrightarrow & K_2'(X_l)\otimes\mathbf{Z}_l \\
\big\downarrow{\scriptstyle \delta} & & \big\downarrow & & \big\downarrow & & {\scriptstyle \delta}\big\downarrow \\
K_3^{\mathrm{ind}}(X_l)/l^m & \longrightarrow & A_3(m) & \longrightarrow & B_3(m) & \longrightarrow & \big(K_2(X_l)/l^m\big)' \\
\big\uparrow & & \big\uparrow & & \big\uparrow & & {\scriptstyle \pm1}\big\uparrow \\
\mathbf{H}^0\big(X_l\,;C_{l,m}(2)\big) & \longrightarrow & A_4(m) & \longrightarrow & B_4(m) & \longrightarrow & \big(K_2(X_l)/l^m\big)'
\end{array}
$$

In the diagram of Section 4.20 the upper two 2-extensions are derived from the pull-backs of those of Theorem 3.2, after tensoring with $\mathbf{Z}_l$. The second, third and fourth rows come from the diagram of Section 4.13 and the last two come from the diagrams of Section 4.18, Section 4.19.

4.21

Consider now the 2-extension of Section 4.12

$$K_3^{\mathrm{ind}}(L)/l^m \longrightarrow A(m) \longrightarrow B(m) \longrightarrow \big(K_2(L)/l^m\big)$$

and its pullback, in Section 4.19,

$$K_3^{\mathrm{ind}}(X_l)/l^m \longrightarrow A_3(m) \longrightarrow B_3(m) \longrightarrow \big(K_2(X_l)/l^m\big)'$$

which appears as the fourth row in the diagram of Section 4.20.

We wish to analyse the effect of taking the inverse limit over m. The cases of these two 2-extensions are similar so we shall concentrate on the first one. This sequence is obtained by truncating the complex

$$R\Gamma_*\big(L,\mathbf{Z}/l^m(2)\big) = \lim_{\substack{\rightarrow \\ E/\mathbf{Q}}} \mathrm{Hom}_{G(E/L)}\big(B_*G(E/\mathbf{Q}),\mathbf{Z}/l^m(2)\big).$$

The homomorphism

$$\mathrm{Hom}_{G(E/L)}\big(B_iG(E/\mathbf{Q}),\mathbf{Z}/l^m(2)\big) \longrightarrow \mathrm{Hom}_{G(E/L)}\big(B_iG(E/\mathbf{Q}),\mathbf{Z}/l^{m-1}(2)\big)$$

is surjective and therefore so is the direct limit

$$R\Gamma_i\big(L,\mathbf{Z}/l^m(2)\big) \longrightarrow R\Gamma_i\big(L,\mathbf{Z}/l^{m-1}(2)\big)$$

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which implies that

$$A(m) = R\Gamma_0\big(L, \mathbf{Z}/l^m(2)\big)/d\Big(R\Gamma_{-1}\big(L, \mathbf{Z}/l^m(2)\big)\Big) \longrightarrow A(m-1)$$
$$= R\Gamma_0\big(L, \mathbf{Z}/l^{m-1}(2)\big)/d\Big(R\Gamma_{-1}\big(L, \mathbf{Z}/l^{m-1}(2)\big)\Big)$$

is also surjective. Hence, if we split the 2-extension into two short exact sequences

$$0 \longrightarrow K_3^{\mathrm{ind}}(L)/l^m \longrightarrow A(m) \longrightarrow Y(m) \longrightarrow 0$$

and

$$0 \longrightarrow Y(m) \longrightarrow B(m) \longrightarrow K_2(L)/l^m \longrightarrow 0$$

then $Y(m) \to Y(m-1)$ is also surjective. Therefore $\lim^1_{\substack{\leftarrow \\ m}} Y(m) = 0$ and, since $K_3^{\mathrm{ind}}(L)/l^m$ is finite, we obtain two short exact sequences upon taking the inverse limit over m

$$0 \longrightarrow K_3^{\mathrm{ind}}(L) \otimes \mathbf{Z}_l \longrightarrow \lim_{\substack{\leftarrow \\ m}} A(m) \longrightarrow \lim_{\substack{\leftarrow \\ m}} Y(m) \longrightarrow 0$$

and

$$0 \longrightarrow \lim_{\substack{\leftarrow \\ m}} Y(m) \longrightarrow \lim_{\substack{\leftarrow \\ m}} B(m) \longrightarrow \lim_{\substack{\leftarrow \\ m}} K_2(L)/l^m \longrightarrow 0.$$

Upon taking inverse limits over m, the 2-extensions yield 2-extensions of $\mathbf{Z}_l[G(L/K)]$-modules of the form

$$K_3^{\mathrm{ind}}(L) \otimes \mathbf{Z}_l \longrightarrow \lim_{\substack{\leftarrow \\ m}} A(m) \longrightarrow \lim_{\substack{\leftarrow \\ m}} B(m) \longrightarrow \lim_{\substack{\leftarrow \\ m}} K_2(L)/l^m$$

and

$$K_3^{\mathrm{ind}}(X_l) \otimes \mathbf{Z}_l \longrightarrow \lim_{\substack{\leftarrow \\ m}} A_3(m) \longrightarrow \lim_{\substack{\leftarrow \\ m}} B_3(m) \longrightarrow K_2(X_l) \otimes \mathbf{Z}_l,$$

the latter being the pull-back of the former.

A similar argument applies to the third row of the diagram of Section 4.20. For $A_2(m) = R\Gamma_0\big(L, D_{l,m})\big)/d\big(R\Gamma_{-1}(L, D_{l,m})\big)$ and the map from

$$\mathrm{Hom}_{G(E/L)}\big(B_0G(E/\mathbf{Q}), K_3^{\mathrm{ind}}(E)\big) \oplus \mathrm{Hom}_{G(E/L)}\big(B_1G(E/\mathbf{Q}), \mathbf{Z}/l^m(2)\big)$$

to

$$\mathrm{Hom}_{G(E/L)}\big(B_0G(E/\mathbf{Q}), K_3^{\mathrm{ind}}(E)\big) \oplus \mathrm{Hom}_{G(E/L)}\big(B_1G(E/\mathbf{Q}), \mathbf{Z}/l^{m-1}(2)\big)$$

is multiplication by l on the first summand and is surjective on the second. However, a $\mathbf{Z}[G(E/L)]$-module map on $B_0G(E/\mathbf{Q}) = \mathbf{Z}[G(E/\mathbf{Q})]$ is determined by its values on a set of coset representatives for $G(E/L) \setminus G(E/\mathbf{Q})$—$[L:\mathbf{Q}]$ of them—so that multiplication by l is surjective on $\lim_{\substack{\rightarrow \\ E/\mathbf{Q}}} \mathrm{Hom}_{G(E/L)}\big(B_0G(E/\mathbf{Q}), K_3^{\mathrm{ind}}(E)\big)$, since $K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}}) = \lim_{\substack{\rightarrow \\ E/\mathbf{Q}}} K_3^{\mathrm{ind}}(E))$ is divisible.

Furthermore the inverse limit of the map between the third and fourth rows of the diagram of Section 4.20 defines an equivalence between the resulting inverse limit 2-extensions and we obtain the required commutative diagram of 2-extensions of $\mathbf{Z}_l[G(L/K)]$-modules, which is given in Section 4.6.

5 The Case of Ramification At Infinity

5.1

This section is devoted to the proof of Theorem 1.1 in the case of a Galois extension of number fields, L/K, which is ramified at infinity. The proof depends upon the commutative diagram which was constructed in Theorem 4.2. In the totally real case Theorem 1.1 is an immediate consequence of Theorem 4.2 (see Corollary 4.3) although an alternative proof in that case may be derived from the ramified at infinity case by Galois descent, as explained below.

An alternative proof of the results of this section, using modular Hecke algebras, may be found in [23].

Suppose that $K \subset L \subset E$ is a chain of number fields in which E/K and L/K are Galois. If

$$X \longrightarrow A \longrightarrow B \longrightarrow Y$$

is a 2-extension of finitely generated $\mathbf{Z}[G(E/K)]$-modules in which A and B are cohomologically trivial, then we obtain a commutative diagram of $G(E/L)$-invariants and coinvariants of the following form

$$\begin{array}{ccccccc} & & A_{G(E/L)} & \longrightarrow & B_{G(E/L)} & \longrightarrow & Y_{G(E/L)} \\ & & \cong \big\downarrow N & & \cong \big\downarrow N & & \\ X^{G(E/L)} & \longrightarrow & A^{G(E/L)} & \longrightarrow & B^{G(E/L)} & & \end{array}$$

in which N denotes the norm. In the resulting 2-extension of $\mathbf{Z}[G(L/K)]$-modules

$$X^{G(E/L)} \longrightarrow A^{G(E/L)} \longrightarrow B^{G(E/L)} \longrightarrow Y_{G(E/L)}$$

both $A^{G(E/L)}$ and $B^{G(E/L)}$ are cohomologically trivial. Under the canonical inflation map

$$\mathrm{Inf}_{E/L}\colon \mathcal{CL}\big(\mathbf{Z}[G(E/K)]\big) \longrightarrow \mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big)$$

the Euler characteristic, $[A] - [B]$, maps to $[A^{G(E/L)}] - [B^{G(E/L)}]$.

Lemma 5.2 *(i) Applying the construction of Section 5.1 to $\Omega_1(E/K)$ and $\Omega_1(E/K, 3)$ yields the relations*

$$\mathrm{Inf}_{E/L}\big(\Omega_1(E/K)\big) = \Omega_1(L/K)$$

and

$$\mathrm{Inf}_{E/L}]\big(\Omega_1(E/K), 3\big) = \Omega_1(L/K, 3).$$

(ii) It is sufficient to prove Theorem 1.1 for any totally complex Galois extension, $E/\mathbf{Q}$.

Proof The first of the relations in part (i) is proved in [6] and the second is derived by an argument similar to those given in [20, Section 7.1.57], [22, Section 4.4].

For part (ii), embed L/K into a totally complex Galois extension, $E/\mathbf{Q}$, such that E/K is Galois. The natural map, $\mathrm{Res}\colon \mathcal{CL}\big(\mathbf{Z}[G(E/\mathbf{Q})]\big) \to \mathcal{CL}\big(\mathbf{Z}[G(E/K)]\big)$ is easily seen to satisfy $\mathrm{Res}\big(\Omega_1(E/\mathbf{Q})\big) = \Omega_1(E/K)$ and $\mathrm{Res}\big(\Omega_1(E/\mathbf{Q}, 3)\big) = \Omega_1(E/K, 3)$. Hence if the result holds for $E/\mathbf{Q}$ then it holds for E/K and therefore, by part (i), for L/K.

5.3 Completion of the Proof of Theorem 1.1

By Lemma 5.2, it suffices to prove that $\Omega_1(L/\mathbf{Q}) = \Omega_1(L/\mathbf{Q},3)$ when $L/\mathbf{Q}$ is a Galois extension of number fields in which L is totally complex. Therefore, for the remainder of this section, $L/\mathbf{Q}$ will be assumed to be of this type.

From the construction of $\Omega_1(L/\mathbf{Q},3)$ in Section 2 and Theorem 4.2 we obtain a commutative diagram of $\mathbf{Z}[G(L/\mathbf{Q})]$-modules of the following form.

$$\begin{array}{ccccccc}
K_3^{\mathrm{ind}}(\mathcal{O}_{L,S})' & \longrightarrow & A & \xrightarrow{a} & B & \longrightarrow & K_2'(\mathcal{O}_{L,S}) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle (\pi_2,0)} & & {\scriptstyle \tilde{\pi}_3}\downarrow \\
K_3^{\mathrm{ind}}(\mathcal{O}_{L,S}) & \longrightarrow & \bigoplus_{w\in S_\infty(L)} \tilde{K}_3^{\mathrm{ind}}(L_w) & \xrightarrow{\tilde{\delta}} & C\oplus T_2 & \longrightarrow & \tilde{K}_2'(\mathcal{O}_{L,S})
\end{array}$$

Here the left-hand vertical map is surjective with kernel E_+ and the right-hand vertical map is injective with cokernel E_-. From Corollary 4.9

$$E_+ \cong \bigoplus_{w\in S_\infty(L)} \mathbf{Z} \cong \mathrm{Ind}_{\langle c\rangle}^{G(L/\mathbf{Q})}(\mathbf{Z})$$

in which complex conjugation, c, acts trivially on $\mathbf{Z}$. From Section 2

$$E_- = T_3 \cong \mathrm{Ind}_{\langle c\rangle}^{G(L/\mathbf{Q})}(\mathbf{Z}_-)$$

where $\mathbf{Z}_-$ is $\mathbf{Z}$ acted upon by $c(m) = -m$.

Lemma 5.4 *(i) There is a 3-extension of $\mathbf{Z}[G(L/\mathbf{Q})]$-modules of the form*

$$E_+ \longrightarrow A \xrightarrow{(\pi_1,a)} \Big(\bigoplus_{w\in S_\infty(L)} \tilde{K}_3^{\mathrm{ind}}(L_w)\Big)\oplus B \xrightarrow{(\tilde{\delta},-(\pi_2,0))} C\oplus T_2 \longrightarrow E_-$$

in which the middle three groups are cohomologically trivial, representing

$$\phi \in \mathrm{Ext}^3_{\mathbf{Z}[G(L/\mathbf{Q})]}(E_-,E_+).$$

(ii) There is an Euler characteristic

$$\chi(\phi) \in \mathcal{CL}\big(\mathbf{Z}[G(L/\mathbf{Q})]\big)$$

depending only on ϕ.

(iii) In the notation of parts (i) and (ii)

$$\chi(\phi) = \Omega_1(L/\mathbf{Q}) - \Omega_1(L/\mathbf{Q},3) \in \mathcal{CL}\big(\mathbf{Z}[G(L/\mathbf{Q})]\big).$$

Proof Part (i) is a simple diagram chase and part (ii) is similar to the proof of the analogous fact for 2-extensions (*cf.* [20, Proposition 7.1.31]). For part (iii), construct a commutative diagram of 2-extensions of the form

$$\begin{array}{ccccccc} K_3^{\mathrm{ind}}(\mathcal{O}_{L,S})' & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & \tilde{K}_2'(\mathcal{O}_{L,S}) \\ \downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow & & {\scriptstyle \tilde{\pi}_3}\downarrow \\ K_3^{\mathrm{ind}}(\mathcal{O}_{L,S}) & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & K_2'(\mathcal{O}_{L,S}) \end{array}$$

in which A_1, B_1, A_2, B_2 are finitely generated, cohomologically trivial $\mathbf{Z}[G(L/\mathbf{Q})]$-modules together with a commutative diagram mapping this finitely generated diagram into the diagram of Section 5.3 so as to give equivalences of 2-extensions on the top and bottom layers of the three-dimensional commutative diagram. Then ϕ is clearly also represented by the 3-extension

$$E_+ \longrightarrow A_1 \longrightarrow A_2 \oplus B_1 \longrightarrow B_2 \longrightarrow E_-.$$

Therefore

$$\chi(\phi) = [A_1] - [A_2 \oplus B_1] + [B_2] = ([A_1] - [B_1]) - ([A_2] - [B_2]),$$

which completes the proof of the lemma.

5.5 The next step—identifying 3-extensions

By Lemma 5.4, we wish to show that $\chi(\phi) = 0$. We shall evaluate $\chi(\phi)$ by calculating the class of ϕ in $\mathrm{Ext}^3_{\mathbf{Z}[G(L/\mathbf{Q})]}(E_-, E_+)$ and then giving another realisation of this 3-extension whose Euler characteristic is evidently zero. In order to do this we need a method by which to determine when two 3-extensions are equal.

Let $x_1, \dots, x_r \in \Omega_{\mathbf{Q}}$ be a set of double coset representatives for $\Omega_L \setminus \Omega_{\mathbf{Q}}/\langle c\rangle$ and also denote by $x_i \in G(L/\mathbf{Q}) \cong \Omega_{\mathbf{Q}}/\Omega_L$ the image of x_i. Hence we have $S_\infty(L) = \{(w_\infty)x_i \mid 1 \le i \le r\}$ for some fixed Archimedean embedding, as in Section 2.1. Let $\tau_z = zcz^{-1}$ so that τ_{x_i} generates the decomposition group corresponding to $(w_\infty)x_i$. The group, $E_+ \cong \mathrm{Ind}^{G(L/\mathbf{Q})}_{\langle c\rangle}(\mathbf{Z})$, has a canonical $\mathbf{Z}$-basis consisting of elements of the form $x_i \bigotimes_{\langle c\rangle} 1$ and similarly for the group, $E_- \cong \mathrm{Ind}^{G(L/\mathbf{Q})}_{\langle c\rangle}(\mathbf{Z}_-)$. If z is one of the x_i's then

$$H^i(\langle \tau_z\rangle\, ;\, E_-) \cong \begin{cases} \bigoplus_{\tau_{x_i} = \tau_z \in G(L/\mathbf{Q})} \mathbf{Z}/2\langle x_i \bigotimes_{\langle c\rangle} 1\rangle & \text{for } i \text{ odd,} \\ 0 & \text{for } i \text{ even.} \end{cases}$$

Similarly

$$H^i(\langle \tau_z]\rangle\, ;\, E_+) \cong \begin{cases} \bigoplus_{\tau_{x_i} = \tau_z \in G(L/\mathbf{Q})} \mathbf{Z}/2\langle x_i \bigotimes_{\langle c\rangle} 1\rangle & \text{for } i \text{ even,} \\ 0 & \text{for } i \text{ odd.} \end{cases}$$

There are canonical isomorphisms

$$\begin{aligned}
&\operatorname{Hom}_{\mathbf{Z}[G(L/\mathbf{Q})]}\big(\operatorname{Ind}_{\langle c\rangle}^{G(L/\mathbf{Q})}(M), \operatorname{Ind}_{\langle c\rangle}^{G(L/\mathbf{Q})}(\mathbf{Z})\big)\\
&\qquad\cong \operatorname{Hom}_{\mathbf{Z}[\langle c\rangle]}\big(M, \operatorname{Res}_{\langle c\rangle}^{G(L/\mathbf{Q})} \operatorname{Ind}_{\langle c\rangle}^{G(L/\mathbf{Q})}(\mathbf{Z})\big)\\
&\qquad= \bigoplus_{x_i\in\Omega_L\backslash\Omega_{\mathbf{Q}}/\langle c\rangle} \operatorname{Hom}_{\mathbf{Z}[\langle c\rangle]}\big(M, \operatorname{Ind}_{\langle c\rangle\cap\langle\tau_{x_i}\rangle}^{\langle c\rangle}(\mathbf{Z})\big).
\end{aligned}$$

Similarly there are canonical isomorphisms

$$\begin{aligned}
\operatorname{Ext}^3_{\mathbf{Z}[G(L/\mathbf{Q})]}(E_-, E_+) &\cong \bigoplus_{x_i\in\Omega_L\backslash\Omega_{\mathbf{Q}}/\langle c\rangle} \operatorname{Ext}^3_{\mathbf{Z}[\langle c\rangle]}\big(\mathbf{Z}_-, \operatorname{Ind}_{\langle c\rangle\cap\langle\tau_{x_i}\rangle}^{\langle c\rangle}(\mathbf{Z})\big)\\
&\cong \bigoplus_{x_i\in\Omega_L\backslash\Omega_{\mathbf{Q}}/\langle c\rangle} \operatorname{Ext}^4_{\mathbf{Z}[\langle c\rangle]}\big(\mathbf{Z}, \operatorname{Ind}_{\langle c\rangle\cap\langle\tau_{x_i}\rangle}^{\langle c\rangle}(\mathbf{Z})\big)\\
&\cong \bigoplus_{\tau_{x_i}=c\in G(L/\mathbf{Q})} H^4(\langle c\rangle\,;\mathbf{Z})\\
&\cong \bigoplus_{\tau_{x_i}=c\in G(L/\mathbf{Q})} \mathbf{Z}/2\\
&\cong \operatorname{Ext}^3_{\mathbf{Z}_2[G(L/\mathbf{Q})]}(E_-\otimes\mathbf{Z}_2, E_+\otimes\mathbf{Z}_2).
\end{aligned}$$

We shall now describe how to determine the equivalence class of a 3-extension of finitely generated $\mathbf{Z}[G(L/\mathbf{Q})]$-modules of the form

$$\epsilon\colon E_+ \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow E_-.$$

For x_i such that $\tau_{x_i} = c \in G(L/\mathbf{Q})$ the homomorphism of $\mathbf{Z}[\langle c\rangle]$-modules, $\tilde\lambda_i\colon \mathbf{Z}_- \to E_-$, sending 1 to $x_i \bigotimes_{\langle c\rangle} 1$, induces a $\mathbf{Z}[G(L/\mathbf{Q})]$-module endomorphism, λ_i, of E_-. On the other hand $\tilde\lambda_i$ determines a cohomology class, $[x_i \bigotimes_{\langle c\rangle} 1] \in H^1(\langle c\rangle\,; E_-)$, given by the image under $(\tilde\lambda_i)_*$ of the generator of $H^1(\langle c\rangle\,;\mathbf{Z}_-)\cong\mathbf{Z}/2$. Similarly, replacing $\mathbf{Z}_-, E_-$ by $\mathbf{Z}, E_+$, there is a canonical generator, $[x_i \bigotimes_{\langle c\rangle} 1] \in H^4(\langle c\rangle\,; E_+)$ corresponding to x_i such that $\tau_{x_i} = c \in G(L/\mathbf{Q})$.

The endomorphism, λ_i of E_-, induces a chain map of the standard 3-extension into ϵ and the resulting endomorphism of the left-hand module, $\psi_i\colon E_+ \to E_+$, corresponds to a cohomology class

$$[\psi_i] = [\epsilon] \cup \Big[x_i \bigotimes_{\langle c\rangle} 1\Big] \in H^4(\langle c\rangle\,; E_+) \cong \bigoplus_{\tau_{x_i}=\tau_z\in G(L/\mathbf{Q})} \mathbf{Z}/2\Big\langle \Big[x_i \bigotimes_{\langle c\rangle} 1\Big]\Big\rangle$$

given by the cup-product of $[x_i \bigotimes_{\langle c\rangle} 1]$ with the equivalence class of the 3-extension, ϵ.

Under the isomorphism

$$\operatorname{Ext}^3_{\mathbf{Z}[G(L/\mathbf{Q})]}(E_-, E_+) \cong H^4(\langle c\rangle E_+) \cong \bigoplus_{\tau_{x_i}=c\in G(L/\mathbf{Q})} \mathbf{Z}/2\langle [x_i \bigotimes_{\langle c\rangle} 1]\rangle$$

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the equivalence class of ϵ corresponds to the cup-product $[\epsilon] \cup [x_j \bigotimes_{\langle c\rangle} 1]$ where x_j is the representative of the double coset of the identity.

If $E = \mathrm{Ind}_{\{1\}}^{G(L/\mathbf{Q})}(\mathbf{Z}) \cong \mathbf{Z}[G(L/\mathbf{Q})]$ we have a standard 3-extension of the form

$$\epsilon_0 \colon E_+ \longrightarrow E \longrightarrow E \longrightarrow E \longrightarrow E_-$$

obtained by induction from the 3-extension of $\mathbf{Z}[\langle c\rangle]$-modules

$$\mathbf{Z} \longrightarrow \mathbf{Z}[\langle c\rangle] \xrightarrow{1-c} \mathbf{Z}[\langle c\rangle] \xrightarrow{1+c} \mathbf{Z}[\langle c\rangle] \longrightarrow \mathbf{Z}_- .$$

For this extension one shows easily that

$$[\epsilon_0] \cup \Big[x_i \bigotimes_{\langle c\rangle} 1\Big] = \Big[x_i \bigotimes_{\langle c\rangle} 1\Big] \in H^4(\langle c\rangle \,; E_+)$$

for all $1 \le i \le r$.

5.6 The final step

We shall conclude this proof by showing, using the results from the Appendix of Section 7, that

$$[\phi] \cup \Big[x_i \bigotimes_{\langle c\rangle} 1\Big] = \Big[x_i \bigotimes_{\langle c\rangle} 1\Big] \in H^4(\langle c\rangle \,; E_+)$$

for all $1 \le i \le r$ when $[\phi]$ is the class of the 3-extension of Lemma 5.4, constructed from the commutative diagram of Theorem 4.22. This will imply $[\phi] = [\epsilon_0]$ and therefore that the associated Euler characteristic, $\chi(\phi) \in \mathcal{CL}\big(\mathbf{Z}[G(L/\mathbf{Q})]\big)$, is trivial since

$$\begin{aligned} \chi(\phi) &= \chi(\epsilon_0) \\ &= [E] - [E] + [E] - \mathrm{rank}([E] - [E] + [E]) \cdot \mathbf{Z}[G(L/\mathbf{Q})] \\ &= 0. \end{aligned}$$

For this ϕ we know that the cup-product must be an isomorphism

$$([\phi] \cup -) \colon H^1(H \,; E_-) \xrightarrow{\cong} H^4(H \,; E_+)$$

for all $H \subseteq G(L/\mathbf{Q})$, since the middle three modules are cohomologically trivial and the cup-product may be identified by the composition of the three coboundary maps (all isomorphisms) resulting from chopping the 3-extension into three short exact sequences. In particular, we have an isomorphism

$$([\phi] \cup -) \colon H^1\big(G(L/\mathbf{Q}) \,; E_-\big) \cong \mathbf{Z}/2 \xrightarrow{\cong} H^4\big(G(L/\mathbf{Q}) \,; E_+\big) \cong \mathbf{Z}/2.$$

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However, under restriction to the subgroup $\langle c\rangle$, the generator of $H^1\big(G(L/\mathbf{Q})\ ;\ E_-\big)$ maps to

$$\sum_{\tau_{x_i}=c\in G(L/\mathbf{Q})}\Big[x_i\bigotimes_{\langle c\rangle}1\Big]\in H^1(\langle c\rangle\ ;\ E_-).$$

A similar remark holds for E_+ in dimension four so that, since cup-product commutes with restriction,

$$[\phi]\cup\Big(\sum_{\tau_{x_i}=c\in G(L/\mathbf{Q})}\Big[x_i\bigotimes_{\langle c\rangle}1\Big]\Big)=\sum_{\tau_{x_i}=c\in G(L/\mathbf{Q})}\Big[x_i\bigotimes_{\langle c\rangle}1\Big]\in H^4(\langle c\rangle\ ;\ E_+).$$

Therefore it suffices to show that $[\phi]\cup[x_i\bigotimes_{\langle c\rangle}1]=[x_i\bigotimes_{\langle c\rangle}1]$ for all the x_i whose double coset is different from that of the identity. Also, in order to compute these cup-products we may replace E_-, E_+ by $E_-\otimes\mathbf{Z}_2$ and $E_+\otimes\mathbf{Z}_2$, respectively. In this case, such an x_i is given by the element, y, of Section 7 when $z=1$ and $\tau_z=c$. Furthermore $[y\bigotimes_{\langle c\rangle}1]$ corresponds to the element of Section 7.11

$$\sum_{w\in\Omega_L}wy\bigotimes_{\langle c\rangle}1\in\Big(\mathrm{Ind}_{\langle c\rangle}^{\Omega\mathbf{Q}}\big(\mathbf{Z}_2(2)\big)\Big)^{\Omega_L}\cong E_+\otimes\mathbf{Z}_2.$$

In order to compute $[\phi]\cup[y\bigotimes_{\langle c\rangle}1]$ we must first lift $y\bigotimes_{\langle c\rangle}1$ to $\big(0,(1)_{(w_\infty)y}\big)\in(C\oplus T_2)\otimes\mathbf{Z}_2$, where the suffix $(w_\infty)y$ indicates the copy of $\mathbf{Z}_2$ corresponding to $(w_\infty)y$ in $T_2\otimes\mathbf{Z}_2$. Next we must find an element of

$$\Big(\Big(\bigoplus_{w\in S_\infty(L)}\tilde{K}_3^{\mathrm{ind}}(L_w)\Big)\oplus B\Big)\otimes\mathbf{Z}_2$$

which maps to

$$(1+c)\big(0,(1)_{(w_\infty)y}\big)=\big(0,(1+c)_{(w_\infty)y}\big)$$

under $\big(\tilde{\delta},-(\pi_2,0)\big)$. However $B\otimes\mathbf{Z}_2$ is given by the iterated pull-back of the B_i $(1\le i\le 4)$ of Section 4.6 (when $l=2$). Examining the construction of $B\otimes\mathbf{Z}_2$, as an iterated pullback, one finds that the family

$$\Big(\alpha(f_u),\lambda_1\big(\alpha(f_u)\big),(F_{u,m+1},H_{u,m}),H_{u,m},(G_{u,m},H_{u,m})\Big)$$

defines an element, $b_1\in\otimes\mathbf{Z}_2$, such that

$$(\pi_2,0)(b_1)=(\alpha_{(w_\infty)y},0)$$

in the notation of Section 2.15. However, from Section 2.15,

$$\tilde{\delta}(0,1)_{(w_\infty)y}=\big(\alpha_{(w_\infty)y},(1+c)_{(w_\infty)y}\big)$$

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and therefore

$$\big(\tilde{\delta}, -(\pi_2, 0)\big)(\alpha_{(w_\infty)y}, b_1) = \big(0, (1+c)_{(w_\infty)y}\big).$$

There is one technical point worthy of mention here. The family referred to above consists of elements which depend on an element, $u \in L^{\langle c\rangle}$, and lie in a chain complex of the form $R\Gamma_*\big(\mathrm{Spec}(L), -\big)$. However, we require elements lying in a subcomplex equivalent to $R\Gamma_*\big(\mathrm{Spec}(\mathcal{O}_{L,S}[1/2]), -\big)$. Since we are working with only a finite number ($r-1$, in fact) of elements, u, we may choose S large enough to ensure that this technical condition is met.

Next we must find an element, $a_1 \in A \otimes \mathbf{Z}_2$, which maps via (π_1, a) to

$$(1-c)(\alpha_{(w_\infty)y}, b_1) = \big((-1)_{(w_\infty)y} \tilde{\oplus}\, 0, (1-c)b_1\big).$$

This time $A \otimes \mathbf{Z}_2$ is the iterated pull-back of the $A(i)$'s and, by Sections 7.2, 7.6 and 7.9, such an element is given by the family

$$\Big((-1)_{(w_\infty)y}, \lambda_0\big((-1)_{(w_\infty)y}\big), (A_{u,m}, E_{u,m}), E_{u,m}, (X_{u,m}, E_{u,m})\Big).$$

Finally we must pull $(1+c)(a_1)$ back to E_+. However, by Sections 7.8, 7.10 and 7.11, this element is equal to the element defined by the family

$$\big(0, 0, (-d_*, d^*)(-W_{u,m}), d^*(-W_{u,m}), (1+c)X_{u,m}, d^*(-W_{u,m})\big)$$

which is equal to the element defined, in the notation of Section 7.11, by

$$\big(0, 0, (0, 0), 0, (1+c)X_{u,m} - \phi_*(W_{u,m}), 0\big).$$

However, by Section 7.11, this element lies in $E_+ \otimes \mathbf{Z}_2$ and is congruent modulo 2 to $\sum_{w\in\Omega_L} wy \bigotimes_{\langle c\rangle} 1$. This element represents $[y \bigotimes_{\langle c\rangle} 1] \in H^4(\langle c\rangle\,;\, E_+ \otimes \mathbf{Z}_2)$, which completes the proof.

6 Appendix: Another Construction of $\Omega_1(L/K, 3)$

In this section we shall, for completeness, give without proof a useful second method of constructing the invariant, $\Omega_1(L/K, 3)$ of Section 2, from the 2-extension of Corollary 2.10

$$K_3^{\mathrm{ind}}(\mathcal{O}_{L,S}) \longrightarrow \bigoplus_{w\in S_\infty(L)} K_3^{\mathrm{ind}}(L_w) \xrightarrow{\delta} C \longrightarrow K_2'(\mathcal{O}_{L,S}).$$

In the notation of Section 2, let $v \in S'_\infty(K)$ and $w = w(v)$. In the pull-back construction of $\Omega_1(L/K, 3)$, which we are about to describe, one first constructs a diagram of $\mathbf{Z}[G_w]$-modules of the form

$$\text{(1)} \qquad \begin{array}{ccc} \mathbf{Z}_{w,-} & \xrightarrow{\eta_w} & \mathbf{Z}[G_w] \\ \downarrow & & \downarrow \\ K_3^{\mathrm{ind}}(L_w) & \xrightarrow{\delta} & C \end{array}$$

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in the following manner. The left vertical homomorphism sends $1 \in \mathbf{Z}_{w,-}$ to $(-1)_w \in K_3(L_w)$ and is a G_w-cohomology isomorphism. The right vertical homomorphism sends $1 \in \mathbf{Z}[G_w]$ to $\alpha_w \in C$, the element appearing in the proof of Proposition 2.13 (following diagram (1)). The upper horizontal homomorphism is injective and sends $1 \in \mathbf{Z}_{w,-}$ to $(1-\tau_w) \in \mathbf{Z}[G_w]$. The equation, $(1-\tau_w)(\alpha_w) = \delta((-1)_w \in C$, ensures that diagram (1) commutes.

Define $V_{1,w(v)} = \mathbf{Z}_{w(v),-}$ and $V_{2,w(v)} = \mathbf{Z}[G_{w(v)}]$ if $v \in S'_\infty(K)$. Set $V_{1,w(v)} = V_{2,w(v)} = 0$ otherwise. Define $V_i = \bigoplus_{v \in S_\infty(K)} \operatorname{Ind}_{G_{w(v)}}^{G(L/K)} V_{i,w(v)}$ for $i = 1, 2$. Then, via induction from $G_{w(v)}$ to $G(L/K)$, the homomorphisms in (1) give rise to a commutative diagram of $\mathbf{Z}[G(L/K)]$-modules in which the bottom row is a 2-extension.

$$
\begin{array}{ccccccc}
 & & V_1 & \longrightarrow & V_2 & & \\
 & & \downarrow & & \downarrow & & \\
K_3^{\mathrm{ind}}(L) & \longrightarrow & \bigoplus_w K_3^{\mathrm{ind}}(L_w) & \xrightarrow{\delta} & C & \longrightarrow & K_2'(O_{L,S})
\end{array} \tag{2}
$$

Since $K_2'(O_{L,S})$ and $K_3^{\mathrm{ind}}(L)$ are finitely generated, we can successively choose finitely generated, free $\mathbf{Z}[G(L/K)]$-modules, F_0, F_1, F_2 and construct an enlargement of the diagram (2) of the following form, in which the bottom row is a 2-extension and the top row is exact.

$$
\begin{array}{ccccccc}
F_2 & \longrightarrow & F_1 \oplus V_1 & \longrightarrow & F_0 \oplus V_2 & \longrightarrow & K_2'(O_{L,S}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow 1 \\
K_3^{\mathrm{ind}}(L) & \longrightarrow & \bigoplus_w K_3^{\mathrm{ind}}(L_w) & \xrightarrow{\delta} & C & \longrightarrow & K_2'(O_{L,S})
\end{array} \tag{3}
$$

The rows of this diagram are exact and we may require the homomorphism, $F_2 \to K_3^{\mathrm{ind}}(L)$, to be surjective.

Let X_2 be the image of F_2 in $F_1 \oplus V_1$. Then $F_2 \mapsto K_3^{\mathrm{ind}}(E)$ induces an exact sequence

$$0 \longrightarrow P \longrightarrow X_2 \longrightarrow K_3^{\mathrm{ind}}(L) \longrightarrow 0.$$

Theorem 6.1 *The $\mathbf{Z}[G(L/K)]$-module, P, is finitely generated and projective.*
In $\mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big) \subset K_0\big(\mathbf{Z}[G(L/K)]\big)$

$$\Omega_1(E/F, 3) = \operatorname{rank}(P) \cdot \big[\mathbf{Z}[G(L/K)]\big] - [P].$$

7 Appendix: Some Elements

7.1

In this appendix we shall construct the elements which were used in Section 5 to prove Theorem 1.1 in the cases of extensions which are ramified at infinity. This appendix is provided for the reader's convenience, however, since they are straightforward but laborious, the proofs will be omitted. The notation follows that of Section 5. Namely, $L/\mathbf{Q}$ is a Galois extension of number fields with L totally complex. The absolute Galois group of L is

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denoted by Ω_L so that $G(L/\mathbf{Q}) \cong \Omega_{\mathbf{Q}}/\Omega_L$. Let $c \in \Omega_{\mathbf{Q}}$ denote complex conjugation and fix $z \in \Omega_{\mathbf{Q}} - \Omega_L$ so that $\tau_z = zcz^{-1}$ is an involution whose image generates the decomposition group, $\langle \tau_z \rangle = G_{(w_\infty)z} \subseteq G(L/\mathbf{Q})$ of Section 2. Let $x_1, \dots, x_r \in \Omega_{\mathbf{Q}}$ denote a set, containing z, of double coset representatives of $\Omega_L \setminus \Omega_{\mathbf{Q}}/\langle c \rangle$ so that $\{x_i, x_i c \mid 1 \leq i \leq r\}$ is a set of coset representatives for $\Omega_{\mathbf{Q}}/\Omega_L$. In addition, we may choose the x_i's so that they either occur in pairs of the form $\{x_i, \tau_z x_i = x_j \mid i \neq j\}$ or $\tau_z x_i = a_i x_i c$ for some $a_i = \tau_z \tau_{x_i} \in \Omega_L$. The latter set corresponds to double cosets, $\Omega_L x_i \langle c \rangle$, which are fixed by the left action of τ_z, given by $\tau_z(\Omega_L w \langle c \rangle) = \Omega_L \tau_z w \langle c \rangle$. Finally, if $a \in \Omega_{\mathbf{Q}}$ define $g(a) \in \Omega_L$ by the equation $a = g(a) x_i c^e$ for some i, e so that, if $w \in \Omega_L$ and $a \in \Omega_{\mathbf{Q}}$, then $g(wa) = wg(a)$.

Next we choose any double coset representative, say $y = x_1$, which is different from z and such that τ_z fixes $\Omega_L y \langle c \rangle$. Associated to y we choose $u \in L^{\tau_z}$. Since the images of τ_z and τ_{x_i} in $G(L/\mathbf{Q})$ are equal for any x_i corresponding to a double coset fixed by τ_z, the element u will be real at any of the corresponding place, $(w_\infty)x_i$. We choose u to be negative at the place $(w_\infty)y$ and positive at all the other places $(w_\infty)x_i$ fixed by τ_z. For $1 \leq i$ we choose a sequence $\{u_i \in \mathbf{Q}^{\mathrm{sep}}\}$ such that $u_i^2 = u_{i-1}$ and $u_1^2 = u$. If $\Omega_L x_i \langle c \rangle$ is fixed by τ_z then $\tau_{x_i}(u_1) = -u_1$ for $x_i = x_1 = y$ and $\tau_{x_i}(u_1) = u_1$ otherwise.

Associated to $u \in L^{\tau_z}$, we are going to construct the following families of inter-related elements for each integer, $m \geq 1$:

$$\alpha(f_u) \in C = \big(K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})_+\big)^{\Omega_L} = \Big(\mathrm{Ind}_{\langle c \rangle}^{\Omega_{\mathbf{Q}}}\big(K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big)/im(\phi)\Big)^{\Omega_L},$$

$$F_{u,m+1} \in \mathrm{Hom}_{\Omega_L}\big(B_1\Omega_{\mathbf{Q}}, K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big),$$

$$H_{u,m} \in \mathrm{Hom}_{\Omega_L}\big(B_2\Omega_{\mathbf{Q}}, \mathbf{Z}/2^m(2)\big),$$

$$G_{u,m} \in \mathrm{Hom}_{\Omega_L}\Big(B_1\Omega_{\mathbf{Q}}, \mathrm{Ind}_{\langle c \rangle}^{\Omega_{\mathbf{Q}}}\big(\mathbf{Z}/2^m(2)\big)\Big),$$

$$i_n \in \mathrm{Hom}_{\Omega_L}(B_n\Omega_L, B_n\Omega_{\mathbf{Q}}),$$

$$f_n \in \mathrm{Hom}_{\Omega_L}(B_n\Omega_{\mathbf{Q}}, B_n\Omega_L) \quad (n = 0, 1),$$

$$s_n \in \mathrm{Hom}_{\Omega_L}(B_n\Omega_{\mathbf{Q}}, B_{n+1}\Omega_{\mathbf{Q}}) \quad (n = 0, 1),$$

$$A_{u,m} \in \mathrm{Hom}_{\Omega_L}\big(B_0\Omega_{\mathbf{Q}}, K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big),$$

$$E_{u,m} \in \mathrm{Hom}_{\Omega_L}\big(B_1\Omega_{\mathbf{Q}}, \mathbf{Z}/2^m(2)\big),$$

$$W_{u,m} \in \mathrm{Hom}_{\Omega_L}\big(B_0\Omega_{\mathbf{Q}}, \mathbf{Z}/2^m(2)\big),$$

$$X_{u,m} \in \mathrm{Hom}_{\Omega_L}\Big(B_0\Omega_{\mathbf{Q}}, \mathrm{Ind}_{\langle c \rangle}^{\Omega_{\mathbf{Q}}}\big(\mathbf{Z}/2^m(2)\big)\Big).$$

Each of these groups should be interpreted as the direct limit, over E, of the corresponding groups in which Ω_L, $\Omega_{\mathbf{Q}}$ and $\mathbf{Q}^{\mathrm{sep}}$ are replaced by $G(E/L)$, $G(E/\mathbf{Q})$ and E, respectively, for some large Galois extension, $E/\mathbf{Q}$, containing L. However, to simplify the notation, already complicated enough, we shall persist in the use of Ω_L, $\Omega_{\mathbf{Q}}$ and $\mathbf{Q}^{\mathrm{sep}}$. In particular, an element of $\big(K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})_+\big)^{\Omega_L}$ will be represented by the coset (modulo $im(\phi)$) of a formal sum of the type $\sum_{w \in \Omega_L} \sum_{i=1}^{r} wx_i \bigotimes_{\langle c \rangle} v(w, i)$ where $v(w, i) \in K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})$.

Proposition 7.2 *Let* $[f] \in H^1\big(L\,; (\mathbf{Q}/\mathbf{Z})(2)\big)$ *be represented by a continuous* 1*-cocycle*

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$$f\colon \Omega_L \longrightarrow (\mathbf{Q}/\mathbf{Z})(2) \subset K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})$$

and define a formal sum

$$\alpha(f) = \sum_{w\in\Omega_L}\sum_{i=1}^{r} wx_i \bigotimes_{\langle c\rangle} x_i^{-1} f(w^{-1}) \in K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})_+.$$

Then

$$\alpha(f) \in \big(K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})_+\big)^{\Omega_L}.$$

If u, u_1 are as in 7, define a continuous 1-cocycle, f_u, by the formula

$$f_u(g) = \xi_2 \otimes \big(g(u_1)/u_1\big),$$

for all $g \in \Omega_L$. Hence f_u takes values in the subgroup of order two in $(\mathbf{Q}/\mathbf{Z})(2)$. Then

$$(1-\tau_z)\alpha(f_u) = \sum_{w\in\Omega_L} wy \bigotimes_{\langle c\rangle}(\xi_2\otimes\xi_2),$$

where $y = x_1$ is the double coset representative chosen in 7 and ξ_n denotes a primitive n-th root of unity.

Proposition 7.3 *For $a, b \in \Omega_{\mathbf{Q}}$ and $m \geq 0$, define*

$$F_{u,m+1}(a[b]) = g(a)\xi_{2^{m+1}} \otimes \frac{g(ab)(u_{m+1})}{g(a)(u_{m+1})} \in \mathbf{Z}/2^{m+1}(2)$$

which lies in $(\mathbf{Q}/\mathbf{Z})(2) \subset K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})$.

Then

(i) $F_{u,m+1} \in \operatorname{Hom}_{\Omega_L}\big(B_1\Omega_{\mathbf{Q}}, K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big)$,

(ii) $2F_{u,m+1} = F_{u,m}$,

and

(iii) if $\lambda_1\colon \big(K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})_+\big)^{\Omega_L} \to \operatorname{Hom}_{\Omega_L}\big(B_1\Omega_{\mathbf{Q}}, K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big)$ is the homomorphism of Section 3 then

$$\lambda_1\big(\alpha(f_u)\big) = F_1 = 2^m F_{u,m+1}.$$

Proposition 7.4 *For $a, b, c \in \Omega_{\mathbf{Q}}$, define*

$$H_{u,m}(a[b|c]) = \frac{g(ab)\xi_{2^{m+1}}}{g(a)\xi_{2^{m+1}}} \otimes \frac{g(abc)(u_{m+1})}{g(ab)(u_{m+1})}.$$

Then

(i) $H_{u,m} \in \operatorname{Hom}_{\Omega_L}\big(B_2\Omega_{\mathbf{Q}}, \mathbf{Z}/2^m(2)\big)$,

(ii) if $d\colon \mathbf{Z}/2^m(2) \to K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})$ is the canonical inclusion of Section 3

$$d^*F_{u,m+1} = d_*H_{u,m} \in \operatorname{Hom}_{\Omega_L}\big(B_2\Omega_{\mathbf{Q}}, K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big)$$

and

(iii) $d^*H_{u,m} = 0$.

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Proposition 7.5 *For $a, b \in \Omega_{\mathbf{Q}}$ define*

$$G_{u,m}(a[b]) = \sum_{w \in \Omega_L} \sum_{i=1}^{r} wx_i \bigotimes_{\langle c \rangle} x_i^{-1} H_{u,m}([w^{-1}a|b]) \in \mathrm{Ind}_{\langle c \rangle}^{\Omega_{\mathbf{Q}}}\big(\mathbf{Z}/2^m(2)\big).$$

Then
(i) $G_{u,m} \in \mathrm{Hom}_{\Omega_L}\Big(B_1\Omega_{\mathbf{Q}}, \mathrm{Ind}_{\langle c \rangle}^{\Omega_{\mathbf{Q}}}\big(\mathbf{Z}/2^m(2)\big)\Big)$,
(ii) $\phi_* H_{u,m} = d^* G_{u,m}$ *for each m,*
(iii) The element,

$$(G_{u,m}, H_{u,m}) \in \mathrm{Hom}_{\Omega_L}\Big(B_1\Omega_{\mathbf{Q}}, \mathrm{Ind}_{\langle c \rangle}^{\Omega_{\mathbf{Q}}}\big(\mathbf{Z}/2^m(2)\big)\Big) \oplus \mathrm{Hom}_{\Omega_L}\big(B_2\Omega_{\mathbf{Q}}, \mathbf{Z}/2^m(2)\big),$$

is in the kernel of the differential in $R\Gamma_*$.

Proposition 7.6 *For $a, b \in \Omega_{\mathbf{Q}}$ define*

$$E_{u,m}(a[b]) = g(ab)\xi_{2^{m+1}} \otimes \frac{g(\tau_z ab)(u_{m+1})}{g\big(\tau_z g(ab)\big)(u_{m+1})} + g(a)\xi_{2^{m+1}} \otimes \frac{g(ab)(u_{m+1}) g\big(\tau_z g(a)\big)(u_{m+1})}{g(a)(u_{m+1}) g(\tau_z ab)(u_{m+1})}.$$

and

$$A_{u,m}(a) = g(a)\xi_{2^{m+1}} \otimes \frac{g\big(\tau_z g(a)\big)(u_{m+1})}{g(\tau_z a)(u_{m+1})}.$$

Then
(i) $E_{u,m} \in \mathrm{Hom}_{\Omega_L}\big(B_1\Omega_{\mathbf{Q}}, \mathbf{Z}/2^m(2)\big)$,
(ii) $A_{u,m} \in \mathrm{Hom}_{\Omega_L}\big(B_0\Omega_{\mathbf{Q}}, K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})\big)$,
(iii) $d^* E_{u,m} = (1 - \tau_z) H_{u,m}$,
(iv) If $d \colon \mathbf{Z}/2^m(2) \to K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})$ is the canonical injection then

$$d^* A_{u,m} + d_* E_{u,m} = (1 - \tau_z) F_{u,m+1}.$$

Proposition 7.7 *The family of homomorphisms of Proposition 7.5*

$$G_{u,m} \in \mathrm{Hom}_{\Omega_L}\Big(B_1\Omega_{\mathbf{Q}}, \mathrm{Ind}_{\langle c \rangle}^{\Omega_{\mathbf{Q}}}\big(\mathbf{Z}/2^m(2)\big)\Big),$$

satisfies

$$(1 - \tau_z) G_{u,m}(a[b]) = \sum_{w \in \Omega_L} \sum_{i=1}^{r} wx_i \bigotimes_{\langle c \rangle} x_i^{-1} \left(\frac{\big(g(w^{-1}a)(\xi_{2^{m+1}})\big)}{\xi_{2^{m+1}}} \otimes \frac{g(w^{-1}ab)(u_{m+1}) g(w^{-1}\tau_z a)(u_{m+1})}{g(w^{-1}\tau_z ab)(u_{m+1}) g(w^{-1}a)(u_{m+1})} \right).$$

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Proposition 7.8 *For $a \in \Omega_{\mathbf{Q}}$ define*

$$W_{u,m}(a) = g(a)\xi_{2^{m+1}} \otimes \frac{g\big(\tau_z g(a)\big)(u_{m+1})g\big(\tau_z g(\tau_z a)\big)(u_{m+1})}{g(\tau_z a)(u_{m+1})g(a)(u_{m+1})}.$$

Then
(i) $W_{u,m} \in \mathrm{Hom}_{\Omega_L}\big(B_0\Omega_{\mathbf{Q}}, \mathbf{Z}/2^m(2)\big)$.
(ii) If $d\colon \mathbf{Z}/2^m(2) \to K_3^{\mathrm{ind}}(\mathbf{Q}^{\mathrm{sep}})$ *is the canonical injection then*

$$(1+\tau_z)A_{u,m} = d_* W_{u,m},$$

and
(iii) $(1+\tau_z)E_{u,m} = -d^* W_{u,m}$.

Proposition 7.9 *For $a \in \Omega_{\mathbf{Q}}$ define*

$$X_{u,m}(a) = \sum_{w\in\Omega_L}\sum_{i=1}^{r} wx_i \bigotimes_{\langle c\rangle} x_i^{-1}\Bigg(\xi_{2^{m+1}} \otimes \frac{g(\tau_z w^{-1}a)(u_{m+1})}{g(w^{-1}a)(u_{m+1})} - g(w^{-1}a)\xi_{2^{m+1}} \otimes \frac{g(w^{-1}\tau_z a)(u_{m+1})}{g\big(w^{-1}\tau_z g(a)\big)(u_{m+1})}\Bigg).$$

Then
(i) $X_{u,m} \in \mathrm{Hom}_{\Omega_L}\Big(B_0\Omega_{\mathbf{Q}}, \mathrm{Ind}_{\langle c\rangle}^{\Omega_{\mathbf{Q}}}\big(\mathbf{Z}/2^m(2)\big)\Big)$,
and
(ii) $(1-\tau_z)G_{u,m} - \phi_* E_{u,m} = d^* X_{u,m}$.

Corollary 7.10

$$(1+\tau_z)X_{u,m}(a) = \phi_* W_{u,m}(a) + \sum_{w\in\Omega_l}\sum_{\tau_z\Omega_L x_i\langle c\rangle = \Omega_L x_i\langle c\rangle} wx_i \bigotimes_{\langle c\rangle} x_1^{-1} \left(\xi_{2^{m+1}} \otimes \frac{g(\tau_z w^{-1}a)(u_{m+1})g(\tau_z\tau_{x_i} w^{-1}a)(u_{m+1})}{g(w^{-1}a)(u_{m+1})g(\tau_{x_1} w^{-1}a)(u_{m+1})}\right)$$

in $\mathrm{Ind}_{\langle c\rangle}^{\Omega_{\mathbf{Q}}}\big(\mathbf{Z}/2^m(2)\big)$.

Proposition 7.11 *Let $L/\mathbf{Q}$ denote the totally complex Galois extension of Section 7. Let $x_1, \dots, x_r \in \Omega_{\mathbf{Q}}$ be a set of double coset representatives of $\Omega_L \setminus \Omega_{\mathbf{Q}}/\langle c\rangle$ with $1 \neq y = x_1$ and let $y \neq z \in \{x_1, \dots, x_r\}$ be such that $\tau_z(\Omega_L y\langle c\rangle) = \Omega_L y\langle c\rangle$. Then, in the notation of Section 7, the reduction modulo 2 of*

$$(1+\tau_z)X_{u,m} - \phi_* W_{u,m} \in \mathrm{Hom}_{\Omega_L}\Big(B_0\Omega_{\mathbf{Q}}, \mathrm{Ind}_{\langle c\rangle}^{\Omega_{\mathbf{Q}}}\big(\mathbf{Z}/2^m(2)\big)\Big)$$

is given by

$$a \mapsto \sum_{w\in\Omega_L} wy \bigotimes_{\langle c\rangle} 1$$

for all $a \in \Omega_{\mathbf{Q}}$.

References

[1] D. Burns and M. Flach, *Motivic L-functions and Galois module structure invariants*. Math. Ann. **305**(1996), 65–102.
[2] ———, *On Galois structure invariants associated to Tate motives*. Amer. J. Math. **120**(1998), 1343–1397.
[3] S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives*. Grothendieck Festschrift, Progress in Math. (86) **1**, Birkhäuser, Boston, 1990, 333–400.
[4] T. Chinburg, *Exact sequences and Galois module structure*. Ann. of Math. **121**(1985), 351–376.
[5] T. Chinburg, M. Kolster, G. Pappas and V. P. Snaith, *Galois structure of K-groups of rings of integers*. C. R. Acad. Sci. Paris Séries I Math. **320**(1995), 1435–1440.
[6] ———, *Galois structure of K-groups of rings of integers*. K-Theory (4) **14**(1998), 319–369.
[7] ———, *Quaternionic Exercises in K-theory Galois Module Structure*. Proc. Great Lakes K-theory Conf., 1996. Fields Insitute Communications **16**, Amer. Math. Soc. Publ. 1997, 1–29.
[8] ———, *Quaternionic Exercises in K-theory Galois Module Structure II*. Proc. Algebraic K-theory Conf. I.C.T.P. Trieste, World Scientific Press, 1999, 337–369.
[9] D. R. Grayson, *On the K-theory of fields*. Proc. Conf. algebraic K-theory and algebraic number theory, Honolulu, 1987, Amer. Math. Soc. Contemp. Math. **83**(1989), 31–55.
[10] P. J. Hilton and U. Stammbach, *A Course in Homological Algebra*. Graduate Texts in Math. **4**, Springer-Verlag, 1971.
[11] B. Kahn, *Descente Galoisienne et K_2 des corps de nombres*. K-Theory (1) **7**(1993), 55–100.
[12] K. Kato, *Iwasawa theory and p-adic Hodge theory*. Kodai Math. J. (1) **16**(1993), 1–31.
[13] S. Lang, *Algebraic Number Theory*. Addison-Wesley 2nd. ed., 1984.
[14] M. Levine, *The indecomposable K_3 of a field*. Ann. Sci. École Norm. Sup. **22**(1989), 255–344.
[15] A. S. Merkurjev, *K_2 of fields and the Brauer group*. Applications of algebraic K-theory to geometry and number theory, Part II, Contemp. Math. **55**(1986), 529–546.
[16] J. S Milne, *Étale cohomology*. Princeton Mathematical Series **33**, Princeton Univ. Press, 1980.
[17] A. S. Merkurjev and A. A. Suslin, *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism*. Izv. Akad. Nauk. SSSR **46**(1982), 1011–1046; English trans. Math. USSR **21**(1983), 307–340.
[18] ———, *The K_3 group of a field*. Izv. Akad. Nauk. SSSR Ser. Mat. **54**(1990), 339–356; English trans. Math. USSR **36**(1990), 541–565.
[19] V. P. Snaith, *Galois Module Structure*. Fields Institute Monographs **2**, Amer. Math. Soc., 1995.
[20] ———, *Explicit Brauer Induction (with applications to algebra and number theory)*. Cambridge Studies in Advanced Math. **42**, Cambridge University Press, 1994.
[21] ———, *Topological Methods in Galois Representation Theory*. Canadian Math. Soc. Monographs, Wiley, 1989.
[22] ———, *Local fundamental classes derived from higher-dimensional K-groups I,II*. Proc. Great Lakes K-theory Conf., 1996, Fields Insitute Communications **16**, Amer. Math. Soc. Publ., 1997, 285–323 and 325–344.
[23] ———, *Hecke algebras and class-group Invariants*. Canad. J. Math. (6) **49**(1997), 1265–1280.
[24] C. Soulé, *K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale*. Invent. Math. **55**(1979), 251–295.
[25] ———, *Groupes de Chow et K-théorie de variétés sur un corps fini*. Math. Ann. **268**(1984), 317–345.

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