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LEVEL CROSSINGS OF A RANDOM TRIGONOMETRIC POLYNOMIAL WITH DEPENDENT COEFFICIENTS

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Abstract

This paper provides an asymptotic estimate for the expected number of K-level crossings of the random trigonometric polynomial $g_1 \cos x + g_2 \cos 2x + \ldots + g_n \cos nx$ where $g_j (j = 1, 2, \ldots, n)$ are dependent normally distributed random variables with mean zero and variance one. The two cases of ρ_{jr}, the correlation coeffiecient between the j-th and r-th coefficients, being either (i) constant, or (ii) $\rho^{|j-r|}$ $j \neq r, 0 < \rho < 1$, are considered. It is shown that the previous result for $\rho_{jr} = 0$ still remains valid for both cases.

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1. Introduction

Suppose that $g_1(\omega), g_2(\omega), \ldots, g_n(\omega)$ is a sequence of random variables defined on a probability space (Ω, A, P), each normally distributed with mathematical expectation zero and variance one, and that $N_{n,K}(a, b) \equiv N(a, b)$ is the number of real roots of the equation $T(\theta) = K$ where

$$T(\theta) \equiv T_n(\theta, \omega) = \sum_{j=1}^{n} g_j(\omega) \cos j\theta. \tag{1.1}$$

Dunnage [2] has shown that in the case of independent coefficients in the interval $0 \leq \theta \leq 2\pi$ all save a certain exceptional set of equations $T(\theta) = 0$ have $2n/\sqrt{3} + O\{n^{11/13}(\log n)^{3/13}\}$ roots, when n is large. The measure of his exceptional set does not exceed $(\log n)^{-1}$. For $K \neq 0$ such that $K = o(\sqrt{n})$ Farahmand [3] and [4] has shown that the mathematical expectation of $N(0, 2\pi)$, denoted by $EN(0, 2\pi)$, is

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asymptotic to $2n/\sqrt{3}$. Here we consider the effect of coefficients being dependent on $EN(a,b)$. We show that the above asymptotic formula persists whenever the correlation coefficient between any two coefficients g_j and g_r, denoted by ρ_{jr} is either (i) constant, or (ii) $\rho^{|j-r|} j \neq r$, $0 < \rho < 1$. For the case of $K = 0$ the same result has been obtained separately by Sambandham [8] and Renganathan and Sambandham [5]. We prove the following results in this paper.

THEOREM 1. *If the coefficients of $T(x)$ in* (1.1) *are normally distributed random variables with mean zero, variance one and ρ_{jr}, the correlation coefficients between j-th and r-th coefficients, are either* (i) *constant, or* (ii) $\rho^{|j-r|} j \neq r$, $0 < \rho < 1$, *then for all sufficiently large n and any constant K, the expected number of real roots of the equation $T(\theta) = K$ satisfies*

$$EN(0, 2\pi) = 2n/\sqrt{3} + O(n^{3/4}) \qquad \text{if } K = O(n^{3/8})$$

and

$$EN(0, 2\pi) = 2n/\sqrt{3} + o(n) \qquad \text{if } K = o(\sqrt{n}).$$

2. Preliminary Analysis

Let

$$A^2 = \sum_{j=1}^{n} \cos^2 j\theta, \qquad B^2 = \sum_{j=1}^{n} j^2 \sin^2 j\theta$$

and

$$C = \sum_{j=1}^{n} j \sin j\theta \cos j\theta.$$

By using the expected number of level crossings given by Cramér and Leadbetter [1, page 285] for the equation $T(\theta) - K$ we can obtain

$$EN(a,b) = \int_a^b \frac{\Delta^{1/2}}{\alpha} \phi\left(-\frac{K}{\alpha^{1/2}}\right) \left[\phi\left(\frac{K\gamma}{\alpha^{1/2}\Delta^{1/2}}\right) + K\gamma \left\{2\Phi\left(\frac{K\gamma}{\alpha^{1/2}\Delta^{1/2}}\right) - 1\right\}\right] d\theta \tag{2.1}$$

where

$$\alpha = \operatorname{var}\{T(\theta)\} = A^2 + \sum_{j \neq r}^{n} \sum^{n} \rho_{jr} \cos j\theta \cos r\theta, \tag{2.2}$$

$$\beta = \operatorname{var}\{T'(\theta)\} = B^2 + \sum_{j \neq r} \sum \rho_{jr} jr \cos j\theta \sin r\theta, \tag{2.3}$$

$$\gamma = \operatorname{Cov}\{T(\theta), T'(\theta)\} = -C - \sum_{j \neq r} \sum \rho_{jr} j \cos j\theta \sin r\theta, \tag{2.4}$$

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$$\Delta = \alpha\beta - \gamma^2, \qquad \Phi(t) = (2\pi)^{-1/2}\int_{-\infty}^{t} \exp(-y^2/2)\,dy \qquad \text{and}$$
$$\phi = \Phi'(t) = (2\pi)^{-1/2}\exp(-t^2/2).$$

From (2.1) and since $\Phi(t) = 1/2 + (\pi)^{-1/2}\mathrm{erf}(t/\sqrt{2})$ we have the extension of the Kac-Rice formula [5]

$$EN(a,b) = \int_a^b (\Delta^{1/2}/\pi\alpha)\exp(-K^2\beta/2\Delta)\,d\theta$$
$$+(\sqrt{2}/\pi)\int_a^b |K\gamma|\alpha^{-3/2}\exp(-K^2/2\alpha)\mathrm{erf}(|k\gamma|/\sqrt{2}\alpha^{1/2}\Delta^{1/2})\,d\theta$$
(2.5) $$= I_1(a,b) + I_2(a,b),$$

say.

Let $S(\theta) = \sin(2n+1)\theta/\sin\theta$ then the terms A^2, B^2 and C appearing in (2.5) can all be written as a function of $S(\theta)$ as follows. Since

(2.6) $$S(\theta) = 1 + 2\sum_{j=1}^{n}\cos 2j\theta,$$

we have

(2.7) $$A^2 = (1/2)\sum_{j=1}^{n}(1+\cos 2j\theta) = n/2 + (1/4)\{S(\theta) - 1\}.$$

Also since from (2.6)

$$S''(\theta) = -8\sum_{j=1}^{n} j^2\cos 2j\theta = 4\sum_{j=1}^{n} j^2(2\sin^2 j\theta - 1),$$

we have

(2.8) $$B^2 = n(n+1)(2n+1)/12 + (1/8)S''(\theta).$$

From (2.7) we also obtain

$$C = (-1/2)\frac{d}{d\theta}(A^2) = (-1/8)S'(\theta).$$

As they will be required later, we define

$$D = \sum_{j=1}^{n}\cos j\theta \qquad \text{and} \qquad E = \sum_{j=1}^{n} j\sin j\theta.$$

From (2.6) we can show that

(2.9) $$D = (1/2)\{S(\theta/2) - 1\}$$

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and

$$E = (-1/4)S'(\theta). \tag{2.10}$$

As $S(\theta)$ occurs frequently, we collect together some related inequalities. From (2.6) it is obvious that as $n \to \infty$,

$$S^{(\nu)}(\theta) = O(n^{\nu+1})$$

uniformly in θ. Stricter inequalities can be obtained by confining θ to the intervals $\epsilon \leq \theta \leq \pi - \epsilon$ and $\imath + \epsilon \leq \theta \leq 2\pi - \epsilon$, where ϵ is any positive constant smaller than π. Then, since $|S(\theta)| < 1/\sin\epsilon$, we can obtain

$$S(\theta) = O(1/\epsilon).$$

Also

$$S'(\theta) = (2n+1)\cos(2n+1)\theta/\sin\theta - \cot\theta S(\theta) = O(n/\epsilon)$$

and

$$\begin{aligned} S''(\theta) = -(2n+1)^2 S(\theta) - (2n+1)\cos\theta\cos(2n+1)\theta\sin^{-2}\theta \\ - \cot\theta S'(\theta) - \operatorname{cosec}^2\theta S(\theta) \\ = O(n^2/\epsilon). \end{aligned}$$

These together with (2.7) - (2.10) give

$$A^2 = n/2 + O(1/\epsilon), \tag{2.11}$$

$$B^2 = n^3/6 + O(n^2/\epsilon), \tag{2.12}$$

$$C = O(n/\epsilon), \tag{2.13}$$

$$D = O(1/\epsilon) \quad \text{and} \tag{2.14}$$

$$E = O(n/\epsilon). \tag{2.15}$$

3. Proof of the Theorem

We shall divide the roots of $T(\theta) - K = 0$ into two groups: (i) those lying in the intervals $(0, \epsilon)$, $(\pi - \epsilon, \pi + \epsilon)$ and $(2\pi - \epsilon, 2\pi)$, and (ii) those lying in the intervals $(\epsilon, \pi - \epsilon)$ and $(\pi + \epsilon, 2\pi - \epsilon)$. For the roots of type (i) which, it so happens, are negligible, we need some modification to apply Dunnage's [2] approach. Those roots which make a significant contribution to the final result are of type (ii) and their expected number is found by using the Kac-Rice formula (2.5). The choice of ϵ is important. It must not be too large, so that we can deal easily with type (i) roots; but if it is too small the approximation for type (ii) will become inadequate. We will see $\epsilon = n^{-1/4}$ is sufficient for both requirements.

First we consider the case of $\rho_{jr} \equiv \rho$ (constant). From (2.2), (2.11) and (2.14) We have

$$\alpha = A^2 + \rho \sum_{j\neq r}^{n} \sum^{n} \cos j\theta \cos r\theta = A^2 + \rho \left\{ \left(\sum_{j=1} \cos j\theta \right)^2 + \sum_{j=1}^{n} \cos^2 j\theta \right\}$$

$$= (1-\rho)A^2 + \rho D^2 = n(1-\rho)/2 + O(\epsilon^{-2}). \tag{3.1}$$

Similarly from (2.3), (2.4) and (2.12) we can obtain

$$\beta = n^3(1-\rho)/6 + O(n^2\epsilon^{-2}), \quad \text{and} \tag{3.2}$$

$$\gamma = O(n\epsilon{-}2). \tag{3.3}$$

Hence from (3.1)–(3.3) we have

$$\Delta = n^4(1-\rho)^2/12 + O(n^3\epsilon^{-2}). \tag{3.4}$$

So from (2.5) and (3.1)–(3.4) we can write

$$I_1(\epsilon, \pi - \epsilon) = (n\sqrt{3})\{1 + O(\epsilon)\} \exp\{-2K^2/n(1-\rho) + O(K^2/n^2\epsilon^2)\} \tag{3.5}$$

and

$$I_2(\epsilon, \pi - \epsilon) = O(K^3 n^{-3/2} \epsilon^{-2}). \tag{3.6}$$

Hence for $K = O(n^{3/8})$ from (2.5), (3.5) and (3.6) we have

$$EN(\epsilon, \pi - \epsilon) = n/\sqrt{3} + O(n^{3/4}), \tag{3.7}$$

and for $K = o(\sqrt{n})$,

$$EN(\epsilon, \pi - \epsilon) = n/\sqrt{3} + o(\sqrt{n}). \tag{3.8}$$

Now we turn to the intervals $(0, \epsilon)$, $(\pi - \epsilon, \pi + \epsilon)$ and $(2\pi - \epsilon, 2\pi)$, and we show that the equation has a negligible expected number of real roots in these intervals. By periodicity, the expected number of real roots in $(0, \epsilon)$ and $(2\pi - \epsilon, 2\pi)$ is the same as the expected number in $(-\epsilon, \epsilon)$. We shall therefore confine ourselves to this last interval; the interval $(\pi - \epsilon, \pi + \epsilon)$ can be treated in exactly the same way to give the same result. The idea, due to Dunnage [2], is to consider the random integral function $T(z, \omega) - K$ of the complex variable z. The number of real roots between $\pm\epsilon$ does not exceed the number in the circle $|z| < \epsilon$. Let $N(r) \equiv N(r, \omega, K)$ denote the number of real roots of $T(z, \omega) - K = 0$ in $|z| < r$. Assuming that $T(0) \neq K$ then by Jensen's theorem [9, page 125] or [7, page 332] we have

$$N(\epsilon)\log 2 \leq (2\pi)^{-1}\int_0^{2\pi} \log|\{T(2\epsilon e^{i\theta}, \omega) - K\}/\{T(0) - K\}\, d\theta. \tag{3.9}$$

Let $\Lambda^2 = n + \rho n(n-1)$; then, by standard probability theory, the distribution function of $T(O, \omega) = \sum_{j=1}^{n} g_j(\omega)$ is

$$G(x) = (2n\Lambda^2)^{-1/2}\int_{-\infty}^{x} \exp(-t^2/2\Lambda^2)\, dt,$$

from which, for any positive ν, we can see that $|T(0, \omega) - K| > e^{-\nu}$ except for sample functions in an ω-set of measure not exceeding

$$(2\pi\Lambda^2)^{-1/2}\int_{K-e^{-\nu}}^{K+e^{-\nu}} \exp(-t^2/2\Lambda^2)\, dt < 2(2\pi\Lambda^2)^{-1/2}e^{-\nu}. \tag{3.10}$$

Also since $|\cos(2n\epsilon e^{i\theta})| \leq 2e^{2n\epsilon}$ we have

$$|T(2\epsilon e^{i\theta})| \leq 2e^{2n\epsilon}(|g_1| + |g_2| + \ldots + |g_n|) \leq 2ne^{2n\epsilon}\max|g_j| \tag{3.11}$$

where the maximum is taken over $1 \leq j \leq n$. The distribution function of $|g_j|$ is

$$F(x) = \begin{cases} \sqrt{2/\pi}\int_0^x \exp(-t^2/2)\, dt & x \geq 0 \\ 0 & x < 0. \end{cases}$$

Now if $\max|g_j| > ne^\nu$ then $|g_j| > ne^\nu$ for at least one value of $j \leq n$, so that

$$\begin{aligned} \mathrm{Prob}(\max|g_j| > ne^\nu) &\leq \sum_{j=1}^{n}\mathrm{Prob}(|g_j| > ne^\nu) \leq n\,\mathrm{Prob}(|g_1| > ne^\nu) \\ &= n\sqrt{2/\pi}\int_{ne^\nu}^{\infty} \exp(-t^2/2)\, dt \sim \sqrt{2/\pi}\exp(-\nu - n^2e^{2\nu}/2) \end{aligned} \tag{3.12}$$

for all sufficiently large n. Therefore from (3.11) and (3.12) except for sample functions in an ω-set of measure not exceeding $(2/\pi)^{1/2}\exp(-\nu - n^2e^{2\nu}/2)$,

$$|T(2\epsilon e^{i\theta}) - K| < |n^2\exp(2n\epsilon + \nu) - K|. \tag{3.13}$$

Combining (3.10) with (3.13) and since for both $K = O(n^{3/8})$ and $K = o(\sqrt{n})$

$$|n^2\exp(2n\epsilon + \nu) - K| < 2n^2\exp(2n\epsilon + \nu),$$

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for all θ we get

$$|\{T(2\epsilon e^{i\theta}, \omega) - K\}/\{T(0, \omega) - K\}| < 2n^2 \exp(2n\epsilon + 2\nu) \tag{3.14}$$

except for sample functions in an ω-set of measure not exceeding

$$2(2\pi \Lambda^2)^{-1/2} e^{-\nu} + (2/\pi)^{1/2} \exp(-\nu - n^2 e^{2\nu}/2).$$

Therefore from (3.9) and (3.14) we can show that outside the exceptional set

$$N(\epsilon) \leq (\log 2 + 2 \log n + 2n\epsilon + 2\nu)/\log 2. \tag{3.15}$$

Since $\epsilon = n^{-1/4}$ from (3.15) and for all sufficiently large n

$$\begin{aligned} \text{Prob}\{N(\epsilon) > 3n\epsilon + 2\nu\} < {} & 2(2\pi \Lambda^2)^{-1/2} e^{-\nu} \\ & + (2/\pi)^{1/2} \exp(-\nu - n^2 e^{2\nu}/2). \end{aligned} \tag{3.16}$$

Let n' be the integer part of $3\sqrt{n}$. Then from (3.16) and for n sufficiently large we have

$$\begin{aligned} EN(\epsilon) &= \sum_{j>0} \text{Prob}\{N(\epsilon) \geq j\} \\ &= \sum_{1 \leq j \leq n'} \text{Prob}\{N(\epsilon) > j\} + \sum_{j \geq 1} \text{Prob}\{N(\epsilon) > n' + j\} \\ &\leq n' + 2(2\pi \Lambda^2)^{-1/2} \sum_{j \geq 1} e^{-j/2} + (2/\pi)^{1/2} \sum_{j \geq 1} \exp\{-j/2 - n^2 e^j/2\} \\ &= O(n^{3/4}). \end{aligned} \tag{3.17}$$

This gives an upper bound for the number of real roots of $T(\theta)$ in the interval $(-\epsilon, \epsilon)$, which together with (3.6) and (3.7) completes proof of the theorem for the case of $\rho_{jr} \equiv \rho$.

Now we consider the case of $\rho_{jr} = \rho^{|j-r|}$, $j \neq r$, $0 < \rho < 1$. To avoid repetition we only point out the adjustment necessary in the calculation of α, β, γ. For $\epsilon < 0 < \pi - \epsilon$ from (2.14) we have

$$\sum_{j \neq r}\sum \rho^{|j-r|} \cos j\theta \cos r\theta \leq D^2 = O(\epsilon^{-2})$$

which together with (2.2) and (2.11) gives

$$\alpha = A^2 + \sum_{j \neq r}\sum \rho^{|j-r|} \cos j\theta \cos r\theta = n/2 + O(\epsilon^{-2}). \tag{3.18}$$

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Similarly, since from (2.14) and (2.15)

$$\sum\sum_{j\neq r} jr\rho^{|j-r|}\sin j\theta\sin r\theta < E^2 = O(n^2\epsilon^{-2})$$

and

$$\sum\sum_{j\neq r} j\rho^{|j-r|}\sin\theta\sin r\theta < DE = O(n\epsilon^{-2}),$$

we obtain

$$\beta = n^3/6 + O(n^2\epsilon^{-2}) \tag{3.19}$$

and

$$\gamma = O(n^2\epsilon^{-2}). \tag{3.20}$$

Now (3.18)–(3.20) are sufficient for obtaining (3.7) and (3.8). For the intervals $(0, \epsilon)$, $(\pi-\epsilon, \pi+\epsilon)$ and $(2\pi-\epsilon, 2\pi)$ the same argument remains valid if we replace A^2 in (3.10) by

$$\Lambda'^2 = n + \sum\sum_{j\neq r}\rho^{|j-r|}.$$

Then since $\Lambda'^2 < \Lambda^2$ from (3.17) we can obtain $EN(\epsilon)$, which completes the proof of the theorem for the case $\rho_{jr} = \rho^{|j-r|}$.

References

[1] H. Cramér and M. R. Leadbetter, *Stationary and related stochastic processes* (Wiley, New York, 1967).

[2] J. E. A. Dunnage, 'The number of real zeros of a random trigonometric polynomial', *Proc. London Math. Soc.* **16** (1966), 53–84.

[3] K. Farahmand, 'On the average number of level crossings of a random polynomial', *Ann. Probab.* **18** (1990), 1403–1409.

[4] ———, 'Level crossings of a random trigonometric polynomial', *Proc. Amer. Math. Soc.* **111** (1991), 551–557.

[5] N. Renganathan and M. Sambandham, 'On the average number of real zeros of a tandom trigonometric polynomial with dependent coefficients', *Indian J. Pure Appl. Math.* **15** (1984), 951–956.

[6] S. O. Rice, 'Mathematical theory of random noise', *Bell System Tech.* **25** (1945), 46–156.

[7] W. Rudin, *Real and complex analysis*, 2nd edition (McGraw-Hill, New York, 1974).

[8] M. Sambandham, 'On random trigonometric polynomial', *Indian J. Pure Appl. Math.* **7** (1976), 841–849.

[9] E. C. Titchmarsh, *The theory of functions*, 2nd edition (Oxford University Press, Oxford, 1939).

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