A SYMMETRIC PROOF OF THE RIEMANN-ROCH THEOREM, AND A NEW FORM OF THE UNIT THEOREM

S. BEATTY AND N. D. LANE

Introduction. Let $F(z, u)$ denote

$$u^n + u^{n-1}F_1(z) + \ldots + F_n(z), \tag{1}$$

where $F_1(z), \ldots, F_n(z)$ are rational functions of z with complex coefficients. We shall speak of $F(z, u) = 0$ as the fundamental algebraic equation and shall adopt z as the independent variable and u as the dependent, except in § 4, where we use x and y instead of them, and where it is understood that x and y are connected birationally with z and u. We shall say that the fundamental equation is reducible in the field obtained by adjoining z to the totality of complex numbers, provided $F(z, u)$ is the product of two factors $H(z, u)$ and $K(z, u)$ of the same form as itself and beginning with the terms u^h and u^k of positive degrees. If no such factorization of $F(z, u)$ exists, we shall say that the fundamental equation is irreducible in the stated field. Nothing is lost by excluding the case where $F(z, u)$ has repeated factors.

The form of the Riemann-Roch Theorem [8, Chap. 19, § 5, I] in which z is adopted as the independent variable is

$$N(\tau) + \tfrac{1}{2}\sum\sum\tau\nu = N(\bar{\tau}) + \tfrac{1}{2}\sum\sum\bar{\tau}\nu. \tag{2}$$

In § 4 we shall show that the theorem has invariant character, in the sense that it continues to hold when any rational function of (z, u), say x, is used as a substitute for z in playing the role of independent variable, provided x is non-constant for each of the irreducible equations contained within the fundamental algebraic equation.

In the meantime, it is necessary to state what is meant by the various items appearing in (2). We shall speak of an order basis (τ) or a divisor A. A given individual order τ refers to a given u-branch at a given z-value. The z-value is the primary thing, since it alone is responsible for the division of the expansions of a rational function of (z, u) into cycles. The only point in mentioning u derives from the fact that its expansions are certainly all different, and so it is easy to recognize from them what the cycle distribution is. If the u-branch belongs to a u-cycle made up of ν branches altogether, then the order τ must refer to each of these u-branches and must be a multiple of $1/\nu$. The totality of all individual orders τ taken for all u-branches at all z-values is the order basis (τ), where it must be understood that only a finite number of the individual orders τ are different from zero. Our concern is with rational functions of (z, u) taking orders

Received October 12, 1950; in revised form October 31, 1951.

for all u-branches at all z-values equal to or greater than the corresponding orders in (τ). Such rational functions of (z, u) are said to be built on the order basis (τ). The number of linearly independent rational functions of (z, u) built on the order basis (τ) is denoted by $N(\tau)$, or dimension (A), which is positive except in the case where 0 is the only rational function of (z, u) built on the order basis (τ). The sum $\sum\sum \tau\nu$ of all the individual orders τ making up the order basis (τ) is, of course, an integer and is denoted by $-n(A)$. The primary element of the summation is τ, which refers to a u-branch at a z-value, while the secondary element is $\tau\nu$, which refers to a u-cycle at a z-value. These secondary elements $\tau\nu$ are then summed for all u-cycles at all z-values. Thus far, we have indicated what is meant by the (τ) side of (2), namely

(3) $$N(\tau) + \tfrac{1}{2}\sum\sum\tau\nu,$$

which we shall denote by $RR(\tau)$ and refer to as the Riemann-Roch expression for the order basis (τ). As far as the $(\bar{\tau})$ side of (2) is concerned, it is only necessary to state how $(\bar{\tau})$ is derived from (τ). Order bases (τ) and $(\bar{\tau})$ are said to be complementary, to the level of a rational function $S(z, u)$ not identically zero for any u-branch at any z-value, provided

(4) $$\tau + \bar{\tau} = \sigma - 1 + 1/\nu$$

for all u-branches at all finite z-values, while

(5) $$\tau + \bar{\tau} = \sigma + 1 + 1/\nu$$

for all u-branches at the infinite z-value, where (σ) is the order basis composed of the exact orders of $S(z, u)$ for all u-branches at all z-values. The value of the $(\bar{\tau})$ side of (2) is the same for all such rational functions $S(z, u)$, since both $N(\bar{\tau})$ and $\sum\sum \bar{\tau}\nu$ are the same. A first important special choice of $S(z, u)$ is the function $\partial F(z, u)/\partial u$. Since $F(z, u)$, has no repeated factors, the function $\partial F(z, u)/\partial u$ supplies an order basis, to be denoted by (μ). This is the choice of $S(z, u)$ that will be made in §§ 1, 2, 3. It owes its importance to its presence in the Lagrange Interpolation Formula for reducing a rational function $R(z, u)$, as given by

(6) $$\sum_i \left(R(z,u)\Big/\frac{\partial F(z,u)}{\partial u}\right)_{u=u_i} \frac{F(z, u)}{u-u_i}.$$

A second important special choice of $S(z, u)$ is a function which is a constant different from zero for all of the irreducible equations making up $F(z, u) = 0$. It supplies by its orders the order basis (0). This is the choice of $S(z, u)$ to be made use of in § 4. It identifies the order basis $(\bar{\tau})$ with the divisor W/A, where W is the divisor equivalent to the order basis made up of orders $-1 + 1/\nu$ for all u-branches at all finite z-values, and orders $1 + 1/\nu$ for all u-branches at the infinite z-value. A natural source of each of these individual orders is well known. Indeed, for an individual u-branch at an individual z-value the individual order presents itself as the least order we can permit a rational function of (z, u) to have there, if, on multiplying the function by dz and integrating the

https://doi.org/10.4153/CJM-1952-014-7 Published online by Cambridge University Press

resulting differential, we insist on the integral being finite there. We shall write (2) from here on in the form

$$RR(\tau) = RR(\bar{\tau}). \tag{7}$$

In [7], formula (7) was established directly for assigned complementary order bases (τ) and $(\bar{\tau})$. In [**2**] and [**4**], the issue turned on the result of depressing (τ) to (t) and correspondingly raising $(\bar{\tau})$ to $(\bar{t})$ until a stage was reached when $N(t)$ became known and $N(\bar{t})$ became zero. The present paper treats both complementary order bases (τ) and $(\bar{\tau})$ in the same general way. Indeed, we shall show in § 1 how to pass from complementary order bases (τ) and $(\bar{\tau})$ to complementary order bases (t) and $(\bar{t})$ in such a way that we can count on the equations

$$N(t) = N(\bar{t}) = 0 \text{ and } RR(t) - RR(\bar{t}) = RR(\tau) - RR(\bar{\tau})$$

holding. The resulting combination of complementary order bases (t) and $(\bar{t})$ is called the 0-0 case. In § 2, we shall obtain a lower bound for the value of $N(\tau)$ and make use of it in § 3, in combination with the results of § 1, to complete the proof of the Riemann-Roch Theorem and set up a new form of the Unit Theorem equivalent to it.

1. The 0-0 case. We have already indicated in the introduction what we aim to do in this section. Given complementary order bases (τ) and $(\bar{\tau})$, it is to find complementary order bases (t) and $(\bar{t})$ such that

$$N(t) = N(\bar{t}) = 0 \text{ and } RR(t) - RR(\bar{t}) = RR(\tau) - RR(\bar{\tau}).$$

There is nothing to do if $N(\tau) = N(\bar{\tau}) = 0$. We may suppose, therefore, that $N(\tau)$ and $N(\bar{\tau})$ are not both 0, and we shall show that a finite number of applications of a certain typical process gives us the complementary order bases (t) and $(\bar{t})$ that we are after. It will be enough to describe the first application of the process. Taking $N(\tau)$ to be positive, let us select a u-branch at some z-value so that the corresponding order τ of the order basis (τ) is taken by some of the rational functions of (z, u) built on the order basis (τ). Suppose the u-branch selected belongs to a u-cycle of ν branches. Let us replace each of these orders τ as they appear in the order basis (τ) by $\tau + 1/\nu$ to get a new order basis differing from the original order basis (τ) only in respect of the u-branch selected and the remaining u-branches of the same u-cycle, and then simply by being $1/\nu$ greater. The Riemann-Roch expression for the new order basis is, therefore, less than $RR(\tau)$ by $\frac{1}{2}$, seeing that its first term is less than $N(\tau)$ by 1, while its second term is greater than $\frac{1}{2}\sum\sum \tau\nu$ by $\frac{1}{2}$. We can be certain that the Riemann-Roch expression for the order basis complementary to the new order basis is less than $RR(\bar{\tau})$ by $\frac{1}{2}$, if we can satisfy ourselves that its first term is exactly $N(\bar{\tau})$, while its second term is $\frac{1}{2}$ less than $\frac{1}{2}\sum\sum \bar{\tau}\nu$. It is clear that the latter of these statements is true. We shall now show that the former one is also. There is no rational function of (z, u) built on the order basis complementary to the new order basis but not built on the order basis $(\bar{\tau})$. For if so, let it be multi-

https://doi.org/10.4153/CJM-1952-014-7 Published online by Cambridge University Press

plied by a rational function of (z, u) built on the order basis $\boldsymbol{(}\tau\boldsymbol{)}$ but taking the exact order τ for the u-branch selected at the z-value, and let the product be divided by $\partial F(z, u)/\partial u$. It is clear that this would give us a rational function of (z, u) whose order for the u-branch selected would be -1, or 1, according as the z-value was finite, or infinite. Indeed, this would be the case for all the other u-branches in the same u-cycle as well. In short, we should end up by having a rational function of (z, u) with only a single residue, which we know to be impossible. To sum up, we can say that complementary order bases $\boldsymbol{(}\tau\boldsymbol{)}$ and $\boldsymbol{(}\bar{\tau}\boldsymbol{)}$ were replaced by a new order basis and its complementary order basis, and that $N\boldsymbol{(}\tau\boldsymbol{)}$, $N\boldsymbol{(}\bar{\tau}\boldsymbol{)}$, and $RR\boldsymbol{(}\tau\boldsymbol{)} - RR\boldsymbol{(}\bar{\tau}\boldsymbol{)}$ were replaced by $N\boldsymbol{(}\tau\boldsymbol{)} -1$, $N\boldsymbol{(}\bar{\tau}\boldsymbol{)}$, and $RR\boldsymbol{(}\tau\boldsymbol{)} - RR\boldsymbol{(}\bar{\tau}\boldsymbol{)}$. Hence, after a finite number of applications of the typical process described above we arrive at complementary order bases $\boldsymbol{(}t\boldsymbol{)}$ and $\boldsymbol{(}\bar{t}\boldsymbol{)}$ with the properties:

$$N\boldsymbol{(}t\boldsymbol{)} = N\boldsymbol{(}\bar{t}\boldsymbol{)} = 0 \text{ and } RR\boldsymbol{(}t\boldsymbol{)} - RR\boldsymbol{(}\bar{t}\boldsymbol{)} = RR\boldsymbol{(}\tau\boldsymbol{)} - RR\boldsymbol{(}\bar{\tau}\boldsymbol{)},$$

as required in the 0-0 case.

2. A lower bound for $N\boldsymbol{(}\tau\boldsymbol{)}$. The result of the present section contrasts with that of § 1 as positive with negative, in the sense that it derives from adopting a degree of generality which the functions involved succeed in reaching, whereas in § 1 a degree of generality presented itself which was never attained by any of the functions involved. In particular, we aim to show that

(8) $$N\boldsymbol{(}\tau\boldsymbol{)} \geqslant n + \sum\sum \alpha.$$

The fundamental exponents $\boldsymbol{(}\alpha\boldsymbol{)}$ derive from the order basis $\boldsymbol{(}\tau\boldsymbol{)}$ and from the use of u as dependent variable. Indeed, the derivation takes place locally, that is for each z-value taken by itself. In other words, (α) derives from (τ), where the single brackets in each case refer to the individual z-value we wish to consider. In particular, let (τ) be the part of $\boldsymbol{(}\tau\boldsymbol{)}$ that refers to the individual z-value $z = a$. Moreover, let

(9) $$w_{n-1}(z,u), \ldots , w_0(z,u)$$

be a local function basis for all the rational functions of (z, u) built on (τ), or, in other words, let all these rational functions of (z, u) be just those of the form

$$P_{n-1}(z)w_{n-1}(z,u) + \ldots + P_0(z)w_0(z,u),$$

where $P_{n-1}(z), \ldots ,P_0(z)$ are rational functions of z regular at $z = a$. Indeed, the local function basis in (9) may be normalized so that the functions in order may be of degrees $n - 1, \ldots , 0$ in u, and, furthermore, so that the leading coefficient is just a power of $z - a$ in each case. This makes the functions in (9) start off with the terms

(10) $$\frac{u^{n-1}}{(z - a)^{\alpha_{n-1}}}, \ldots , \frac{1}{(z - a)^{\alpha_0}},$$

which puts in evidence the local fundamental exponents

(11) $$\alpha_{n-1}, \ldots ,\alpha_0,$$

which we denote collectively by (α). When the infinite z-value is chosen instead of $z = a$ as the individual z-value, the corresponding discussion applies, with, however, $1/z$ replacing $z - a$ as element.

It is natural to exhibit a local function basis (9) as a matrix. Each row of the matrix has as elements the coefficients of $u^{n-1}, \ldots, 1$ for any one of the functions in (9), while each column has as elements the coefficients of the functions in (9) for any one of $u^{n-1}, \ldots, 1$. By considering the matrices of equivalent local function bases corresponding to a local order basis (τ), it is proven in [4] that

$$\sum \tau\nu + \sum \alpha = \tfrac{1}{2} \sum (u - 1 + 1/\nu)\nu, \tag{12}$$

which, on being quoted for all z-values simultaneously and the results totalled up, enables us to write (8) in the form

$$N(\tau) \geqslant n - \sum\sum \tau\nu - \tfrac{1}{2} \sum\sum (\nu - 1). \tag{13}$$

For convenience of proof, however, (8) is to be preferred to (13).

We shall first show that the proof of (8) may be reduced to the case where u is without poles for all finite z-values. Given an order basis (τ), let (α) denote the fundamental exponents resulting from the use of u as dependent variable. When $u(z - a)$ is used as dependent variable instead of u, the fundamental exponents remain the same as before, except for the finite z-value $z = a$ and the infinite z-value. For $z = a$ they have to be increased by $n - 1, \ldots, 0$ over what they were originally, whereas for the infinite z-value they have to be decreased by these same amounts. In other words, the total $\sum\sum \alpha$ is not changed. Hence, the adoption of $u(z - a)$ as dependent variable instead of u produces no change in either side of (8). We can say, therefore, that if (8) is valid in either case, so is it in the other as well. It suffices, therefore, to prove (8) on the assumption that u is without poles for all finite z-values, since the preceding argument can be applied until a dependent variable is obtained whose only poles are at the infinite z-value.

Most of the simplification involved in dealing with the case where u is without poles for all finite z-values takes place at the local level and is due to special properties of the normalized function basis in (9) corresponding to the local order basis (τ) for the z-value $z = a$. These properties, which will be found fully discussed in [4], are first that $\alpha_{n-1}, \ldots, \alpha_0$ are monotone decreasing, and second that when $w_i(z, u)$ in (9) is normalized and written in the form

$$\frac{u^i + u^{i-1}H^i_{i-1}(z) + \ldots + H^i_0(z)}{(z - a)^{\alpha_i}}, \tag{14}$$

each $H^i_j(z)$ is regular at $z = a$ and may be taken to be a polynomial of degree less than $\alpha_i - \alpha_j$, in which case its form will be unique. In the sequel, we shall always make use of the unique polynomial form for $H^i_j(z)$.

Local function bases corresponding to local order bases at all finite z-values as they are involved in (τ) can be combined to form a function basis

$$W_{n-1}(z,u), \ldots, W_0(z,u) \tag{15}$$

for all rational functions of (z, u) built simultaneously on all the constituent local order bases (τ) of the order basis $\boldsymbol{(\tau)}$ at all finite z-values. All but a finite number of these local function bases will be $u^{n-1}, \ldots, 1$. The exceptions will be associated with individual finite z-values $z = a$, $z = b, \ldots$ and will have the form of the local function basis in (9), or will be patterned after it, with the element $z - a$ being replaced by $z - b, \ldots$. Naturally, this combined function basis (15) serves locally just as well as the local function basis $u^{n-1}, \ldots, 1$ wherever this applies, or as the local function basis in (9), or others patterned after it, wherever these apply. This is due to the fact that rational functions of (z, u) exist taking simultaneously the precise orders of $\boldsymbol{(\tau)}$ at all finite z-values, which is made possible through the circumstance that we have left complete freedom as to orders at the infinite z-value.

The coefficients of $u^{n-1}, \ldots, 1$ in $W_{n-1}(z, u)$ are all of fixed orders at the infinite z-value. In particular, the order of the coefficient of u^{n-1} at the infinite z-value is $\sum' \alpha_{n-1}$ exactly, where the prime implies that the summation ranges over all the finite z-values but does not extend to the infinite z-value. The same type of remark applies to each of the remaining functions in (15). The final one is that $W_0(z, u)$ has the exact order $\sum' \alpha_0$ at the infinite z-value, where the prime applies as already stated.

The rational function of (z, u),

$$P_{n-1}(z) W_{n-1}(z,u) + \ldots + P_0(z) W_0(z,u), \tag{16}$$

in which $P_{n-1}(z), \ldots, P_0(z)$ are arbitrary polynomials of suitably limited degrees, will serve as a sufficiently general rational function of (z, u) built simultaneously on all the constituent local order bases (τ) of the order basis $\boldsymbol{(\tau)}$ at all finite z-values. This general function (16) can, of course, be converted into the general rational function of (z, u) built on the order basis $\boldsymbol{(\tau)}$ by applying to it the conditions necessary to insure that it is also built on the constituent local order basis (τ) of the order basis $\boldsymbol{(\tau)}$ at the infinite z-value. We are safe in limiting in suitable fashion the degrees of the polynomials $P_{n-1}(z), \ldots, P_0(z)$, seeing that even the general rational function of (z, u) built on the constituent local order basis (τ) of the order basis $\boldsymbol{(\tau)}$ at the infinite z-value does not contain arbitrarily large powers of z. In other words, since the application of the conditions insuring that the function is built on the constituent local order basis (τ) of the order basis $\boldsymbol{(\tau)}$ at the infinite z-value will require the coefficients of all powers of z beyond a certain degree to vanish, we run no risk in taking them to be zero at the start. With this in mind, we shall choose

$$P_{n-1}(z), \ldots, P_0(z) \tag{17}$$

as arbitrary polynomials of degrees

$$\delta_{n-1} + \sum' \alpha_{n-1}, \ldots, \delta_0 + \sum' \alpha_0,$$

where the dash implies that each summation ranges over all finite z-values but does not extend to the infinite z-value. The number of arbitrary constants

https://doi.org/10.4153/CJM-1952-014-7 Published online by Cambridge University Press

appearing in (17), or, what is the same thing, in the parent function (16), is seen to be

$$\sum \delta + n + \sum\sum{}' a, \tag{18}$$

where the prime implies that the double summation ranges over all finite z-values but does not extend to the infinite z-value.

Up to the present, we have merely said that the numbers (δ) need not be chosen arbitrarily large. It is necessary, however, to indicate how we propose to limit them. This we shall attend to in two separate stages.

In the first place, we observe that

$$P_{n-1}(z)W_{n-1}(z,u), \ldots, P_0(z)W_0(z,u)$$

are of degrees $\delta_{n-1}, \ldots, \delta_0$ in z, and, indeed, that these degrees attach to the coefficients of

$$u^{n-1} \text{ in } P_{n-1}(z)W_{n-1}(z,u), \ldots, \ u^0 \text{ in } P_0(z)W_0(z,u)$$

respectively. We wish to be able to write (16) in the form

$$z^{\delta_{n-1}}Q_{n-1}(1/z)u^{n-1} + \ldots + z^{\delta_0}Q_0(1/z), \tag{19}$$

in which $Q_{n-1}(1/z), \ldots, Q_0(1/z)$ are all rational functions of z regular at the infinite z-value and all containing arbitrary constants as their initial terms when they are expanded in powers of the element $1/z$. It is clear that this is achieved by insuring that the degree in z in the coefficient of u^i in $P_i(z)W_i(z, u)$, namely, δ_i, exceeds the degree in z of the coefficient of u^i in

$$P_{n-1}(z)W_{n-1}(z,u) + \ldots + P_{i+1}(z)W_{i+1}(z,u),$$

and this for all the cases $i = n - 2, \ldots, 0$. These inequalities range, therefore, from the first of them, namely, that δ_{n-2} should exceed the degree in z of the coefficient of u^{n-2} in $P_{n-1}(z)W_{n-1}(z, u)$ to the last of them, namely, that δ_0 should exceed the degree in z of the coefficient of u^0 in

$$P_{n-1}(z)W_{n-1}(z,u) + \ldots + P_1(z)W_1(z,u).$$

Let now the local function basis equivalent to the constituent local order basis (τ) of the order basis $\boldsymbol{(\tau)}$ at the infinite z-value be denoted by

$$w^{\infty}_{n-1}(z,u), \ldots, w^{\infty}_0(z,u). \tag{20}$$

This local function basis follows the pattern of the local function basis in (9), with, however, $1/z$ replacing $z - a$ as element. Let the fundamental exponents associated with this local function basis and with the use of u as dependent variable be denoted by

$$a^{\infty}_{n-1}, \ldots, a^{\infty}_0.$$

The form of the general rational function of (z, u) built on the constituent local order basis (τ) of the order basis $\boldsymbol{(\tau)}$ at the infinite z-value is, therefore,

$$P^{\infty}_{n-1}(1/z)w^{\infty}_{n-1}(z,u) + \ldots + P^{\infty}_0(1/z)w^{\infty}_0(z,u), \tag{21}$$

https://doi.org/10.4153/CJM-1952-014-7 Published online by Cambridge University Press

in which

$$P^{\infty}_{n-1}(1/z), \ldots, P^{\infty}_{0}(1/z)$$

are all rational functions of z regular at the infinite z-value. We wish to be able to write (19), and hence (16) also, in the form (21) and must prepare for it by taking account of a second group of inequalities involving the numbers (δ). The identification of (19) with (21) proceeds naturally and simply if we take, in the first place,

$$\delta_i \geqslant \alpha^{\infty}_i \qquad (i = n-1, \ldots, 0)$$

and, in the second place, δ_i greater than the maximum degree in z of the coefficient of u^i in any of the functions

$$z^{\delta_{n-1}-\alpha^{\infty}_{n-1}} w^{\infty}_{n-1}(z,u), \ldots, z^{\delta_{i+1}-\alpha^{\infty}_{i+1}} w^{\infty}_{i+1}(z,u)$$

for $i = n-2, \ldots, 0$.

We are now in a position to take the final step in the proof of inequality (8). The number of linearly independent conditions involved in identifying (19) with (21) is at most

$$\sum \delta - \sum \alpha^{\infty}.$$

The proof is based on the fact that (19), even without conditions, can be written in the form of (21), except that where the rational functions

$$P^{\infty}_{n-1}(1/z), \ldots, P^{\infty}_{0}(1/z)$$

of (21) are without poles at the infinite z-value the corresponding rational functions of (19) can have poles of orders

$$\delta_{n-1} - \alpha^{\infty}_{n-1}, \ldots, \delta_0 - \alpha^{\infty}_0$$

at most. It is these poles that have to be made to disappear in the process of identifying (19) with (21). Hence, the number of linearly independent conditions required to effect this disappearance is not more than

$$\sum \delta - \sum \alpha^{\infty},$$

and when it is subtracted from the number in (18) of arbitrary constants involved in (19) at the outset we have left over not less than

$$n + \sum\sum \alpha$$

arbitrary constants. However, since we have exactly $N(\tau)$ arbitrary constants left over, we must conclude that

$$N(\tau) \geqslant n + \sum\sum \alpha,$$

which is inequality (8).

3. The Riemann-Roch Theorem and a new form of the Unit Theorem. Where (α) is the set of fundamental exponents associated with a local order basis (τ) at a given z-value and based on the use of u as dependent variable and where $(\bar{\alpha})$ is the set of fundamental exponents associated with the complementary local order basis $(\bar{\tau})$ at the given z-value and based on the use of u as dependent variable, it follows from (12) and the corresponding result for $(\bar{\tau})$ that

$$\sum \alpha + \sum \bar{\alpha} = 0, \tag{22}$$

where the equations relating complementary orders to one another are of the form

$$\tau + \bar{\tau} = \mu - 1 + 1/\nu.$$

If, however, the local order basis $(\bar{\tau})$ is exactly 2 more than enough in each of its orders to be complementary in the above sense to the local order basis (τ), the right side of (22) becomes $-2n$. But, this is precisely what does happen when (τ) and $(\bar{\tau})$ are local order bases taken from complementary order bases (τ) and $(\bar{\tau})$ at the infinite z-value, always supposing that $\partial F(z, u)/\partial u$ is the level function made use of to relate the orders of (τ) and $(\bar{\tau})$. Where (α) and $(\bar{\alpha})$ are the fundamental exponents associated with complementary order bases (τ) and $(\bar{\tau})$, we have, therefore,

$$\sum\sum \alpha + \sum\sum \bar{\alpha} = -2n. \tag{23}$$

Let us now apply inequality (8) to the 0-0 case, or, what is the same thing, let us make a joint application of §§ 1 and 2. Beginning with complementary order bases (τ) and $(\bar{\tau})$, let us make use of § 1 to obtain complementary order bases (t) and $(\bar{t})$ which it provides, and let us denote the fundamental exponents of (t) and $(\bar{t})$ by (α) and $(\bar{\alpha})$. Now applying inequality (8) of § 2 to (t) and $(\bar{t})$ separately, we find that

$$\begin{cases} 0 \geqslant n + \sum\sum \alpha, \\ 0 \geqslant n + \sum\sum \bar{\alpha}. \end{cases} \tag{24}$$

When the inequalities in (24) are added and the result compared with (23), we see that the equality sign applies in both cases in (24). In other words,

$$\sum\sum \alpha = -n = \sum\sum \bar{\alpha}. \tag{25}$$

But, we also have

$$\sum\sum \alpha + \sum\sum t\nu = \tfrac{1}{2} \sum\sum (\mu - 1 + 1/\nu)\nu = \sum\sum \bar{\alpha} + \sum\sum \bar{t}\nu, \tag{26}$$

as appears from quoting (12) for all the individual constituent local order bases of (t) and $(\bar{t})$ separately and adding up the results in each case. It follows, therefore, from (25) and (26), that

$$\sum\sum t\nu = \sum\sum \bar{t}\nu$$

and from this, in turn, that

$$RR(t) = RR(\bar{t}).$$

Hence, by § 1, we have that

$$RR(\tau) = RR(\bar{\tau}),$$

which is the statement of the Riemann-Roch Theorem for complementary order bases (τ) and $(\bar{\tau})$.

The new form of the Unit Theorem replaces the inequality (8) by an equation, namely by

$$N(\tau) = n + \sum\sum \alpha + N(\bar{\tau}), \tag{27}$$

where (α) denotes the fundamental exponents associated with (τ) and depending on the use of u as dependent variable. We see at once that

$$n + \sum\sum \alpha = \tfrac{1}{2} \sum\sum \bar{\tau}\nu - \tfrac{1}{2} \sum\sum \tau\nu, \tag{28}$$

since each side reduces to

$$\tfrac{1}{2} \sum\sum \alpha - \tfrac{1}{2} \sum\sum \bar{\alpha}.$$

The Unit Theorem says that the difference

$$N(\tau) - N(\bar{\tau})$$

is equal to the left side of (28), while the Riemann-Roch Theorem says that it equals the right side. The two theorems are, therefore, equivalent.

The original form of the Unit Theorem was that

$$N(\tau_-) - N(\tau) + N(\bar{\tau}) - N(\bar{\tau}_+) = 1. \tag{29}$$

Here (τ) and $(\bar{\tau})$ were complementary order bases, and (τ_-) and $(\bar{\tau}_+)$ were also. It was understood that (τ_-) was obtained from (τ) by depressing a single one of its individual orders by the minimum amount $1/\nu$, while $(\bar{\tau}_+)$ was obtained from $(\bar{\tau})$ by raising the corresponding one of its individual orders by $1/\nu$. It follows from the new form of the Unit Theorem that for a decrease of 1 in any cycle order in (τ) there is either an increase of 1 in $N(\tau)$ but no change in $N(\bar{\tau})$ or no change in $N(\tau)$ but a decrease of 1 in $N(\bar{\tau})$, since (12) shows that $\sum\sum \alpha$ increases by 1. In other words, the new form of the Unit Theorem implies the original form. But conversely, the original form implies the new form, since it certainly implies the Riemann-Roch Theorem, which is equivalent to the new form.

The new form of the Unit Theorem is significant, in that it associates quantities (α) determined by considering all z-values one at a time with quantities $N(\tau)$ and $N(\bar{\tau})$ determined by considering all z-values simultaneously. For that matter, the same remark applies to the Riemann-Roch Theorem itself, where it is (τ) instead of (α) which is determined by dealing with all z-values one at a time.

It is clear from (23) that the statement in (27) of the new form of the Unit Theorem simply repeats itself when the roles of (τ) and $(\bar{\tau})$ are interchanged.

4. The invariant character of the Riemann-Roch Theorem. If the fundamental algebraic equation $F(z, u) = 0$ is irreducible, and if x is a non-constant rational function of (z, u), there is a well-known routine for setting up a rational function of (z, u), say y, so that z and u are both expressible as rational functions of (x, y), and, moreover, the algebraic equation obtained by eliminating z and u, say $G(x, y) = 0$, is irreducible. If, however, $F(z, u) = 0$ is reducible, and if x is a non-constant for each of the ρ irreducible equations making up $F(z, u) = 0$, it is clear that y can still be found so that the new pair (x, y) are birationally equivalent to the original pair (z, u), and, moreover, the algebraic equation obtained by eliminating z and u, say $G(x, y) = 0$, breaks up into ρ irreducible equations. To see this it is only necessary to make use of

$$\sum_j \left(R(z,u) \Big/ \frac{F(z,u)}{f_j(z,u)}\right)_{f_j(z,u) = 0} \frac{F(z,u)}{f_j(z,u)} \tag{30}$$

as the reduced form of a given rational function $R(z, u)$, where $F(z, u)$ is the product of ρ irreducible factors

$$f_1(z,u), \ldots, f_\rho(z,u),$$

all different from one another. Here the first factor of the typical summand in (30) denotes the polynomial in u, with coefficients rational functions of z, obtained on reducing

$$R(z,u) \Big/ \frac{F(z,u)}{f_j(z,u)}$$

with respect to the irreducible equation $f_j(z, u) = 0$. That is, (30) is composed of ρ summands, formed as j ranges over $1, \ldots, \rho$. Each summand is identically 0 for $\rho - 1$ of the irreducible equations making up $F(z, u) = 0$ but ordinarily is not identically 0 for the particular irreducible equation involved in the reduction of its first factor.

Before we can say that the Riemann-Roch Theorem is invariant, [6, § 25] we have to see that it applies to $G(x, y) = 0$ as much as to $F(z, u) = 0$. That is there is to be no change in the Riemann-Roch expression when we shift from an order basis (τ) relative to $F(z, u) = 0$ to the corresponding order basis (t) relative to $G(x, y) = 0$. Besides, when we shift from complementary order bases (τ) and $(\bar{\tau})$ relative to $F(z, u) = 0$ to order bases (t) and $(\bar{t})$ relative to $G(x, y) = 0$, this latter pair of order bases is to be complementary relative to $G(x, y) = 0$, the maintenance if the complementary property involving nothing beyond a natural change from the function used as level for $F(z, u) = 0$ to the one used as level for $G(x, y) = 0$. It will be convenient to adopt x as the independent variable and y as the dependent variable when we are making use of $G(x, y) = 0$ as the fundamental algebraic equation.

When we speak of a cycle about a z-value, we shall continue to use ν generically to denote the number of its branches, and this whether we refer to a finite z-value $z = a$ or the infinite z-value as centre. In the same way, when we speak of a cycle about an x-value, we shall use ω generically to denote the number of its branches, and this whether we refer to a finite x-value $x = \alpha$ or the infinite x-value as centre. All types of correspondence between cycles about z-values and cycles about x-values are given, to a first approximation, by the following:

(31)
$$(z-a)^{\omega} = K(x-\alpha)^{\nu},$$
$$(z-a)^{\omega} = K\left(\frac{1}{x}\right)^{\nu},$$
$$\left(\frac{1}{z}\right)^{\omega} = K(x-\alpha)^{\nu},$$
$$\left(\frac{1}{z}\right)^{\omega} = K\left(\frac{1}{x}\right)^{\nu}.$$

In each case K is, to a first approximation, a constant different from zero.

We shall now verify that the first requirement for the invariance of the Riemann-Roch Theorem is met. In the first case of (31), let a rational function have order $\tau = \lambda/\nu$ for each of the ν branches of the cycle in question about the finite z-value $z = a$. This same rational function will have order $t = \lambda/\omega$ for each of the ω branches of the cycle in question about the finite x-value $x = \alpha$. In other words, from $\tau = \lambda/\nu$ for each of the ν branches of the cycle in question about the finite z-value $z = a$ we deduce $t = \lambda/\omega$ for each of the ω branches of the cycle in question about the finite x-value $x = \alpha$, and we observe that $\tau\nu = t\omega$. The same type of discussion applies to the remaining cases of (31). An order basis (τ) relative to $F(z, u) = 0$ converts, therefore, into an order basis (t) relative to $G(x, y) = 0$, and, besides,

$$\sum\sum \tau\nu = \sum\sum t\omega.$$

Furthermore, on account of the birationality connecting the pairs (z, u) and (x, y) with one another, we see that $N(\tau)$ relative to $F(z, u) = 0$ is the same thing as $N(t)$ relative to $G(x, y) = 0$. That is, the first requirement is met, since $RR(\tau)$ relative to $F(z, u) = 0$ has the same value as $RR(t)$ relative to $G(x, y) = 0$.

We must still see that the second requirement for the invariance of the Riemann-Roch Theorem is also met. Let us suppose that order bases (τ) and $(\bar{\tau})$ are complementary relative to $F(z, u) = 0$, to the level of a function $S(z, u)$, which is a constant different from zero for each of the ρ irreducible equations making up $F(z, u) = 0$, which ρ non-zero constants may be chosen to range all the way from being all the same to being all different. We wish to show that order bases (t) and $(\bar{t})$ are complementary relative to $G(x, y) = 0$ to the level of the function $S(z, u)dx/dz$. A direct verification of this can be easily made. (*Cf.* [5, Chap. 2, § 5].) In the first case of (31) let $\tau = \lambda/\nu$ and $\bar{\tau} = -1 + (1-\lambda)/\nu$ be the individual orders taken from (τ) and $(\bar{\tau})$ for the cycle in question about the finite z-value $z = a$. Then

$$t = \frac{\lambda}{\omega} \text{ and } \bar{t} = -\frac{\nu}{\omega} + \frac{1-\lambda}{\omega},$$

while the order of dx/dz is $1 - \nu/\omega$, all relative to the cycle in question about the finite x-value $x = \alpha$. In other words,

$$t + \bar{t} = \left(1 - \frac{\nu}{\omega}\right) - 1 + \frac{1}{\omega},$$

which means that t and $\bar{t}$ have the complementary property for the cycle in question about the finite x-value $x = \alpha$ to the level of the function $S(z, u)dx/dz$. The same sort of verification can be given for the remaining cases of (31). That is, the second requirement for the invariance of the Riemann-Roch Theorem is also met.

References

1. S. Beatty, *Derivation of the Complementary Theorem from the Riemann-Roch Theorem*, Amer. J. Math., vol. 39 (1917), 257–262.
2. ———, *The algebraic theory of algebraic functions of one variable*, Proc. London Math. Soc. (2), vol. 20 (1920), 435–449.
3. ———, *Integral bases for an algebraic function field*, J. London Math. Soc., vol. 4 (1928), 13–17.
4. S. Beatty and Miss M. J. Fisher, *Theory of algebraic functions based on a study of fundamental exponents*, Scripta Math., vol. 5 (1938), 187–194, 239–245.
5. C. Chevalley, *Introduction to the theory of algebraic functions of one variable*, Math. Surveys, vol. 6 (1951); in particular, 28–33.
6. R. Dedekind and H. Weber, *Theorie der algebraischen Funktionen einer Veränderlichen*, J. Reine Angew. Math., vol. 92 (1882), 181–290.
7. J. C. Fields, *Theory of the algebraic functions of a complex variable* (Berlin, 1906).
8. K. Hensel and G. Landsberg, *Theorie der algebraischen Funktionen einer Variabeln* (Leipzig, 1902).

University of Toronto
Carleton College