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GILLIS'S RANDOM WALKS ON GRAPHS

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Abstract

We consider a random walker on a d-regular graph. Starting from a fixed vertex, the first step is a unit step in any one of the d directions, with common probability $1/d$ for each one. At any later step, the random walker moves in any one of the directions, with probability q for a reversal of direction and probability p for any other direction. This model was introduced and first studied by Gillis (1955), in the case when the graph is a d-dimensional square lattice. We prove that the Gillis random walk on a d-regular graph is recurrent if and only if the simple random walk on the graph is recurrent. The Green function of the Gillis random walk will be also given, in terms of that of the simple random walk.

Keywords: Random walk; generating function; Green function; correlated random walk; recurrence vs. transience

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1. Introduction and results

Let $\Gamma = (V, E)$ be a locally finite graph, with vertex set V and edges E, that may have loops. The degree $\deg(x)$ of a vertex x is equal to the number of incident edges, namely

$$\deg(x) = \operatorname{card}\{e \in E \mid e = (xy) \text{ for some } y \in V\}.$$

When $\deg(x) = d$ for all vertices x, the graph is called d-regular (for example the homogeneous trees $\mathcal{T}_m$ and $\mathbb{Z}^m$). We say that two vertices x and y are adjacent, and write $x \sim y$, if they are connected by an edge. Assume that Γ is a d-regular graph. A Gillis Γ-random walk $(X_n)_{n\geq 0}$, starting from a fixed vertex $0 \in \Gamma$, is the random process defined by $X_0 \equiv 0$;

$$\mathrm{P}(X_1 = x) = \frac{1}{d} \quad \text{for all } x \sim 0;$$

and, for every $x \in \Gamma$ and all $n \geq 1$,

$$\mathrm{P}(X_{n+1} = y \mid X_n = x; X_{n-1} = z) = \begin{cases} q & \text{if } y \sim x,\ y = z, \\ p & \text{if } y \sim x,\ y \neq z, \\ 0 & \text{otherwise,} \end{cases}$$

where $(d-1)p + q = 1$.

Let $\delta = dp - 1 = p - q$; in the sequel, we shall assume that $|\delta| \neq 1$. Let us define the probability of return to the origin after n steps, for every $n \geq 0$, as

$$R_{n,d}(\delta) = \mathrm{P}(X_n = 0).$$

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For $\Gamma = \mathbb{Z}^d$, Gillis (1955) investigated some of the properties of this correlated random walk and, in particular, proved that if $d = 1$ or if d is even, then

$$R_{2n,d}(\delta) \sim \left(\frac{1-\delta}{1+\delta}\right)^{d/2} R_{2n,d}(0), \tag{1.1}$$

and conjectured that (1.1) should hold for any integer d. Gillis's aim was to obtain the following (so-called Gillis) conjecture.

Conjecture 1.1. *The Gillis random walk on $\mathbb{Z}^d$ is recurrent if $d = 1, 2$, and transient if $d \geq 3$.*

This conjecture was first proved by Domb and Fisher (1958). They even considered a more general model called the Gillis–Domb–Fisher random walk, allowing the walker to move with different probabilities along the different directions. Chen and Renshaw (1992), (1994) used characteristic function methods to study the Gillis–Domb–Fisher random walk and, in particular, they gave a new proof of Conjecture 1.1. Another proof of this conjecture was given by Iossif (1986), who established, using a renewal argument, that $R_{2n,d}(\delta) = \mathcal{O}(n^{-d/2})$, which implies the transience of the Gillis random walk for any $d \geq 3$. All the methods used up to now in order to prove Gillis's conjecture are based on an estimation of the probability of return to the origin for the Gillis random walk. We give a new and simple proof of this conjecture using powerful combinatorial results recently obtained by Bartholdi (1999). The method is quite efficient since it can be applied to *any* d-regular graph. It is worth noting that these correlated random walk models have important applications in physics (scattering of waves), biology (rooting patterns of special trees, animal diffusion) and polymer chemistry. The reader can find interesting discussions and references for applications of correlated random walks in Chen and Renshaw (1992), (1994). Our main result is the following theorem.

Theorem 1.1. *The Gillis random walk on a d-regular graph Γ is recurrent if and only if the simple random walk on Γ is recurrent.*

Let us denote by $R_{d,\delta}$ the generating function of the sequence $(R_{n,d}(\delta))_{n\geq 0}$, as follows:

$$R_{d,\delta}(z) = \sum_{n=0}^{\infty} R_{n,d}(\delta) z^n.$$

When $\delta = 0$, $R_{d,0}(\cdot)$ is the Green function of the simple random walk evolving on Γ.

Theorem 1.2. *The following relation holds:*

$$R_{d,\delta}(z) = \frac{1}{1+\delta}\left\{\delta + \frac{1-\delta^2 z^2}{1+\delta z^2} R_{d,0}\left(\frac{(1+\delta)z}{1+\delta z^2}\right)\right\}. \tag{1.2}$$

From this, we easily deduce that Conjecture 1.1 holds.

In Sections 2 and 3, we introduce the result of Bartholdi (1999) and use it to prove Theorems 1.1 and 1.2. In Section 4, we discuss some characteristics of Gillis random walks on graphs (namely the asymptotic probability of returns to the starting point, and the estimation of the expected number of returns to the origin) using (1.2).

2. Counting paths in graphs

Bartholdi (1999) extended Grigorchuk's formula (see Grigorchuk (1978), (1980)), relating co-growth and the spectral radius of random walks on graphs, as follows. Let 0 be a fixed vertex of a locally finite graph $\Gamma = (V, E)$, which might have multiple edges and loops; let $C(n)$ be the number of circuits (i.e. closed sequences of edges) of length n at 0; and let $C(n, m)$ be the number of paths of length n with m backtrackings, i.e. with m occurrences of an edge being followed twice in a row.

Consider the formal power series

$$F(u, t) = \sum_{m,n=0}^{\infty} C(n, m)u^m t^n \quad \text{and} \quad G(t) = \sum_{n=0}^{\infty} C(n)t^n.$$

Let us note that $F(0, t)$ is the generating function of proper circuits, i.e. those without backtracking, and that $F(1, t) = G(t)$.

Theorem 2.1. (Bartholdi.) *When Γ is a d-regular graph, the following relation holds:*

$$F(1 - u, t) = \frac{1 - u^2t^2}{1 + u(d - u)t^2} G\left(\frac{t}{1 + u(d - u)t^2}\right).$$

Two different proofs of this theorem can be found in Bartholdi (1999).

3. Proofs of Theorems 1.1 and 1.2

Let N_n be the number of backtrackings in a circuit of length n. The random variable N_n takes its values in $\{0, \dots, n - 1\}$. Then, we obtain

$$R_{n,d}(\delta) = \mathrm{P}(X_n = 0) = \sum_{m=0}^{n-1} \mathrm{P}(X_n = 0;\, N_n = m).$$

By definition of the random walk $(X_n)_{n\geq 0}$, each circuit of length n, $n \geq 1$, with m backtrackings is assigned the relative probability $q^m p^{n-m-1}/d$ and, if $C(n, m)$ denotes the number of such circuits, we then obtain $\mathrm{P}(X_n = 0;\, N_n = m) = C(n, m)q^m p^{n-m-1}/d$. We can rewrite this as

$$R_{n,d}(\delta) = \frac{1}{pd} \sum_{m=0}^{n-1} C(n, m)q^m p^{n-m}.$$

The generating function $R_{d,\delta}$ of the sequence $(R_{n,d}(\delta))_{n\geq 0}$ is then given by

$$\begin{aligned}
R_{d,\delta}(z) &= 1 + \sum_{n=1}^{\infty} R_{n,d}(\delta)z^n \quad (\text{since } \mathrm{P}(X_0 = 0) = 1) \\
&= 1 + \frac{1}{pd} \sum_{n=1}^{\infty} \left(\sum_{m=0}^{n-1} C(n, m)\right)\left(\frac{q}{p}\right)^m (pz)^n \\
&= 1 + \frac{1}{pd} \sum_{m=0,n=1}^{\infty} C(n, m)\left(\frac{q}{p}\right)^m (pz)^n \\
&= 1 + \frac{1}{pd} F\left(1 - \frac{\delta}{p}, pz\right) - \frac{F(0, 0)}{pd} \\
&= \frac{1}{1 + \delta}\left\{\delta + F\left(1 - \frac{\delta}{p}, pz\right)\right\}, \qquad (3.1)
\end{aligned}$$

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since $F(0,0) = 1$ and $\delta = dp - 1 = p - q$. We use the convention that $C(n,m) = 0$ when $m \geq n$. When $\delta = 0$, i.e. $p = q = 1/d$, the Gillis random walk we consider is just the simple random walk evolving on Γ and

$$R_{d,0}(z) = \sum_{n=0}^{\infty} R_{n,d}(0) z^n = \sum_{n=0}^{\infty} C(n) \left(\frac{z}{d}\right)^n = G\left(\frac{z}{d}\right). \tag{3.2}$$

Since the graph Γ is d-regular (see Theorem 2.1), we obtain the following relation with $u = \delta/p$ and $t = pz$:

$$F\left(1 - \frac{\delta}{p}, pz\right) = \frac{1 - \delta^2 z^2}{1 + \delta(pd - \delta) z^2} G\left(\frac{pz}{1 + \delta(pd - \delta) z^2}\right).$$

From (3.1) and (3.2), we obtain

$$R_{d,\delta}(z) = \frac{1}{1+\delta} \left\{ \delta + \frac{1 - \delta^2 z^2}{1 + \delta(pd - \delta) z^2} R_{d,0}\left(\frac{dpz}{1 + \delta(pd - \delta) z^2}\right) \right\},$$

which can be simplified, since $pd - \delta = 1$, as follows:

$$R_{d,\delta}(z) = \frac{1}{1+\delta} \left\{ \delta + \frac{1 - \delta^2 z^2}{1 + \delta z^2} R_{d,0}\left(\frac{(1+\delta) z}{1 + \delta z^2}\right) \right\}. \tag{3.3}$$

This completes the proof of Theorem 1.2. Theorem 1.1 is easily deduced from the above relation, since the function

$$\phi \colon z \mapsto \frac{(1+\delta) z}{1 + \delta z^2}$$

is continuous at the point $z = 1$ ($\delta \neq -1$), and $\phi(1) = 1$. Consequently, we can let $z \to 1^-$ in (3.3), and so obtain

$$\lim_{z \to 1^-} R_{d,\delta}(z) = \frac{1}{1+\delta} \left\{ \delta + (1 - \delta) \lim_{z \to 1^-} R_{d,0}(z) \right\}.$$

It is well known that the simple random walk is recurrent if and only if

$$\lim_{z \to 1^-} R_{d,0}(z) = \infty,$$

and transient if and only if

$$\lim_{z \to 1^-} R_{d,0}(z) < \infty.$$

4. Some applications of Theorem 1.2

4.1. Application 1

From (1.2) we can compute the Green function of any Gillis random walk evolving on a regular graph when the Green function of the simple random walk is known. For instance, let $d \geq 2$ and consider $\mathcal{T}_d$, the homogeneous tree of degree d. Now, the Green function of the simple random walk on $\mathcal{T}_d$ is given, for $|z| < (d/2)(d-1)^{-1/2}$, by

$$R_{d,0}(z) = \frac{2(d-1)}{d - 2 + \sqrt{d^2 - 4(d-1) z^2}}$$

(see Kesten (1959)). A simple manipulation gives the following expression for the Green function of the Gillis random walk on $\mathcal{T}_d$:

$$R_{d,\delta}(z) = \frac{1}{1+\delta}\left\{\delta + \frac{2(d-1)(1-\delta^2 z^2)}{(d-2)(1+\delta z^2) + \sqrt{d^2(1+\delta z^2)^2 - 4(d-1)(1+\delta)^2 z^2}}\right\}.$$

In particular, if $d=2$ then $\mathcal{T}_2 = \mathbb{Z}$ and

$$\begin{aligned} R_{2,\delta}(z) &= \frac{1}{1+\delta}\left\{\delta + \frac{(1-\delta^2 z^2)}{\sqrt{(1+\delta z^2)^2 - (1+\delta)^2 z^2}}\right\} \\ &= \frac{1}{1+\delta}\left\{\delta + \frac{1-\delta^2 z^2}{\sqrt{(1-z^2)(1-\delta^2 z^2)}}\right\} \\ &= \frac{1}{1+\delta}\left\{\delta + \sqrt{\frac{1-\delta^2 z^2}{1-z^2}}\right\}. \end{aligned}$$

We recover Gillis (1955, Equation 3.17), and an estimate of the probability of returning to the origin at time $2n$ for the Gillis $\mathbb{Z}$-random walk is then found to be

$$R_{2n,2}(\delta) \sim \sqrt{\frac{1-\delta}{1+\delta}}\frac{1}{\sqrt{\pi n}}$$

for large n (see Gillis (1955) for details). Let us mention that the probability of returning to the origin at time $2n$ for the Gillis $\mathbb{Z}^m$-random walk ($m \geq 2$) can also be asymptotically evaluated using complex analysis methods developed in Gillis (1955).

We can also consider the half-line $\mathbb{N}$ with edges $[i, i+1]$ and add a loop at 0 in order to get a two-regular graph. The Green function of the simple random walk evolving on this graph is given by

$$R_{2,0}(z) = \frac{2}{1-z+\sqrt{1-z^2}}$$

(see Woess (2000)), and the Green function of the Gillis random walk on this particular graph is

$$R_{2,\delta}(z) = \frac{1}{\delta+1}\left\{\delta + 2\frac{\sqrt{1-\delta z}(1+\delta z)}{(1-z)\sqrt{1-\delta z} + \sqrt{(1-z^2)(1+\delta z)}}\right\}.$$

4.2. Application 2

Let us consider a transient Gillis random walk $(X_n)_{n\geq 0}$ on a d-regular graph Γ, with $d \geq 3$ and $|\delta| \neq 1$. Let us denote by $N_\infty(\delta)$ the number of returns to a fixed point 0 in Γ for the walk starting from 0. The expected number of returns to the origin for this random walk is finite and equal to $R_{d,\delta}(1)$, so (1.2) gives us the relation

$$\mathrm{E}(N_\infty(\delta)) = \frac{1-\delta}{1+\delta}\,\mathrm{E}(N_\infty(0)) + \frac{\delta}{1+\delta}. \tag{4.1}$$

The function $\delta \mapsto \mathrm{E}(N_\infty(\delta))$ is continuous and strictly decreasing for $\delta \in\,]-1, 1/(d-1)]$. If $\delta \in\,]-1, 0]$ then

$$\mathrm{E}(N_\infty(\delta)) \geq \mathrm{E}(N_\infty(0))$$

and, if $\delta \in [0, 1/(d-1)]$,

$$\mathrm{E}\left(N_\infty\left(\frac{1}{d-1}\right)\right) \leq \mathrm{E}(N_\infty(\delta)) \leq \mathrm{E}(N_\infty(0)).$$

This means that when $\delta \in]-1, 0[$, the Gillis random walk is less transient than the simple one, and that when $\delta \in]0, 1/(d-1)]$, the Gillis random walk is more transient than the simple one. In the most transient case, i.e. $\delta = 1/(d-1)$, we have the following relation:

$$\mathrm{E}\left(N_\infty\left(\frac{1}{d-1}\right)\right) = \frac{d-2}{d}\,\mathrm{E}(N_\infty(0)) + \frac{1}{d}.$$

In the particular case of d-regular graphs ($d \geq 3$) without cycles (i.e. homogeneous trees $\mathcal{T}_d$, $d \geq 3$), we can compute $\mathrm{E}(N_\infty(\delta))$ for any $\delta \in]-1, 1/(d-1)]$, since $\mathrm{E}(N_\infty(0)) = (d-1)/(d-2)$, and we obtain

$$\mathrm{E}(N_\infty(\delta)) = \frac{1}{1+\delta}\left(1 + \frac{1-\delta}{d-2}\right).$$

When the graph has cycles, it is much more difficult to estimate $\mathrm{E}(N_\infty(0))$. Let us consider the case of $\mathbb{Z}^m$, $m \geq 3$ (here, $d = 2m$). The expected number of returns to the origin for the simple $\mathbb{Z}^m$-random walk has been extensively investigated, both numerically and analytically, and explicit analytic expressions can be derived in a variety of ways. For example, we have

$$\mathrm{E}(N_\infty(0)) = R_{2m,0}(1) = \int_0^\infty \mathrm{e}^{-x} I_0^m\left(\frac{x}{m}\right) \mathrm{d}x,$$

where I_0 is the modified Bessel function given by

$$I_0(x) = \sum_{k=0}^\infty \frac{1}{(k!)^2}\left(\frac{x^2}{4}\right)^k.$$

There is no way to evaluate the above integral for $m \geq 3$: it can only be numerically estimated. Numerical estimates of $\mathrm{E}(N_\infty(0))$ for $3 \leq m \leq 10$ can be found in Bender *et al.* (1994), which permits us to give numerical estimations of the expected number of returns to the origin for a Gillis $\mathbb{Z}^m$-random walk for any $m \in \{3, \dots, 10\}$ and for any $\delta \in]-1, 1/(2m-1)]$. It is worth noting that, given any $\delta \in]0, 1/(2m-1)]$, $\mathrm{E}(N_\infty(-\delta))$ can be obtained from $\mathrm{E}(N_\infty(\delta))$ using the following formula, which can in turn be deduced from (4.1):

$$(1+\delta)\,\mathrm{E}(N_\infty(\delta)) + (1-\delta)\,\mathrm{E}(N_\infty(-\delta)) = 2\,\mathrm{E}(N_\infty(0)).$$

Results for three particular values of δ, namely $\pm 1/(2m-1)$, and $-1+10^{-3}$, are displayed in Table 1.

In the most transient case, corresponding to $\delta = 1/(2m-1)$, i.e. $q = 0$ and $p = 1/(2m-1)$, the Gillis random walk never comes back in one step to the last visited site, but loops are allowed. Therefore, returns to the origin are possible, but are less numerous than in the simple random walk. For $\delta = -1/(2m-1)$, i.e. $q = 1/m$ and $p = (m-1)/m(2m-1)$, the Gillis random walk is less transient than the simple one, and the number of returns to the origin is larger than for the simple one. In the limiting case $\delta = -1$, i.e. $q = 1$ and $p = 0$, the Gillis random walk starts from 0, chooses one of its neighbours – each with the same probability $1/2m$ – and will

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TABLE 1.

m	$\mathrm{E}(N_\infty(0))$	$\mathrm{E}(N_\infty(1/(2m-1)))$	$\mathrm{E}(N_\infty(-1/(2m-1)))$	$\mathrm{E}(N_\infty(-1+10^{-3}))$
3	1.516 386 059	1.177 590 706	2.024 579 089	2032.255 732
4	1.239 467 122	1.054 600 342	1.485 956 163	1478.694 777
5	1.156 308 124	1.025 046 499	1.320 385 155	1312.459 94
6	1.116 963 374	1.014 136 145	1.240 356 049	1233.809 785
7	1.093 906 315	1.009 062 556	1.192 890 701	1187.718 724
8	1.078 647 012	1.006 316 136	1.161 310 871	1157.215 377
9	1.067 746 087	1.004 663 188	1.138 714 348	1135.424 428
10	1.059 543 752	1.003 589 377	1.121 715 28	1119.027 96

alternately visit the selected site and 0; this is evidently recurrent and the number of returns to the origin is infinite. A natural question is to ask how the expected number of returns to the origin for a Gillis random walk with $\delta > -1$ increases when δ tends to -1. What is interesting in (4.1) is that we can determine the speed of this convergence: for any $n \geq 1$,

$$\mathrm{E}(N_\infty(-1+10^{-n})) = 10^n(2\,\mathrm{E}(N_\infty(0)) - 1) + 1 - \mathrm{E}(N_\infty(0)).$$

Therefore, the expected number of returns to the origin for a Gillis random walk is (modulo only a constant) inversely proportional to the speed with which it approaches -1. In Table 1, the values of $\mathrm{E}(N_\infty(-1+10^{-n}))$ are given for $n = 3$ and $m \in \{3, \dots, 10\}$.

5. Conclusion

Our main result is that the Gillis random walk on a d-regular graph Γ is recurrent if and only if the simple random walk on Γ is recurrent. The proof is based on the use of a very nice formula recently obtained by Bartholdi (1999). In particular, we have established a relation between the Green function of any Gillis random walk evolving on a d-regular graph and the Green function of the simple random walk on this graph. Some applications of this relation are given in Section 4. From this formula, it should be possible to derive new results about the characteristics of Gillis random walks on any d-regular graph.

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