THE STATES OF A BANACH ALGEBRA GENERATE THE DUAL

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In this paper we prove that the states of a unital Banach algebra generate the dual Banach space as a linear space (Theorem 2). This is a result of R. T. Moore (**4**, Theorem 1(*a*)) who uses a decomposition of measures in his proof. In the proof given here the measure theory is replaced by a Hahn-Banach separation argument. We shall let A denote a unital Banach algebra over the complex field, and $D(1)$ denote $\{f \in A'\colon \|f\| = f(1) = 1\}$ where A' is the dual of A. The motivation of Moore's results is the theorem that in a C^*-algebra every continuous linear functional is a linear combination of four states (the *states* are the elements of $D(1)$) (see (**2**, 2.6.4, 2.1.9, 1.1.10)).

Recall that the *numerical index* $n(A)$ of A is defined by

$$n(A) = \inf\{v(a)\colon a \in A, \| a \| = 1\}$$

where $v(a) = \sup\{|f(a)|\colon f \in D(1)\}$ [**1**, Definition 4.9]. We show that the closed balanced convex hull of the states of a unital Banach algebra contains the dual ball of radius the numerical index (Corollary 4). We denote the convex hull of a subset F of a linear space by co F, and the closed unit ball

$$\{x \in X\colon \| x \| \leqq 1\}$$

in a Banach space X by X_1.

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1. Lemma. *Let F be a finite set of complex numbers each of modulus* 1, *and let η be the radius of the largest disc centre the origin that is contained in the convex hull of F. Then* co $\{\beta D(1)\colon \beta \in F\}$ *is a $\sigma(A', A)$-compact subset of A' containing $\eta n(A)A'_1$.*

Proof. Let $F = \{\beta_1, \ldots, \beta_n\}$. If each $\beta_j D(1)$ has the $\sigma(A', A)$-topology, and if the product

$$\beta_1 D(1) \times \ldots \times \beta_n D(1) \times [0, 1] \times \ldots \times [0, 1]$$

has the product topology, then the product is compact. The subset E of the product consisting of those elements $(\beta_1 f_1, \ldots, \beta_n f_n, \alpha_1, \ldots, \alpha_n)$ such that $\alpha_1 + \ldots + \alpha_n = 1$ is a closed subset of the product. The map θ from E with the product topology into $(A', \sigma(A', A))$ given by

$$\theta(\beta_1 f_1, \ldots, \beta_n f_n, \alpha_1, \ldots, \alpha_n) = \alpha_1\beta_1 f_1 + \ldots + \alpha_n\beta_n f_n$$

is continuous. The image of θ, which is equal to $\text{co}\{\beta D(1)\colon \beta \in F\}$, is thus $\sigma(A', A)$-compact.

Let f be in $\eta n(A)A_1'$, and suppose that f is not in $\text{co}\{\beta D(1)\colon \beta \in F\}$. By a separation form of the Hahn-Banach Theorem (**3**, Theorem V.2.10, p. 417), there is an x in A (**3**, Theorem V.3.9, p. 421) and an $\varepsilon > 0$ such that $\|x\| = 1$ and $\text{Re} f(x) - \varepsilon \geqq \text{Re}\, g(x)$ for all g in $\text{co}\{\beta D(1)\colon \beta \in F\}$. Since co F contains the disc centre the origin with radius η in the complex plane,

$$\sup\{\text{Re}\, g(x)\colon g \in \text{co}\{\beta D(1)\colon \beta \in F\}\} \geqq \eta . \sup\{|g(x)|\colon g \in D(1)\}.$$

Therefore

$$\text{Re} f(x) \geqq \varepsilon + \eta v(x) \geqq \varepsilon + \eta n(A)$$

which proves that f is not in $\eta n(A)A_1'$. This gives a contradiction and completes the proof.

Let $H(A')$ be the real linear subspace of A' generated by $D(1)$. The elements of $H(A')$ are called *hermitian functionals* [**4**].

2. Theorem. *Let A be a complex unital Banach algebra. Then*

$$A' = H(A') + iH(A'),$$

and $H(A')$ is a real Banach space under the norm

$$|f| = \inf\{\alpha + \beta\colon \alpha \geqq 0, \beta \geqq 0, f = \alpha g - \beta h;\ g, h \in D(1)\}.$$

Proof. An application of Lemma 1 with $F = \{1, -1, i, -i\}$ proves that $A' = H(A') + iH(A')$. Since $D(1)$ is convex the subset

$$\{\alpha g - \beta h\colon \alpha, \beta \in \mathbf{R}, \alpha \geqq 0, \beta \geqq 0;\ g, h \in D(1)\}$$

of A' is a real linear subspace, and so is equal to $H(A')$. We next prove that if f is in $H(A')$, then there are $\alpha, \beta \geqq 0$ and g, h in $D(1)$ such that

$$|f| = \alpha + \beta \text{ and } f = \alpha g - \beta h. \tag{1}$$

Let G be the subset of

$$E = D(1) \times D(1) \times [0, |f| + 1] \times [0, |f| + 1]$$

of those (g, h, α, β) that satisfy $f = \alpha g - \beta h$. Then G, which is the intersection of the $\sigma(A', A)$-closed subsets

$$\{(g, h, \alpha, \beta)\colon (g, h, \alpha, \beta) \in E, f(x) = \alpha g(x) - \beta h(x)\}$$

of E as x runs over A, is a compact subset of E. The function $(g, h, \alpha, \beta) \to \alpha + \beta$ is continuous on G, and therefore the infimum is attained. This proves (1). From (1) and the inequality $\|f\| \leqq |f|$ for f in $H(A')$ it follows that $(H(A'), |\cdot|)$ is a normed space.

To prove that $(H(A'), |\cdot|)$ is complete it is sufficient for us to show that if $f_0 = 0$, and if $f_n \in H(A')$ satisfy $|f_{n+1} - f_n| \leqq 2^{-n}$ for $n = 1, 2, \ldots$, then there is an f in $H(A')$ with $|f_n - f|$ tending to zero. Since $\|\cdot\| \leqq |\cdot|$ on $H(A')$, the series $\Sigma(f_{n+1} - f_n)$ converges in A' to an element we denote by f. By (1) there are $\alpha_n, \beta_n \geqq 0$ and g_n, h_n in $D(1)$ such that

$$f_{n+1} - f_n = \alpha_n g_n - \beta_n h_n \text{ and } |f_{n+1} - f_n| = \alpha_n + \beta_n.$$

For $m = 1, 2, \ldots$, let

$$\gamma_m = \sum_{n=m}^{\infty} \alpha_n \quad \text{and} \quad \zeta_m = \sum_{n=m}^{\infty} \beta_n.$$

With convergence in the $\|\cdot\|$-topology we now have

$$f - f_m = \sum_{n=m}^{\infty} (\alpha_n g_n - \beta_n h_n)$$

$$= \gamma_m \sum_{n=m}^{\infty} \alpha_n \gamma_m^{-1} g_n - \zeta_m \sum_{n=m}^{\infty} \beta_n \zeta_m^{-1} h_n.$$

Further $\sum_{n=m}^{\infty} \alpha_n \gamma_m^{-1} g_m$ and $\sum_{n=m}^{\infty} \beta_n \zeta_m^{-1} h_n$ are in $D(1)$, because $D(1)$ is a $\|\cdot\|$-closed convex subset of A'. Therefore f is in $H(A')$, and $|f - f_m| \leqq \gamma_m + \zeta_m$ for all m. This shows that $|f - f_m|$ tends to 0 as m tends to infinity, and completes the proof.

3. Remarks. In proving Theorem 2 we showed that if f is a hermitian functional, then there are $\alpha, \beta \geqq 0$ and g, h in $D(1)$ such that

$$f = \alpha g - \beta h \text{ and } |f| = \alpha + \beta.$$

If A is a C^*-algebra, then α, β, g, h are uniquely specified by these properties (**2**, Corollaire 12.3.4, p. 245). Solving the equations $\alpha + \beta = |f|$ and $\alpha - \beta = f(1)$ shows that α and β are unique. We now give an example to show that g and h are not unique.

Let A be the complex algebra generated by 1 and x satisfying $x^3 = 0$, and let A have the $\|\cdot\|_1$-norm with $\{1, x, x^2\}$ as the basis for A. Let e_1, e_2, e_3 be the continuous linear functionals on A that are 1 at 1, x, x^2 (respectively) and zero on the other basis elements. In A' we have

$$2e_2 + e_3 = (e_1 + e_2 + \alpha e_3) - (e_1 - e_2 + (\alpha - 1)e_3)$$

for all α with $0 \leqq \alpha \leqq 1$. Since the norm in A' is the $\|\cdot\|_\infty$-norm, it follows that $e_1 + e_2 + \alpha e_3$ and $e_1 - e_2 + (\alpha - 1)e_3$ are in $D(1)$ for $0 \leqq \alpha \leqq 1$. Thus $2e_2 + e_3$ is in $H(A')$, and $|2e_2 + e_3| = \|2e_2 + e_3\| = 2$. This gives the required example.

4. Corollary. *Let A be a complex unital Banach algebra, and let B be the closed convex hull of $\bigcup\{\beta D(1) : \beta \in \mathbf{C}, |\beta| = 1\}$. Then $n(A)A'_1 \subseteq B \subseteq A'_1$.*

Proof. The corollary will follow if we show that f in A' with $\|f\| < n(A)$ implies that f is in B. There is a finite set F of complex numbers of modulus 1 whose convex hull contains the disc of radius $\|f\|/n(A)$ centre the origin in the complex plane. By Lemma 1, f is in co $\{\beta D(1) : \beta \in F\}$ which is contained in B. This completes the proof.

5. Remarks. (*a*) Corollary 4 is best possible in the sense that $rA'_1 \subseteq B$ implies that $r \leqq n(A)$. We prove this as follows. Let f be in A' with $\|f\| \leqq r$, and let x be in A. Since f may be approximated by convex sums from

$$\bigcup\{\beta D(1) : \beta \in \mathbf{C}, |\beta| = 1\},$$

we obtain $|f(x)| \leqq v(x)$. Thus for each f in A' of unit norm and each x in A we have $|f(x)| \leqq v(x)/r$. An application of the Hahn-Banach Theorem implies that $\| x \| \leqq v(x)/r$ for each x in A, and therefore $r \geqq n(A)$.

(*b*) When is $|\cdot|$ on $H(A')$ equivalent to the dual norm $\|\cdot\|$ of A' restricted to $H(A')$? R. T. Moore (letter to the author) has shown that $|\cdot|$ and $\|\cdot\|$ are equivalent on $H(A')$ if, and only if,

$$H(A') = \{f \in A' : f(h) \in \mathbf{R} \text{ for } h \text{ hermitian} \in A\}.$$

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