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FREE SURFACE WAVES OVER A DEPRESSION

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Steady waves at the free surface of an incompressible fluid passing over a depression are considered. By studying a KdV equation with negative forcing term, new types of solutions are discovered numerically and a new cut-off value of the Froude number, above which unsymmetric solitary-wave-like wave solutions exist, is also found.

1. INTRODUCTION

The purpose of this paper is to study steady surface waves on a two dimensional incompressible and invisid fluid flow passing over a small depression on a flat bottom. We assume that the depth H^* and the speed c of the fluid flow far upstream are constants and an upstream Froude number F is defined by $F = c/(gH)^{1/2}$. Steady solutions of one layer fluid for a positive obstruction have been studied numerically by Wu and Wu [**9**], Forbes and Schwartz [**3**], Vanden-Broeck [**8**], Forbes [**2**], and others and asymptotically by Cole [**1**], Shen *et al* [**7**], S.P Shen and M.C Shen [**6**], Gong and Shen [**4**] and others. It was found in these papers that there exist a cut-off value of Froude number , $F_1 > 1$, above which two supercritical stationary solitary-wave-like waves appear and there is a critical value of Froude number, $F_2 < 1$ at which a hydraulic fall solution connecting an upstream subcritical flow to a downstream supercritical flow appears. However, up to now, solutions for a forced Korteweg-de Vries equation with a negative forcing have not been completely studied [**5**]. In this paper F is assumed to be near the critical value 1, that is $F = 1 + \varepsilon\lambda$ and the same forced Korteweg-de Vires equation derived in [**6**] is used as our model equation, but we assume that the obstruction is negative and finite, and generates a negative forcing in the forced Korteweg-de Vires equation. Two cut-off values λ_1, λ_2 of the Froude number, where $0 < \lambda_1 < \lambda_2$, are found. Two positive symmetric solitary-wave-like solutions appear for $\lambda > \lambda_1$ and four positive symmetric or unsymmetric solitary-wave-like solutions appear for $\lambda > \lambda_2$. At the discrete values of positive λ's, another type of solitary-wave-like solution, which is zero ahead of the depression and a part of a solitary wave behind the depression, is also discovered. We also find positive symmetric solutions for discrete values of $\lambda < 0$ and a negative solitary-wave-like solution for $\lambda > 0$.

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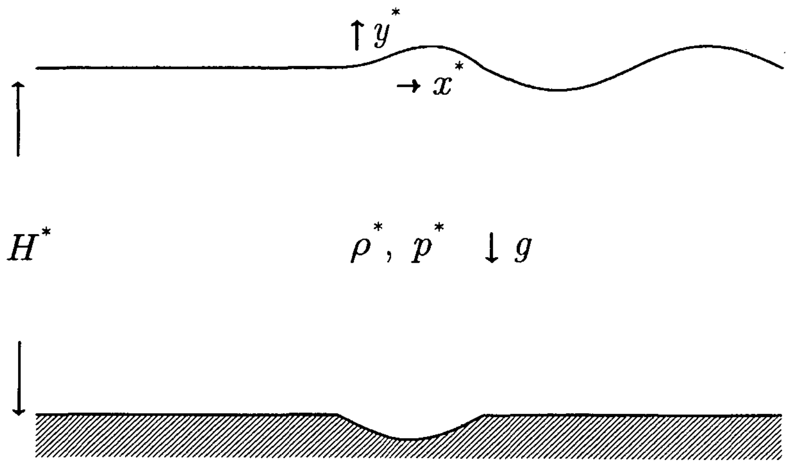

Figure 1: Fluid Domain

2. Formulation and numerical results

The problem considered here concerns steady two dimensional interfacial waves of a fluid with constant density passing over a depression with compact support (Figure 1). The governing equations and boundary conditions are as follows:

(1) $$u^*_{x^*} + v^*_{y^*} = 0,$$

(2) $$u^* u^*_{x^*} + v^* u^*_{y^*} = p^*_{x^*}/\rho^*,$$

(3) $$u^* v^*_{x^*} + v^* u v^*_{y^*} = p^*_{y^*}/\rho^* - g,$$

at the free surface $y^* = \eta^*(x^*)$,

(4) $$u^* \eta *_{x^*} - v^* = 0, \; p^* = 0,$$

at the rigid lower boundary $y^* = h^*(x^*)$,

(5) $$v^* - u^* h^*_{x^*} = 0,$$

where (u^*, v^*) are velocities, p^* is the constant density of the fluid, g is the gravitational acceleration constant, and $h^*(x^*) = -H^* + b^*(x^*)$, where H^* is the constant depth of the fluid at equilibrium state, and $b^*(x^*)$ stands for the obstruction with finite support on the rigid bottom.

We define the following nondimensional variables:

$$\varepsilon = (H^*/L)^{1/2}, \; \eta = \varepsilon^{-1}\eta^*/H^*, \; x = \varepsilon^{1/2}x^*/H^*, \; y = y^*/H^*, p = p^*/gH^*\rho^*,$$
$$(u, v) = (gH^*)^{-1/2}(u^*, \varepsilon^{-1}v^*), \; h(x) = h^*(x)/H^*, \; b(x) = b(x)(H^*\varepsilon^2)^{-1},$$

where L is the horizontal length scale.

In the term of the above nondimensional variables and by assumimg that u, v, p and η possess an asymptotic expansion of the form

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots$$

with $u_0 = 1$, $v_0 = 0$, $p_0 = -y + 1$, and the upstream Froude number $F = C/(gH^*)^{1/2} = 1 + \varepsilon\lambda$, a system of differential equations and boundary conditions for successive approximations are obtained according to the order of ε. Then, by solving the resulting equation with the assumption $\eta(-\infty) = \eta_{1x}(-\infty) = 0$, we can derive the following forced Korteweg-de Vires equation (7),

$$-\frac{1}{3}\eta_{1xxx} - 3\eta_1\eta_{1x} + 2\lambda\eta_{1x} = b_x. \tag{6}$$

Integrating(6) from $-\infty$ to x yields

$$\eta_{1xx} = -\frac{9}{2}\eta_1^2 + 6\lambda\eta_1 - 3b(x). \tag{7}$$

When $b(x) = 0$ and $\lambda > 0$, equation (7) can be solved directly and two solutions which vanish as x tends to $\pm\infty$ are given as follows:

$$\eta_1 = 2\lambda\,\text{sech}^2\big((6\lambda)^{1/2}(x-\delta)/2\big), \tag{8}$$

$$\eta_1 = -2\lambda\,\text{cosech}^2\big((6\lambda)^{1/2}(x-\gamma)/2\big), \tag{9}$$

where δ and γ are phase shifts. Equation (8) is the well-known solitary wave solution. Equation (9) is unbounded and has a singularity at γ. In what follows we shall call (9) an unbounded solitary wave solution.

Next we shall find a periodic solution of (7) without forcing. Assume $b(x) = 0$ and η_1 and η_{1x} are given at some point$x = x_0$. Let $\eta_1(x_0) = \alpha$ and $\eta_{1x}(x_0) = \beta$. Multiplying η_{1x} to (7) with $b(x) = 0$ and integrating the resulting equation from x_0 to $x > x_0$, we have

$$(\eta_{1x})^2 = -3\eta_1^3 + 6\lambda\eta_1^2 + d = f(\eta_1), \tag{10}$$

where $d = \beta^2 + 3\alpha^3 - 6\lambda\alpha^2$. Let c_1, c_2 and c_3 be three roots of $f(\eta_1)$. If all c_1, c_2 and c_3 are real and assume $c_1 < c_2 < c_3$ then (10) has the following periodic solution,

$$\eta_1 = c_2 + (c_3 - c_2)\text{Sn}^2\Big(\sqrt{3}M(x-\delta), k\Big), \tag{11}$$

where $k^2 = (c_3 - c_2)/(c_3 - c_1)$, $M^2 = (c_3 - c_1)/4$, δ is a phase shift and Sn is the Jacobian Elliptic Function. As $c_2 \uparrow c_3$, (11) tends to a constant solution $\eta_1 = c_2$ and as $c_1 \uparrow c_2$, (11) tends to the following non-periodic solution,

$$\eta_1 = c_2 + (c_3 - c_2)\,\text{sech}^2\Big(\big(3(c_3 - c_2)\big)^{1/2}(x-\delta)/2\Big). \tag{12}$$

Here δ is also a phase shift. In particular, if $c_2 = 0$ so that $d = 0$ in (10), then (12) becomes (8). If two c_i's, $i = 1, 2, 3$ are not real, the solution of (10) diverges.

In the following we shall assume $b(x)$ has a compact support [-1,1] and consider two cases, $\lambda > 0$ and $\lambda < 0$.

CASE 1. Supercritical Case $\lambda > 0$.

Since $\eta_1(-\infty) = 0$ is assumed, only two types of solutions, $\eta_1 = 0$ and either bounded or unbounded solitary wave solutions, can appear for $x < -1$. We assume $b(x) = -(1 - x^2)^{1/2}$ for $|x| \leqslant 1$ and $b(x) = 0$ for $|x| \geqslant 1$ for the numerical solutions for (7) to follow. We consider the case of $\eta_1 = 0$ for $x \leqslant -1$ and the case of solitary waves and unbonded waves for $x \leqslant -1$ separately.

1. Solitary waves for $x \leqslant -1$.

We first construct positive solitary-wave-like solution numerically. Let

$$\eta_1 = 2\lambda \operatorname{sech}^2\big((6\lambda)^{1/2}(x - \delta_1)/2\big), \tag{13}$$

for $x \leqslant -1$, and

$$\eta_1 = 2\lambda \operatorname{sech}^2\big((6\lambda)^{1/2}(x - \delta_2)/2\big), \tag{14}$$

for $x \geqslant 1$, where δ_1 and δ_2 are phase shifts. To find a solution in $|x| < 1$, we use a shooting method and the phase shifts δ_1 and δ_2 are determined by (13) and (14) for $x \leqslant -1$ and $x \geqslant 1$ respectively.

Negative solitary-wave-like solutions are numerically constructed by a similar method for the case of positive solitary-wave-like solution. We use a shooting method for (15)

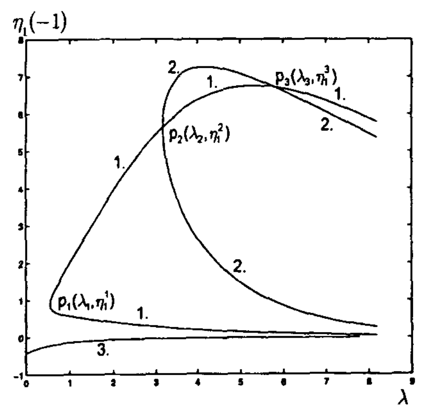

Figure 2: The relation between λ and η_1(-1), λ_1=0.55192, λ_2=3.16415. 1. Positive symmetric solitary-wave-like solution. 2. Postive unsymmetric solitary-wave-like solution. 3. Negative solitary-wave-like solution.

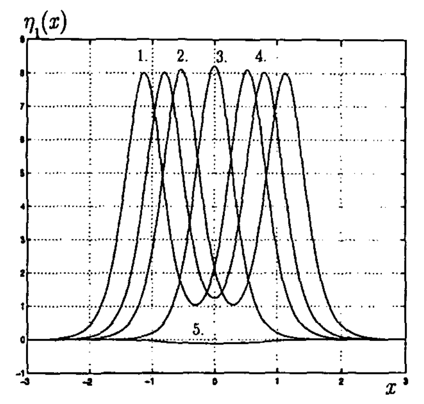

Figure 3: Typical solitary-wave-like solutions, λ=4. 1. and 2. Unsymmetric positive solitary-wave-like solution. 3. and 4. Symmetric positive solitary-wave-like solution. 5. Symmetric negative solitary-wave-like solution.

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and (16) as follows:

$$\eta_1 = -2\lambda \operatorname{cosech}^2\big((6\lambda)^{1/2}(x-\gamma_1)/2\big), \text{ for } x \leqslant -1, \tag{15}$$

$$\eta_1 = -2\lambda \operatorname{cosech}^2\big((6\lambda)^{1/2}(x-\gamma_2)/2\big), \text{ for } x \geqslant -1, \tag{16}$$

where γ_i, $i = 1, 2$ are also phase shifts.

The numerical results are given in Figures 2 and 3. The relation between λ and $\eta_1(-1)$ for solitary-wave-like solutions is given in Figure 2. Two critical points $P_1(\lambda_1, \eta_1^1)$ and $P_2(\lambda_2, \eta_2^1)$ are given in Figure 2. For $\lambda > \lambda_1$, two positive symmetric solitary-wave-like solutions appear and, for $\lambda > \lambda_2$, two positive unsymmetric solitary-wave-like solutions appear. We note that the $\eta_{1x}(P_3)$ corresponding to a positive symmetric solitary-wave-like solution is positive and the $\lambda > \lambda_1$, $\eta_{1x}(P_3)$ corresponding to a positive unsymmetric solitary-wave-like solution is negative. Hence, two positive symmetric solitary-wave-like solutions and two positive unsymmetric solitary-wave-like solutions appear at $\lambda = \lambda_3$. A negative symmetric solitary-wave-like solution appears for any positive value of λ. We also note that negative symmetric solitary-wave-like solution and the cut-off point for the appearance of positive unsymmetric solitary-wave-like solutions do not occur if $b(x)$ is of the positive semicircular form [7]. Figure 3 shows two positive symmetric solitary-wave-like solutions, two unsymmetirc solitary-wave-like solutions, and one negative symmetric solitary-wave-like solution when $\lambda = 4$. since we have derived the possible solutions of (7) in $[1, \infty)$ for any value of $\eta_1(1)$ and $\eta_{1x}(1)$, we can solve (7) by Runge-Kutta Method using (13) for $(-\infty, -1]$. We present a typical periodic wave solution of this case in Figure 4.

(2) $\eta_1 = 0$ for $x \leqslant -1$.

We assume $\eta_1 = 0$ for $x \leqslant -1$ and solve (7) numerically. The numerical results are given in Figures 5 to 7. Figure 5 shows a typical periodic wave and Figures 5 and 7 show the two unsymmetric solitary-wave-like solutions for critical values of λ's. The solutions η_1 in Figure 6 and Figure 7 are 0 for $x \leqslant -1$ and determined by (8) for $x > 1$.

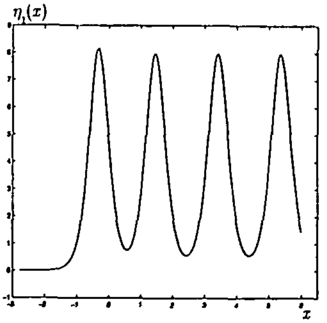

Figure 4: Typical periodic wave solution, λ=4, $\eta_1(-1)$=1.

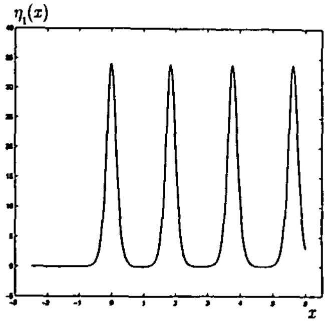

Figure 5: Typical periodic wave solution, η_1=0 for $x \leqslant -1$, λ=17.

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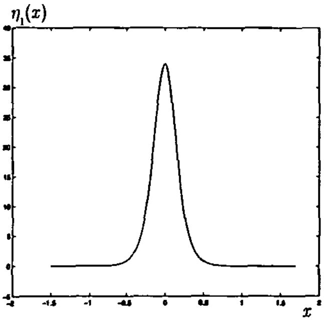

Figure 6: Unsymmetric solitary-wave-like solution, η_1=0 for $x \leqslant -1$, λ=16.961718125

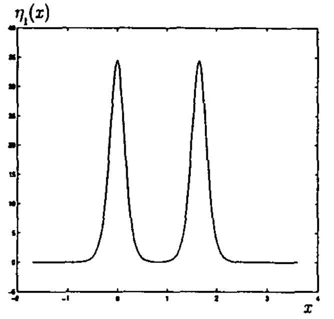

Figure 7: Unsymmetric solitary-wave-like solution, η_1=0 for $x \leqslant -1$, λ=17.2243026835

For $0 < \lambda < 16.9617$ and $\lambda > 17.2243$,only unbounded solutions appear and periodic solutions appear for $16.9617 < \lambda < 17.2243$. We note that the two unsymmetric solitary-wave-like solutions are the limiting cases of the periodic solutions.

CASE 2. Subcritical Case $\lambda < 0$.

In this caes, only $\eta_1 = 0$ can appear for $x < -1$ since we assumed $\eta_1(-\infty) = 0$. We solve (7) by Runge-Kutta Method again. The numerical results of this case are given in Figure 8 to 11. In Figure 8, we present a hydraulic fall solution which is a limiting solution of periodic solutions. This solution appears at $\lambda = -0.79169272 = \lambda_s$ and the solution diverges for $\lambda > \lambda_s$. We show a typical periodic solution in Figure 9. As λ decreases, symmetric one hump solution appears as another type a limiting solution of periodic solutions and is shown in Figure 10. Multi-hump solutions take place for discrete value of λ's. We present a symmetric two humps solution in Figure 11.

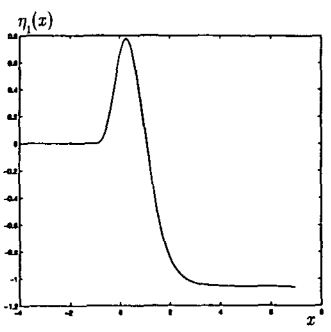

Figure 8: Hydrauric fall solution, η_1=0 for $x \leqslant -1$, $\lambda = -0.79169267$.

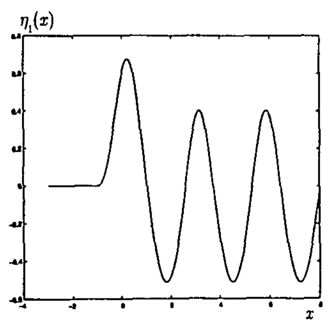

Figure 9: Typical periodic wave solution, η_1=0 for $x \leqslant -1$, $\lambda = -1$.

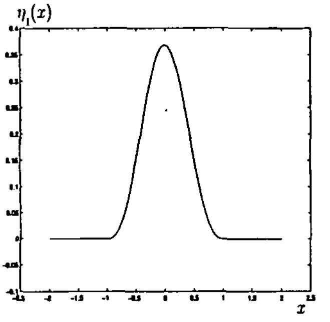

Figure 10: Symmetric wave solution with one hump, $\eta_1(-1)=0$ for $x \leqslant -1$, $\lambda = -2.14430583$.

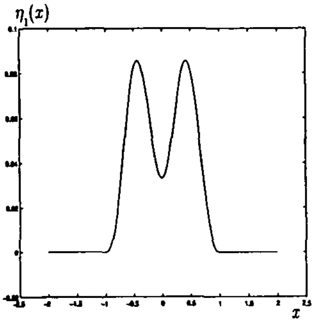

Figure 11: Symmetric wave solution with one humps, $\eta_1(-1)=0$ for $x \leqslant -1$, $\lambda = -6.1258501$.

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