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Differential equations and expansions for quaternionic modular forms in the discriminant 6 case

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Abstract

We study the differential structure of the ring of modular forms for the unit group of the quaternion algebra over $\mathbb{Q}$ of discriminant 6. Using these results we give an explicit formula for Taylor expansions of the modular forms at the elliptic points. Using appropriate normalizations we show that the Taylor coefficients at the elliptic points of the generators of the ring of modular forms are all rational and 6-integral. This gives a rational structure on the ring of modular forms. We give a recursive formula for computing the Taylor coefficients of modular forms at elliptic points and, as an application, give an algorithm for computing modular polynomials.

1. *Introduction*

1.1. *Background*

The differential structure of the ring of modular forms for $\mathrm{SL}_2(\mathbb{Z})$ has been studied extensively, with many applications, see, for example, [**6**, pp. 1–103]. The ring of holomorphic modular forms can be extended in different ways to get differential rings: one is the ring of quasimodular forms, which is closed under differentiation, another is the ring of nearly holomorphic modular forms, which is closed under the so called Maass derivative. These two differential rings are isomorphic and it turns out that an elegant theory of Taylor expansions becomes available in the second differential ring. By choosing appropriate normalizations of modular forms at CM points, one gets Taylor expansions of modular forms with algebraic coefficients.

In the context of Shimura curves, the absence of cusps means that there are no Fourier expansions of modular forms and so computing Taylor expansions is one of our goals. In this article, we will consider analogues of the Ramanujan differential equations, the Maass derivative, and Taylor expansions on the ring of modular forms for V_6, the Shimura curve of discriminant 6.

The question of expansions of quaternionic modular forms has been of interest in recent years. In [**5**], the authors compute the Schwarzian differential equation for the Shimura curve of discriminant 6, and use it to deduce some results about the hauptmodul j. In [**4**], these results are extended to give some terms of the Taylor expansions of the j-function at the elliptic points. In [**3**], modular forms were computed by restricting Hilbert modular forms to a particular Hirzebruch–Zagier cycle. During the preparation of this manuscript, two articles appeared which also address the issue of expansions of quaternionic modular forms, but with notably different methods. In [**10**] the author outlines a project to compute Taylor expansions at CM points of Shimura curves by various methods, including using Shimizu's explicit version of the Jacquet–Langlands correspondence. In [**13**], the action of Hecke operators using the Taylor expansions is computed by looking at modular forms as hypergeometric series in j.

In this article we use the differential structure of the ring of modular forms on V_6 and properties of triangle maps to get a recursion relation for computing Taylor expansions of modular forms at elliptic points and give some applications.

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1.2. *Outline*

Let $\mathcal{H}$ be the complex upper half plane, with coordinate z, and let $D = d/dz$. The particular ring of modular forms that we will be studying has three generators P, Q, R of weights 4, 6, 12 respectively, with the relation $P^6 + 3Q^4 + R^2 = 0$. We let $j = 16Q^4/(9P^6)$, a weight 0 modular form (see Section 2 for details).

In Section 3 we compute the differential relations for the ring of modular forms. In order to compute the second order differential equations, we use the Schwarzian derivative of j. We note that the differential equations of P, Q, R have integral coefficients. In Section 4 we show that at any CM point, the Chowla–Selberg periods appear as transcendental factors in the values of P, Q, R. We compute the values of these modular forms at the elliptic points explicitly using triangle maps.

In Section 5 we outline the Taylor expansion principle for modular forms, following the strategy described by Zagier in [6]. We then give a recursive formula for the Taylor coefficients and compute the expansions of P, Q, R and j at the elliptic points. We show that these coefficients are 6-integral. We then describe how to compute modular polynomials using the expansion of j.

2. *Preliminaries*

We outline key facts from [2, 3] and [9] that will be used in this article.

2.1. *The fundamental domain*

Consider the quaternion algebra A over $\mathbb{Q}$ generated by μ and ν, where $\mu^2 = -1$, $\nu^2 = 3$ and $\mu\nu + \nu\mu = 0$. Let $\mathrm{nr}(x)$ be the reduced norm of an element $x \in A$. The quaternion algebra A has discriminant 6, that is, A is ramified at the places 2 and 3. A maximal order in A is given by $\Lambda = \mathbb{Z}[\mu, \nu, (1 + \mu + \nu + \mu\nu)/2]$.

Consider the groups

$$\Gamma = \{x \in \Lambda \mid \mathrm{nr}(x) = 1\}/\{\pm 1\},$$

and

$$\Gamma^* = \{x \in A \mid x\Lambda = \Lambda x,\ \mathrm{nr}(x) > 0\}/\mathbb{Q}^\times.$$

We let the groups Γ and Γ^* act discretely on $\mathcal{H}$ by Möbius transformations via the real representation

$$\mu \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \nu \mapsto \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}. \tag{1}$$

Let V_6 denote the quotient $\mathcal{H}/\Gamma$; V_6 is a genus 0 algebraic curve over $\mathbb{Q}$ isomorphic to the conic $x^2 + 3y^2 + z^2 = 0$. The quotient group Γ^*/Γ is isomorphic to the Klein 4 group and contains the non-trivial Atkin–Lehner involutions w_2, w_3 and w_6. Let E_6 denote the curve $\mathcal{H}/\Gamma^*$. Then E_6 is isomorphic to $\mathbb{P}^1(\mathbb{Q})$. In [3], the arithmetic of the curves V_6 and E_6 is related to the rings of modular forms for the groups Γ and Γ^*.

Consider the points $z_4 = i$, $z_4' = (2 - \sqrt{3})i$, $z_3 = (-1 + i)/(1 + \sqrt{3})$, $z_3' = (1 + i)/(1 + \sqrt{3})$, $z_{24} = ((\sqrt{6} - \sqrt{2})/2)i$ and $z_{24}' = (-1 + \sqrt{2}i)/\sqrt{3}$ in $\mathcal{H}$. These points are elliptic points for the group Γ^*, and the hyperbolic triangle with vertices z_4, z_3 and z_3' is the closure of a fundamental domain for Γ^*. The fundamental domain for the group Γ is illustrated in Figure 1 (see [2]).

Consider the uniformizing function

$$j : \mathcal{H} \to \mathbb{P}^1(\mathbb{C}),$$

invariant under Γ^* which is uniquely determined by the conditions that $j(z_3) = \infty$, $j(z_{24}) = -16/27$ and $j(z_4) = 0$. This function maps the interior of the hyperbolic triangle with

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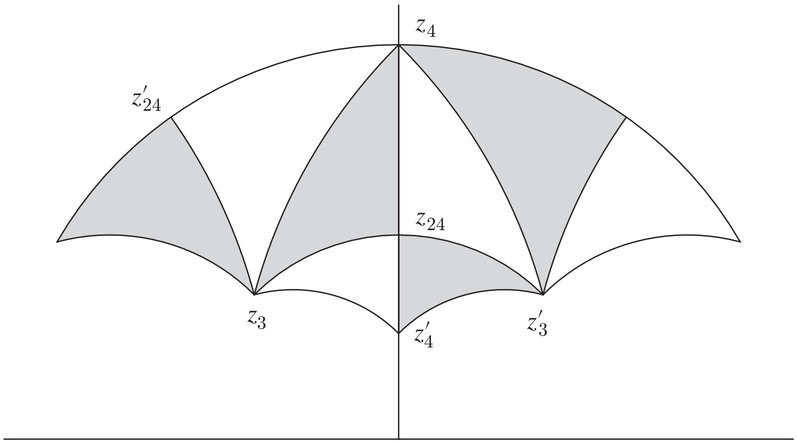

FIGURE 1. *Fundamental domain for the group* Γ. *The shaded regions are mapped by* j *to the upper half plane, the white regions to the lower half plane.*

vertices z_4, z_{24} and z_3 bijectively to the upper half plane. This triangle has angles $\pi/4$ at z_4, $\pi/2$ at z_{24} and $\pi/6$ at z_3. In [3] the map j was constructed as a quotient of modular forms.

2.2. *Modular forms, the* j*-invariant and CM points on* V_6

We identify Γ with a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ using the splitting (1) of A over $\mathbb{R}$. Let k be an even positive integer. A holomorphic modular form of weight k for Γ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that $f(\gamma(z)) = (cz+d)^k f(z)$ for all elements $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ (well-defined since k is even). A meromorphic function f with the same invariance property is called a meromorphic modular form of weight k.

Let $M_k(\Gamma)$ denote the vector space of holomorphic modular forms of weight k. The ring of modular forms

$$M(\Gamma) = \bigoplus_{k=0}^{\infty} M_{2k}(\Gamma),$$

is generated by $P \in M_4(\Gamma)$, $Q \in M_6(\Gamma)$ and $R \in M_{12}(\Gamma)$. These generators satisfy the single relation

$$P^6 + 3Q^4 + R^2 = 0. \tag{2}$$

This was demonstrated in [3]. The action of the Atkin–Lehner group Γ^*/Γ on the ring of modular forms is given by

$$\begin{aligned} w_3(P) &= P = -w_2(P) = -w_6(P), \\ w_2(Q) &= Q = -w_3(Q) = -w_6(Q), \\ w_6(R) &= R = -w_2(R) = -w_3(R). \end{aligned} \tag{3}$$

We conclude that $P(z_3) = 0$, $Q(z_4) = 0$, $R(z_{24}) = 0$.

It was shown in [3] that the function $j(z)$ described in Section 2.1 satisfies the relation $j(z) = 16Q(z)^4/(9P(z)^6)$ and that j defines an isomorphism over $\mathbb{Q}$ between E_6 and $\mathbb{P}^1$. This isomorphism was used to construct an integral model for V_6. The functions $\sqrt{j}$ and $\sqrt{-16-27j}$ lift to meromorphic functions on $\mathcal{H}$ and we choose the lifts by

$$\sqrt{j} = \frac{4}{3}\frac{Q^2}{P^3}, \qquad \sqrt{-16-27j} = 4\frac{R}{P^3}. \tag{4}$$

Let $\Delta < 0$ be the discriminant of an order in an imaginary quadratic field. A CM point $z_{|\Delta|} \in \mathcal{H}$ of discriminant Δ is the common fixed point of the non-zero elements of a quadratic

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suborder of Λ with discriminant Δ. We will drop the absolute value to make the notation more convenient. If $z_\Delta \in \mathcal{H}$ is a CM point, then $j(z_\Delta) \in \bar{\mathbb{Q}}$ (see [**3, 11**]).

2.3. *Triangle maps and the Schwarzian derivative*

Consider a triangle T in the plane with sides that are linear segments or circular arcs and with angles $\alpha\pi, \beta\pi, \gamma\pi$, where $\alpha+\beta+\gamma<1$. Let w be a coordinate on the complex plane. Restricted to the upper half plane $\mathcal{H}$ there is a unique conformal map $f=f(w)$ which maps $\mathcal{H}$ to the interior of T, and which maps 0, ∞ and 1 to the vertices of the triangle. This map can be determined using the Schwarzian derivative [**9**, p. 199] which, for a function $f(w)$, is defined to be

$$S_w(f)=\frac{2f'f'''-3f''^2}{2f'^2},$$

where $'$ denotes differentiation with respect to w. The Schwarzian derivative of the triangle map f is computed to be [**9**, (61)]

$$S_w(f)=\frac{1-\alpha^2}{2w^2}+\frac{1-\gamma^2}{2(w-1)^2}+\frac{\alpha^2+\gamma^2-\beta^2-1}{2w(w-1)}. \tag{5}$$

A particular solution s of (5) can be computed as a quotient of solutions of a hypergeometric differential equation. Let

$$a=\tfrac{1}{2}(1-\alpha-\beta-\gamma), \quad b=\tfrac{1}{2}(1-\alpha+\beta-\gamma), \quad c=1-\alpha.$$

The function

$$s(w)=\frac{w^{1-c}F(a-c+1,b-c+1;2-c;w)}{F(a,b;c;w)} \tag{6}$$

maps the upper half plane to the interior of the triangle with vertices

$$\begin{aligned} s(0)&=0,\\ s(\infty)&=e^{i\pi(1-c)}\frac{\Gamma(b)\Gamma(c-a)\Gamma(2-c)}{\Gamma(c)\Gamma(b-c+1)\Gamma(1-a)},\\ s(1)&=\frac{\Gamma(2-c)\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(1-a)\Gamma(1-b)}, \end{aligned} \tag{7}$$

angles $\alpha\pi$, $\beta\pi$ and $\gamma\pi$ respectively, and with the sides intersecting at $s(0)$ being linear segments and the third side a circular arc [**9**, p. 308]. Mapping this image triangle conformally to the triangle T (using a Möbius transformation) yields the originally sought triangle map $f(w)$.

We can derive a differential equation for the function $j(z)$ using the Schwarzian derivative. The function $w(z)=-27j(z)/16$ maps the hyperbolic triangle in $\mathcal{H}$ with the vertices z_4, z_3, z_{24}, which have the angles $\pi/4, \pi/6, \pi/2$, conformally to $\mathcal{H}$ and the vertices are mapped to 0, ∞ and 1 respectively. Hence $w(z)$ is the inverse of a triangle map $z=f(w)$ in the sense above. Combining (5) with the chain rule $S_j(z)=(dw/dj)^2S_w(z)$ and the inversion formula $S_z(j)=-(Dj)^2S_j(z)$, we obtain

$$S_z(j)=-\frac{960+2004j+2835j^2}{8j^2(27j+16)^2}(Dj)^2. \tag{8}$$

3. *The differential ring of modular forms*

3.1. *The differential equations*

The derivative $DF=dF/dz$ of a non-zero modular form $F\in M_k(\Gamma)$ transforms as

$$(DF)(\gamma z)=(cz+d)^{k+2}(DF)(z)+kc(cz+d)^{k+1}f(z), \tag{9}$$

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for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. So, unless $k = 0$, DF is not modular. If $k > 0$, consider the meromorphic function

$$\varphi = \frac{DF}{kF}, \tag{10}$$

which transforms by $\varphi(\gamma z) = (cz + d)^2 \varphi(z) + c(cz + d)$. In particular, if $\hat{F} \neq 0$ is another modular form of weight $\hat{k} > 0$, then

$$\frac{DF}{kF} - \frac{D\hat{F}}{\hat{k}\hat{F}}$$

is a meromorphic modular form of weight 2.

Proposition 1. *There exists a unique normalization of P, Q and R such that $P(i) \in i\mathbb{R}_+$, $R(i) \in \mathbb{R}_+$, and such that the following linear differential equations hold:*

$$\begin{cases} \dfrac{DQ}{6Q} - \dfrac{DP}{4P} = \dfrac{R^2}{PQR}, \\ \dfrac{DP}{4P} - \dfrac{DR}{12R} = \dfrac{3Q^4}{PQR}, \\ \dfrac{DR}{12R} - \dfrac{DQ}{6Q} = \dfrac{P^6}{PQR}. \end{cases} \tag{11}$$

Furthermore, with this normalization we have that $iP(z), Q(z), R(z) \in \mathbb{R}$ for all $z \in i\mathbb{R}_+$.

Proof. We recall that P and Q do not have any common zeros. The function $DQ/(6Q) - DP/(4P)$ is a meromorphic modular form of weight 2 with poles at exactly the zeros of Q and P. Thus $QP(DQ/(6Q) - DP/(4P))$ is a holomorphic modular form of weight 12, so there are constants a, b, c so that

$$DQ/(6Q) - DP/(4P) = (aP^3 + bQ^2 + cR)/(QP).$$

From (3) it is clear that $DQ/(6Q) - DP/(4P)$ is invariant under the action of the Atkin–Lehner group, which by (3) again implies that $a = b = 0$. Hence, the first equation of (11) holds up to some constant factor on the right hand side. Similarly, we see that the other two equations hold up to constant factors. Adding all three equations, we see that the constants have to coincide, because of the unique relation $P^6 + 3Q^4 + R^2 = 0$. By normalizing P, Q, R appropriately, we can arrange that constant to be 1. The remaining degree of freedom is that one can change the sign of precisely two of them. We know that at the point $z = i$ the modular forms P and R are non-zero. We now claim that $P(i) \in i\mathbb{R}$ and $R(i) \in \mathbb{R}$, from which we conclude that the conditions $P(i) \in i\mathbb{R}_+$ and $R(i) \in \mathbb{R}_+$ will determine P, Q and R uniquely. By using (11) and (4), we get $Dj = 24jR/(PQ)$ and the representations

$$\begin{cases} P = \dfrac{(Dj)^2}{2^4 3 \sqrt{j}^3 \sqrt{-16 - 27j}^2}, \\ Q = \dfrac{(Dj)^3}{2^7 3 \sqrt{j}^4 \sqrt{-16 - 27j}^3}, \\ R = \dfrac{(Dj)^6}{2^{14} 3^3 \sqrt{j}^9 \sqrt{-16 - 27j}^5}. \end{cases} \tag{12}$$

Along the imaginary axis, we know that j takes real values, hence Dj is purely imaginary there. Since furthermore we have $-16/27 \leqslant j \leqslant 0$, it follows from (12) that P is purely imaginary and that Q and R are real on the imaginary axis. $\square$

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Let $\phi = DP/(4P)$, which is a meromorphic function. We claim that $D\phi - \phi^2$ is meromorphic modular of weight 4. We have

$$D\phi - \phi^2 = \frac{4PD^2P - 5(DP)^2}{16P^2}.$$

Take any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Differentiation of the relation $P(\gamma(z)) = (cz+d)^4 P(z)$, yields

$$(DP)(\gamma(z)) = 4c(cz+d)^5 P(z) + (cz+d)^6 (DP)(z),$$

and

$$(D^2P)(\gamma(z)) = 20c^2(cz+d)^6 P(z) + 10c(cz+d)^7 (DP)(z) + (cz+d)^8 (D^2P)(z).$$

Hence we see that

$$(D\phi - \phi^2)(\gamma(z)) = (cz+d)^4 (D\phi - \phi^2)(z),$$

which proves the claim.

Proposition 2. *The function ϕ and the modular forms P, Q and R satisfy the differential equations*

$$\begin{cases} D\phi = \phi^2 + 15\dfrac{Q^2}{P^2}, \\ DP = 4\phi P, \\ DQ = 6\phi Q + 6\dfrac{R}{P}, \\ DR = 12\phi R - 36\dfrac{Q^3}{P}. \end{cases}$$

Proof. The last three equations follow immediately from Proposition 1, so it remains to prove the first equation. Using (12) a straightforward computation yields the identity

$$D\phi - \phi^2 = \frac{1}{2} S_z(j) + \frac{3(3645j^2 + 2592j + 1280)(Dj)^2}{64(27j+16)^2 j^2}.$$

Combining this with the differential equation (8) we get

$$D\phi - \phi^2 = \frac{-15(Dj)^2}{64(27j+16)j},$$

so $D\phi - \phi^2 = 15Q^2/P^2$. □

3.2. *Invariance of differential equations*

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be any matrix in $\mathrm{GL}_2(\mathbb{C})$. Define the operator $|_k\, \gamma$ acting on modular forms $F \in M_k(\Gamma)$ by

$$(F\,|_k\,\gamma)(z) = \frac{(ad-bc)^{k/2}}{(cz+d)^k} F\left(\frac{az+b}{cz+d}\right).$$

We extend the definition of the slash operator to also act on functions φ of the form (10) by

$$(\varphi\,|_2\,\gamma)(z) = \frac{D((F\,|_k\,\gamma)(z))}{k(F\,|_k\,\gamma)(z)} = \frac{ad-bc}{(cz+d)^2}\varphi\left(\frac{az+b}{cz+d}\right) - \frac{c}{cz+d}.$$

It is straightforward to verify the following.

Proposition 3. *The differential equations of Proposition 2 are invariant under slashing with any $\gamma \in \mathrm{GL}_2(\mathbb{C})$, that is, Proposition 2 holds if we replace ϕ, P, Q, R by $\phi\,|_2\,\gamma$, $P\,|_4\,\gamma$, $Q\,|_6\,\gamma$, $R\,|_{12}\,\gamma$ respectively.*

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We also write $F|_k\,\alpha$ for any non-zero $\alpha\in\Lambda$, using implicitly the representation (1). If $\mathrm{nr}(\alpha)>0$, then $F|_k\,\alpha$ is defined in the upper half plane $\mathcal{H}$ and is modular with respect to the subgroup $\Gamma\cap\alpha\Gamma\alpha^{-1}$ of finite index in Γ.

3.3. *The Maass derivative*

Define a differential operator ∂_k on $M_k(\Gamma)$ by (see [8] and [6, p. 51]):

$$\partial_k f(z)=Df(z)+\frac{kf(z)}{2i\,\mathrm{Im}(z)}.$$

One verifies that $(\partial_k f)\,|_{k+2}\,\gamma=\partial_k f$ for every $\gamma\in\Gamma$, hence it makes sense to iteratively apply the Maass derivative. The ring generated by the holomorphic modular forms and the iterations of Maass derivatives is called the ring of nearly holomorphic modular forms.

When no confusion can arise, we will write ∂ instead of ∂_k, and let ∂^n be the composition $\partial_{k+2n-2}\circ\cdots\circ\partial_k$. It follows from [6, p. 51] that for any differentiable function $f:\mathcal{H}\to\mathbb{C}$ and $\alpha=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{GL}_2^+(\mathbb{R})$, we have

$$\partial_k(f\,|_k\,\alpha)=(\partial_k f)\,|_{k+2}\,\alpha. \tag{13}$$

Let $\phi^*=\partial P/4P$, $\eta^*=\partial Q/6Q$ and $\rho^*=\partial R/12R$. It follows from (13) that ϕ^*, η^* and ρ^* transform as modular forms of weight 2 for the group Γ^* and that ϕ^* vanishes at the elliptic points z_4 and z_{24}. Similarly, η^* vanishes at z_3 and z_{24}, ρ^* vanishes at z_3 and z_4. Now we have

$$\begin{aligned}\partial\phi^*(z)-{\phi^*}^2(z)&=D\phi^*(z)+\frac{2\phi^*(z)}{2i\,\mathrm{Im}(z)}-{\phi^*}^2(z)\\&=D\left(\phi(z)+\frac{1}{2i\,\mathrm{Im}(z)}\right)+\frac{2}{2i\,\mathrm{Im}(z)}\left(\phi(z)+\frac{1}{2i\,\mathrm{Im}(z)}\right)\\&\quad-\left(\phi(z)+\frac{1}{2i\,\mathrm{Im}(z)}\right)^2\\&=D\phi(z)-\phi^2(z).\end{aligned}$$

Analogous to the $\mathrm{SL}_2(\mathbb{Z})$ case (cf. [6]), we have the following.

PROPOSITION 4. *The two differential rings* $(\mathbb{C}[\phi,P,Q,R],D)$ *and* $(\mathbb{C}[\phi^*,P,Q,R],\partial)$ *are isomorphic.*

Proof. The differential relations of Proposition 2 can be reformulated using the differential operator ∂ as follows:

$$\begin{cases}\partial\phi^*={\phi^*}^2+15\dfrac{Q^2}{P^2},\\ \partial P=4\phi^*P,\\ \partial Q=6\phi^*Q+6\dfrac{R}{P},\\ \partial R=12\phi^*R-36\dfrac{Q^3}{P}.\end{cases} \tag{14}$$

From these relations, it is clear that the two differential rings are isomorphic. □

We note that relations analogous to (14) hold for η^* and ρ^* too, in particular:

$$\begin{cases}\partial\eta^*={\eta^*}^2+7\dfrac{P^4}{Q^2},\\ \partial\rho^*={\rho^*}^2+39\dfrac{Q^2P^4}{R^2}.\end{cases} \tag{15}$$

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4. *Values of modular forms*

In this section we will compute values of modular forms at elliptic points. We first show that at any CM point z_Δ with Δ a fundamental discriminant, the value of the modular forms P, Q and R are algebraic multiples of a Chowla Selberg period. At the three elliptic points z_4, z_3 and z_{24}, we can compute the value exactly using triangle maps. This will be used to compute Taylor expansions in Section 5.

4.1. *Chowla–Selberg periods and algebraicity*

The Chowla–Selberg formula gives the transcendental part of the value of algebraic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ at a CM point (see for example [**6**, § 6.3]). If Δ is the discriminant of an imaginary quadratic field, define Ω_Δ as

$$\Omega_\Delta = \frac{1}{\sqrt{|\Delta|}}\left(\prod_{j=1}^{|\Delta|-1} \Gamma\left(\frac{j}{|\Delta|}\right)^{\chi_\Delta(j)}\right)^{1/2h'(\Delta)}, \tag{16}$$

where $\chi_\Delta(j)$ is the quadratic character associated to $\mathbb{Q}(\sqrt{\Delta})$, $h'(-3)=1/3$, $h'(-4)=1/2$ and $h'(\Delta)$ is the class number of $\mathbb{Q}(\sqrt{\Delta})$ if $\Delta<-4$. Note that our definition of Ω_Δ differs from other uses in the literature, for example [**6**, (97)], by a factor of $\sqrt{2\pi}$. The reason for this is that our differential operator $D=d/dz$, which determines the normalization of the modular forms, does not contain a factor 2π.

Shimura showed [**12**, Theorem 7.1] that the values, up to algebraic factors, of the derivatives of algebraic modular functions for any quaternion algebra over $\mathbb{Q}$ at any point α with CM by K are equal to the values of particular period symbols $p_K(\zeta,\eta)$. Here, ζ,η are $\mathbb{Z}$-linear combinations of embeddings of K into $\mathbb{C}$ determined by the CM type of α. In the case $K=\mathbb{Q}(\sqrt{\Delta})$, it follows that all of the $p_K(\zeta,\eta)$, for different pairs (ζ,η), are equivalent up to algebraic factors, so we pick one of them and denote it p_K.

Proposition 5. *If Δ is a fundamental discriminant then the numbers $D(j)(z_\Delta)/\Omega_\Delta^2$, $P(z_\Delta)/\Omega_\Delta^4$, $Q(z_\Delta)/\Omega_\Delta^6$ and $R(z_\Delta)/\Omega_\Delta^{12}$ are algebraic.*

Proof. Using the properties of the period symbol p_K proven in [**14**, Theorem 1.2, p. 62], [**14**, (1.4), p. 63] and [**12**, Theorem 7.1], we see that $\pi p_K^2/\Omega_\Delta^2$ is algebraic, so the number $D(j)(z_\Delta)/\Omega_\Delta^2$ is algebraic. Using (12) and the fact that $j(z_\Delta)$ is algebraic, we see that the values of P, Q and R at z_Δ are algebraic multiples of powers of $D(j)(z_\Delta)$. Thus, we conclude that the four numbers are algebraic. □

4.2. *Gamma function identities*

In the next section we will encounter certain products of values of the Gamma function, and we will find the following identities useful.

Lemma 6. *We have*

$$\frac{\Gamma(3/4)\Gamma(13/24)\Gamma(17/24)}{\Gamma(1/4)\Gamma(19/24)\Gamma(23/24)} = 3^{1/4}(\sqrt{6}-2), \tag{17}$$

$$\frac{\Gamma(2/3)^3\Gamma(1/6)\Gamma(7/24)\Gamma(19/24)}{\Gamma(1/3)^3\Gamma(5/6)\Gamma(11/24)\Gamma(23/24)} = 2^{1/6}(\sqrt{6}-\sqrt{2}), \tag{18}$$

$$\frac{\Gamma(5/24)\Gamma(11/24)\Gamma(13/24)\Gamma(19/24)}{\Gamma(1/24)\Gamma(7/24)\Gamma(17/24)\Gamma(23/24)} = (\sqrt{2}-1)^2. \tag{19}$$

Proof. By the reflection formula [1, 6.1.17] and Gauss' multiplication formula [1, 6.1.20], we get

$$\begin{aligned}
&\frac{\Gamma(3/4)\Gamma(13/24)\Gamma(17/24)}{\Gamma(1/4)\Gamma(19/24)\Gamma(23/24)}\\
&\quad= \frac{\Gamma(1/24)\Gamma(13/24)}{\Gamma(1/12)}\frac{\Gamma(5/24)\Gamma(17/24)}{\Gamma(5/12)}\frac{\Gamma(1/12)\Gamma(5/12)\Gamma(3/4)}{\Gamma(1/4)}\\
&\frac{1}{\Gamma(1/24)\Gamma(23/24)}\frac{1}{\Gamma(5/24)\Gamma(19/24)}\\
&\quad= (\sqrt{2\pi}2^{5/12})(\sqrt{2\pi}2^{1/12})(2\pi 3^{1/4})\frac{\sin(\pi/24)}{\pi}\frac{\sin(5\pi/24)}{\pi}\\
&\quad= 3^{1/4}2^{3/2}(\cos(\pi/6)-\cos(\pi/4))\\
&\quad= 3^{1/4}(\sqrt{6}-2),
\end{aligned}$$

which proves (17). The proofs of (18) and (19) are similar. □

4.3. *Values of modular forms at elliptic fixed points*

In this section, we compute the exact values of the modular forms P, Q, R at the three elliptic points. We will use the triangle map (6) associated to the fundamental triangle described in Section 2.1 and the differential equations of Proposition 2.

THEOREM 7. *The values of P, Q and R at the elliptic points z_Δ are given in the following table.*

Δ	$\frac{\mathrm{Im}(z_\Delta)^2 P(z_\Delta)}{\Omega_\Delta^4}$	$\frac{\mathrm{Im}(z_\Delta)^3 Q(z_\Delta)}{\Omega_\Delta^6}$	$\frac{\mathrm{Im}(z_\Delta)^6 R(z_\Delta)}{\Omega_\Delta^{12}}$
-3	0	$\frac{1+i}{2^9 3}$	$-\frac{\sqrt{3}}{2^{17}3^2}$
-4	$\frac{i}{72}$	0	$\frac{1}{72^3}$
-24	$\frac{i\sqrt{3}}{12}$	$-\frac{1}{24}$	0

To prove this, we first need some preliminary results.

LEMMA 8. $(DP)(i) = 2iP(i)$.

Proof. The Atkin–Lehner action of w_2 on P given by (3) can be explicitly written as $(P\,|_4\,\gamma)(z) = -P(z)$, where $\gamma = 1-\mu = \left(\begin{smallmatrix}1 & 1\\ -1 & 1\end{smallmatrix}\right)$, that is,

$$\frac{4}{(1-z)^4}P\left(\frac{1+z}{1-z}\right) = -P(z).$$

Now applying D to this equation and evaluating at $z=i$ yields the result. □

LEMMA 9. *There are positive numbers ω_{-4}, ω_{-3} and ω_{-24}, such that*

$$P(z_4) = i\omega_{-4}^4, \quad Q(z_4)=0, \quad R(z_4) = \omega_{-4}^{12}, \tag{20}$$

$$P(z_3) = 0, \quad Q(z_3) = \frac{1+i}{\sqrt{2}}\omega_{-3}^6, \quad R(z_3) = -\sqrt{3}\omega_{-3}^{12}, \tag{21}$$

$$P(z_{24}) = \sqrt{3}i\omega_{-24}^4, \quad Q(z_{24}) = -\sqrt{3}\omega_{-24}^6, \quad R(z_{24}) = 0. \tag{22}$$

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Proof. The proof of (20) follows from Proposition 1.

To prove (21), let $\rho = e^{3\pi i/4}$, and consider the change of variables $u = \rho^{-1}((z-i)/(z+i))$, that is, $z = \gamma(u)$, where $\gamma = \left(\begin{smallmatrix} \rho i & i \\ -\rho & 1 \end{smallmatrix}\right)$. Let

$$u_{-3} = \rho^{-1}\frac{z_3 - i}{z_3 + i} = \frac{\sqrt{3}-1}{\sqrt{2}}.$$

Define functions $p(u) = -(P\,|_4\,\gamma)(u)$, $q(u) = i(Q\,|_6\,\gamma)(u)$, $r(u) = i(R\,|_{12}\,\gamma)(u)$ and $\varphi(u) = p'(u)/4p(u)$, for $|u| < 1$. Here, $'$ denotes the derivative with respect to u. By Propositions 2 and 3, we conclude that these functions satisfy the differential equations:

$$\begin{cases} \varphi' = \varphi^2 - 15\dfrac{q^2}{p^2}, \\ p' = 4\varphi p, \\ q' = 6\varphi q - 6\dfrac{r}{p}, \\ r' = 12\varphi r - 36\dfrac{q^3}{p}. \end{cases} \tag{23}$$

We have

$$p(u) = \frac{-4i}{(1-\rho u)^4} P\left(i\frac{1+\rho u}{1-\rho u}\right).$$

By Lemma 8 we get $p'(0) = 8\rho((DP)(i) - 2iP(i)) = 0$. We conclude that

$$[\varphi(0), p(0), q(0), r(0)] = [0, 4\omega_{-4}^4, 0, 64\omega_{-4}^{12}].$$

We hence have that the functions $\varphi(u)$, $p(u)$, $q(u)$ and $r(u)$ solve the ordinary differential equations (23) with real initial values at $u = 0$, hence these functions are real for u real. Furthermore, $p(u)$, $q(u)$ and $r(u)$ are non-zero on the interval $(0, u_{-3})$, and $r(0) > 0$ and $q'(0) = -96\omega_{-4}^8 < 0$, so we conclude that

$$r(u_{-3}) > 0, \quad q(u_{-3}) < 0.$$

Tracing the definitions, we get

$$\arg(R(z_3)) = \pi, \quad \arg(Q(z_3)) = \pi/4,$$

and, using (2), (21) follows.

The proof of (22) is analogous. □

Proof of Theorem 7. Consider the case $\Delta = -4$. Let T_{-4} be the hyperbolic triangle in $\mathcal{H}$ with vertices z_4, z_3 and z_{24}. We will define three maps v, τ, and s, which will be related by a commutative diagram.

First, consider the map $v : T_{-4} \to \mathcal{D}$ (where $\mathcal{D}$ is the unit disk with coordinate v) given by $v = v(z) = -(z-i)/(z+i)$. The image of T_{-4} is the hyperbolic triangle in $\mathcal{D}$ with vertices 0, B, C, where $B = e^{i\pi/4}(\sqrt{3}-1)/(\sqrt{2})$ and $C = \sqrt{3} - \sqrt{2}$.

Second, consider the map $\tau : T_{-4} \to \mathcal{H}$,

$$\tau(z) = -\frac{27}{16} j(z) = \frac{-3Q^4(z)}{P^6(z)}.$$

The map $z \mapsto \tau$ maps the interior of T_{-4} bijectively to the upper half plane, and the vertex z_4 maps to 0, z_3 to ∞ and z_{24} to 1.

Third, consider the triangle map $\zeta = s(\tau)$ from $\mathcal{H}$ to the triangle with

$$\alpha = 1/4, \quad \beta = 1/6, \quad \gamma = 1/2.$$

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We have $a = 1/24$, $b = 5/24$ and $c = 3/4$, so by (6),

$$\zeta = s(\tau) = \frac{\tau^{1/4} F(7/24, 11/24; 5/4; \tau)}{F(1/24, 5/24; 3/4; \tau)}. \tag{24}$$

The image of $\mathcal{H}$ is the triangle 0, $s(\infty)$, $s(1)$, where by (7)

$$s(1) = \frac{\Gamma(5/4)\Gamma(17/24)\Gamma(13/24)}{\Gamma(3/4)\Gamma(23/24)\Gamma(19/24)}.$$

Using (16) and (17), we get that $s(1)$ can be expressed as

$$s(1) = 3^{1/4}\sqrt{2}(\sqrt{3} - \sqrt{2})\Omega_{-4}^2.$$

It is clear that there exists a constant k such that the following diagram commutes.

$$\begin{tikzcd} T_{-4} \arrow[r, "v"] \arrow[d, "\tau"'] & \mathcal{D} \arrow[d, "v \mapsto kv"] \\ \mathcal{H} \arrow[r, "s"] & \mathbb{C} \end{tikzcd} \tag{25}$$

Since $v(z_{24}) = ks(1)$ we get

$$k = 3^{1/4}\sqrt{2}\Omega_{-4}^2.$$

Let S denote the inverse of s, which is defined on the interior of the orthogonal circle of the image triangle $s(\mathcal{H})$ (see for example [**9**, Section 5, Chapter VI]). By (24) we have

$$S(\zeta) = \zeta^4 + O(\zeta^8) \quad \text{when } \zeta \to 0. \tag{26}$$

By (25) we have

$$\tau(z) = \frac{-3Q^4(z)}{P^6(z)} = S(kv(z)). \tag{27}$$

Also, $(Dv)(z_4) = i/2$ and, by Proposition 1, $(DQ)(z_4) = 6R(z_4)/P(z_4)$.

Applying the operator D^4 to equation (27) and evaluating the result at the point $z = z_4$, we get (using (26))

$$\frac{-3(DQ)^4(z_4)}{P^6(z_4)} = \frac{k^4}{16},$$

which yields

$$\omega_{-4} = \frac{\Omega_{-4}}{72^{1/4}}.$$

Hence

$$P(z_4) = \frac{i\Omega_{-4}^4}{72}, \qquad Q(z_4) = 0, \qquad R(z_4) = \frac{\Omega_{-4}^{12}}{72^3}.$$

The proofs of the remaining cases are analogous, using (18) and (19). $\square$

5. *Taylor expansions*

In [**6**] Zagier describes the following construction of Taylor expansions of modular forms. Let z_0 be a point on $\mathcal{H}$ and let $w = (z - z_0)/(z - \bar{z}_0)$. Proposition 17 of [**6**] gives the Taylor expansion of a modular form at z_0 as follows.

Proposition 10. *Any modular form f of weight k has an expression of the form*

$$f(z) = (1 - w)^k \sum_{n=0}^{\infty} c_n \frac{w^n}{n!}$$

with $c_n = (2i\,\mathrm{Im}(z_0))^n (\partial^n(f))(z_0)$.

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Assume that z_Δ is a Δ CM point. We renormalize the Taylor expansions of Proposition 10 at the point z_Δ in the following way. Let

$$u = 2\Omega_\Delta^2 \frac{z - z_\Delta}{z - \bar{z_\Delta}},$$
$$a_n = \frac{i^n(\mathrm{Im}(z_\Delta))^{n+k/2}(\partial^n f)(z_\Delta)}{\Omega_\Delta^{k+2n}}, \tag{28}$$

for $n = 0, 1, 2, \ldots$. We get from Proposition 10 that for any modular form f of weight k we have

$$f(z) = (\mathrm{Im}(z_\Delta))^{k/2}\left(\frac{2i\Omega_\Delta}{z - \bar{z}_\Delta}\right)^k \sum_{n=0}^{\infty} a_n \frac{u^n}{n!}. \tag{29}$$

5.1. *Taylor expansion at elliptic points*

We now compute the expansions of P, Q, R and j at the three elliptic points using the values of ϕ^*, P, Q, R given in Theorem 7. Some related work on Taylor expansions of uniformizing functions has been done in [5].

THEOREM 11. *Let $u = 2\Omega_{-4}^2((z-i)/(z+i))$. The modular forms P, Q, and R have the following Taylor expansions at $z = i$:*

$$\begin{aligned}
P(z) &= i\left(\frac{2i\Omega_{-4}}{z+i}\right)^4 \sum_{n=0}^{\infty} p_n \frac{u^n}{n!}\\
&= i\left(\frac{2i\Omega_{-4}}{z+i}\right)^4 \left(\frac{1}{2^3 3^2} + \frac{5}{2^4 3^3}\frac{u^4}{4!} - \frac{5\cdot 17}{2^4 3^4}\frac{u^8}{8!} + \frac{5\cdot 317}{2^5 3^3}\frac{u^{12}}{12!} + \ldots\right),\\
Q(z) &= \left(\frac{2i\Omega_{-4}}{z+i}\right)^6 \sum_{n=0}^{\infty} q_n \frac{u^n}{n!}\\
&= \left(\frac{2i\Omega_{-4}}{z+i}\right)^6 \left(\frac{1}{2^5 3^3}u - \frac{7}{2^5 3^4}\frac{u^5}{5!} - \frac{7}{2^6 3}\frac{u^9}{9!} - \frac{7\cdot 41\cdot 61}{2^5 3^4}\frac{u^{13}}{13!} - \ldots\right),\\
R(z) &= \left(\frac{2i\Omega_{-4}}{z+i}\right)^{12} \sum_{n=0}^{\infty} r_n \frac{u^n}{n!}\\
&= \left(\frac{2i\Omega_{-4}}{z+i}\right)^{12} \left(\frac{1}{2^9 3^6} - \frac{13}{2^{10} 3^6}\frac{u^4}{4!} + \frac{11\cdot 13}{2^9 3^7}\frac{u^8}{8!} - \frac{13\cdot 19\cdot 487}{2^{10} 3^7}\frac{u^{12}}{12!} - \ldots\right),
\end{aligned}$$

where $p_n, q_n, r_n \in \mathbb{Z}[1/6]$ for all n. Furthermore, the Taylor expansion of $j(z)$ is

$$\begin{aligned}
j(z) = \sum_{n=0}^{\infty} j_n \frac{u^n}{n!} &= -2^5 3^{-1}\frac{u^4}{4!} + 2^6 3^{-2} 7\cdot 103\frac{u^8}{8!} - 2^4 11\cdot 67\cdot 797\frac{u^{12}}{12!}\\
&\quad + 2^7 7\cdot 41\cdot 1253951\frac{u^{16}}{16!} - 2^4 3^{-1} 19\cdot 4865571860153\frac{u^{20}}{20!}\\
&\quad + 2^5 3^{-1} 7\cdot 11\cdot 23\cdot 389\cdot 16729\cdot 82187899\frac{u^{24}}{24!} + \ldots,
\end{aligned}$$

where all $j_n \in \mathbb{Z}[1/6]$.

Proof. By (29) we have

$$P(z) = (2i\Omega_{-4}/(z+i))^4 \sum_{n=0}^{\infty} a_n \frac{u^n}{n!}, \quad \text{where } a_n = i^n(\partial^n P)(i)/\Omega_{-4}^{4+2n}.$$

By (14) we conclude that $\partial^n P \in \mathbb{Z}[\phi^*, P, 1/P, Q, R]$ for every n, and by Theorem 7 we have

$$[\phi^*(i), P(i), Q(i), R(i)] = [0, i\Omega_{-4}^4/72, 0, \Omega_{-4}^{12}/72^3],$$

so all $a_n \in \mathbb{Z}[i, 1/6]$. Now by Proposition 1 we have $P(z) \in i\mathbb{R}$ for all z on the imaginary axis, from which we conclude that $a_n \in i\mathbb{Z}[1/6]$. This proves that all $p_n \in \mathbb{Z}[1/6]$. The proofs of the expansions of Q and R are similar.

The 6-integrality of the expansion of $j = 16Q^4/(9P^6)$ follows immediately from the 6-integrality of p_n and q_n and the fact that p_0 is a unit in $\mathbb{Z}[1/6]$. □

REMARK 1. We have not been able to find any way of renormalizing the expansions such that $j_n \in \mathbb{Z}$ for all n without causing the valuations of j_n at 2 and 3 to grow linearly in n.

REMARK 2. Given the numerical data of Theorem 11 we conjecture that for $n \geqslant 2$, the Taylor coefficients of P, Q and R satisfy the congruences $p_n \equiv 0 \pmod 5$, $q_n \equiv 0 \pmod 7$ and $r_n \equiv 0 \pmod{13}$. This conjecture will be generalized and proved in a forthcoming article.

Results analogous to Theorem 11 on Taylor expansions at the other elliptic points z_{24} and z_3 are straightforward to get. We note that using (12), the Taylor expansions of P, Q and R can be reconstructed from the Taylor expansion of j, so it is sufficient to provide the latter.

Let $u = 2\Omega_{-24}^2((z - z_{24})/(z - \bar{z}_{-24}))$. The Taylor series of $j(z)$ at z_{24} is

$$\begin{aligned} j(z) = &-2^4 3^{-3} + 2^7 3^{-1}\frac{u^2}{2!} - 2^8 3^{-1} 103\frac{u^4}{4!} + 2^6 3^{-1} 271 \cdot 521\frac{u^6}{6!} - 2^9 2943869\frac{u^8}{8!} \\ &+ 2^6 3^2 197 \cdot 9130237\frac{u^{10}}{10!} - 2^7 3^2 \cdot 811507890107\frac{u^{12}}{12!} \\ &+ 2^5 3^3 13 \cdot 17 \cdot 19 \cdot 294968136517\frac{u^{14}}{14!} \\ &- 2^{10} 3^5 47 \cdot 461 \cdot 280859014453\frac{u^{16}}{16!} + 2^6 3^6 17 \cdot 2677 \cdot 1221672251883949\frac{u^{18}}{18!} + \dots. \end{aligned}$$

This yields, for example,

$$P(z) = i\sqrt{3}\,\mathrm{Im}(z_{24})^2 \left(\frac{2i\Omega_{-24}}{z - \bar{z}_{-24}}\right)^4 \left(\frac{1}{2^2 3} + \frac{5}{2^2 3}\frac{u^2}{2!} - \frac{3^2 5}{2^3}\frac{u^4}{4!} + \frac{3 \cdot 5 \cdot 37}{2^2}\frac{u^6}{6!} + \dots\right).$$

Finally, the Taylor series for $1/j$ at z_3, where $u = e^{\pi i/12} 2\Omega_{-3}^2((z - z_3)/(z - \bar{z}_{-3}))$, is

$$\begin{aligned} \frac{1}{j(z)} = &\, 2^{-5} 3^5 5\frac{u^6}{6!} + 2^{-7} 3^8 \cdot 5 \cdot 11 \cdot 113\frac{u^{12}}{12!} + 2^{-11} 3^{12} 5 \cdot 17 \cdot 151 \cdot 9521\frac{u^{18}}{18!} \\ &+ 2^{-12} 3^{13} 5 \cdot 11 \cdot 23 \cdot 31 \cdot 13183 \cdot 88397\frac{u^{24}}{24!} \\ &+ 2^{-19} 3^{17} 5^3 29 \cdot 263 \cdot 1181 \cdot 136672684427\frac{u^{30}}{30!} + \dots, \end{aligned}$$

which yields

$$P(z) = e^{\pi i/6}\sqrt{6}\,\mathrm{Im}(z_3)^2 \left(\frac{2i\Omega_{-3}}{z - \bar{z}_{-3}}\right)^4 \left(\frac{1}{2^7 3}u + \frac{3 \cdot 5}{2^{12}}\frac{u^7}{7!} - \frac{3^4 5 \cdot 83}{2^{15}}\frac{u^{13}}{13!} + \dots\right).$$

5.2. *Explicit recursion formulae*

To show how to explicitly compute Taylor expansions, we derive the formulae for getting the Taylor coefficients p_n of $P(z)$ at $z = i$. The other cases are similar.

To do this, we will need yet another differential operator. For details about this we refer to [**6**, § 5]. For positive even integers k, consider the spaces $\hat{M}_k$ of meromorphic modular

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forms $f \in \mathbb{C}[P, P^{-1}, Q, R]$ such that $f|_k \gamma = f$ for all $\gamma \in \Gamma$. Define differential operators $\vartheta = \vartheta_k : \hat{M}_k \to \hat{M}_{k+2}$, by

$$\vartheta_k f = Df - k\phi f.$$

We get

$$\vartheta P = 0, \qquad \vartheta Q = \frac{6R}{P}, \qquad \vartheta R = -36\frac{Q^3}{P}.$$

Furthermore, define $\vartheta^{[n]} : \hat{M}_k \to \hat{M}_{k+2n}$ for $n = 0, 1, 2, \ldots$, by the recursion

$$\vartheta^{[0]} f = f, \qquad \vartheta^{[1]} f = \vartheta f, \qquad \vartheta^{[n+1]} f = \vartheta(\vartheta^{[n]} f) + n(k+n-1)15\frac{Q^2}{P^2}\vartheta^{[n-1]} f.$$

Now, since $\varphi^*(i) = 0$ we get by [6, (68)], that

$$\partial^n P(i) = \vartheta^{[n]} P(i)$$

for all n. Writing $\vartheta^{[n]} P = F_n(P, Q) + G_n(P, Q)R$, we get by (28) that

$$ip_n = \frac{i^n(\vartheta^{[n]}P)(i)}{\Omega_{-4}^{4+2n}} = \frac{i^n}{\Omega_{-4}^{4+2n}}\left(F_n\left(\frac{i\Omega_{-4}^4}{72}, 0\right) + \frac{1}{72^3\Omega_{-4}^{12}} G_n\left(\frac{i\Omega_{-4}^4}{72}, 0\right)\right).$$

By induction we get $G_{2n} = F_{2n+1} = 0$ for all n, and using the homogeneity $F_n(\lambda^4 P, \lambda^6 Q) = \lambda^{4+2n} F_n(P, Q)$, we get the non-zero coefficients $p_{4m} = (-1)^m 72^{-1-2m} F_{4m}(1, 0)$. Introducing the polynomials $f_n(x) = F_n(1, x)$, $g_n(x) = G_n(1, x)$ and working out the recursion, we get

$$p_{4m} = \frac{(-1)^m}{72^{1+2m}} f_{4m}(0),$$

where $f_0(x) = 1$, $f_1(x) = g_0(x) = g_1(x) = 0$ and

$$\begin{aligned} f_{n+1}(x) &= -36x^3 g_n(x) - 6(1+3x^4) g_n'(x) + 15n(n+3)x^2 f_{n-1}(x), \\ g_{n+1}(x) &= 6f_n'(x) + 15n(n+3)x^2 g_{n-1}(x). \end{aligned}$$

5.3. *Modular polynomials*

As an application, we will show how one can compute modular polynomials using Taylor expansions. Let l be a prime not dividing 6. The polynomial $\Phi_l(x, y)$, which satisfies $\Phi_l(j(\alpha z), j(z)) = 0$ for any $\alpha \in \Lambda$ with $\mathrm{nr}(\alpha) = l$, is called the modular polynomial of level l.

Let S_{-4} be the set of primes of the form $x^2 + y^2$, S_{-3} denote the set of primes of the form $x^2 + xy + y^2$, and S_{-24} the set of primes of the form $x^2 + 6y^2$. We see that if $l \in S_\Delta$, then there exists an element $\alpha_l \in \Lambda$ with $\mathrm{nr}(\alpha) = l$ and $\alpha(z_\Delta) = z_\Delta$. To find the modular polynomial $\Phi_l(x, y)$, we compute as many coefficients of the expansion of $j(z)$ and $j(\alpha(z))$ respectively at z_Δ as is necessary to uniquely determine the coefficients of $\Phi_l(x, y)$. For example, if $l \in S_{-4}$, then $l = a^2 + b^2$ for some $a, b \in \mathbb{Z}$, and for $\alpha = a + b\mu$ we have

$$j(\alpha z) = \sum_{n=0}^{\infty} j_n \left(\frac{a - ib}{a + ib}\right)^n \frac{u^n}{n!},$$

with notation as in Theorem 11. The polynomials $\Phi_5(x, y)$ and $\Phi_{13}(x, y)$ are (implicitly) given in [7], but, for example, $\Phi_{17}(x, y)$ can be determined using this method. The actual polynomial

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is too big to print, but we get

$$\begin{aligned}\Phi_{17}(x,x) = {}& 3^{18}2^{42}x^2(64x-81)^2(729x+3136)^2(1728x+2401)(15625x+5184)^2\\ & (91125x-38416)^2(421875x+30976)^2(421875x+11478544)^2\\ & (1000000x-194481)^2(1771561x-248935680)^2(27000000x+312900721)^2\\ & (64000000x-2847396321)^2(1510977796875x+519433145237776)\\ & (20179187015625x+19244645944384)^2\\ & (8703679193x^2+15700809312x+24591257856)\\ & (19098395217x^2-325029024x+4036823296)^2\\ & (2985984000000x^2+480450744000x+120354180241)^2,\end{aligned}$$

giving the j-invariants which are self 17-isogenous. We also get the Kronecker congruence

$$\Phi_{17}(x,y) \equiv (x^{17}-y)(y^{17}-x) \pmod{17}.$$

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