ON THE BALAYAGE FOR LOGARITHMIC POTENTIALS

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To Professor KIYOSHI NOSHIRO on the occasion of his 60th birthday

In this paper, we shall consider the logarithmic potential

$$U^{\mu}(P) = \int \log \frac{1}{PQ} d\mu(Q),$$

where μ is a positive measure in the plane, P and Q are any points and PQ denotes the distance from P to Q. In general, consider the potential

$$K(P, \mu) = \int K(P, Q) d\mu(Q)$$

of a positive measure μ taken with respect to a kernel $K(P, Q)$ which is a continuous function in P and Q and may be $+\infty$ for $P = Q$. A kernel $K(P, Q)$ is said to satisfy the balayage principle if, given any compact set F and any positive measure μ with compact support, there exists a positive measure μ' supported by F such that $K(P, \mu') = K(P, \mu)$ on F with a possible exception of a set of K-capacity zero and $K(P, \mu') \leqq K(P, \mu)$ everywhere. A kernel $K(P, Q)$ is said to satisfy the equilibrium principle if, given any compact set F, there exists a positive measure λ supported by F such that $K(P, \lambda) = V$ (a constant) on F with a possible exception of a set of K-capacity zero and $K(p, \lambda) \leqq V$ everywhere. The logarithmic kernel

$$K(P, Q) = \log \frac{1}{PQ}$$

satisfies the equilibrium principle in the plane, but it does not satisfy the balayage principle in the above form. As is well-known, given any compact set F and any point M of the complement CF of F, there exist a positive measure ε' supported by F with total mass 1 and a non-negative constant γ such that

(1) $U^{\varepsilon'}(P) = \log \dfrac{1}{MP} + \gamma$ on F with a possible exception of a set of logarithmic capacity zero, and

Received July 1, 1966.

(2) $U^{\varepsilon'}(P) \leqq \log \frac{1}{MP} + \gamma$ everywhere.

Here, the constant γ does not always reduce to zero. The balayage for logarithmic potentials has been studied in detail in the book of C. de la Vallée Poussin ([2]). In the present paper, we shall study it in a more general case. Namely we shall try to balayage any positive measure onto any closed set.

We shall deal with the positive measures whose logarithmic potentials are never $-\infty$. The total mass of such a positive measure is naturally finite. The logarithmic potential of such a positive measure is superharmonic in the plane and is harmonic outside the support of the measure. Let us recall the definition of the logarithmic capacity $C(F)$ of a compact set F. Putting

$$V = \inf_{\mu} \sup_{P} U^{\mu}(P) \quad \text{and} \quad W = \inf_{\mu} \int U^{\mu} d\mu$$

for any positive measure μ supported by F with total mass 1, we have always $V = W$. The logarithmic capacity is given by $C(F) = e^{-V} = e^{-W}$ if $V = W < +\infty$ and by $C(F) = 0$ if $V = W = +\infty$.

We have the following theorem.

THEOREM 1. *Given any closed set F containing a compact set of positive logarithmic capacity and any positive measure μ with total mass* 1, *there exist a positive measure μ' supported by F with total mass* 1 *and a non-negative constant γ_{μ} such that*

(1) $U^{\mu'}(P) = U^{\mu}(P) + \gamma_{\mu}$ *on F with a possible exception of a set of logarithmic capacity zero, and*

(2) $U^{\mu'}(P) \leqq U^{\mu}(P) + \gamma_{\mu}$ *everywhere.*

We shall call μ' a balayaged measure of μ onto F. We can construct a balayaged measure such that the reciprocal relation always holds:

(3) $\int (U^{\mu'} - \gamma_{\mu}) d\nu = \int (U^{\nu'} - \gamma_{\nu}) d\mu$ *for any positive measure μ with total mass* 1 *and any positive measure ν of finite logarithmic energy with total mass* 1, *where μ' and ν' are their balayaged measures and γ_{μ} and γ_{ν} are their associated constants.*

Under this additional condition, a balayaged measure is unique.

Proof. We are going to prove the theorem by dividing the proof into several steps.

[I] The case where F is compact and the support of μ is a compact set which has no intersection with F.

https://doi.org/10.1017/S002776300002434X Published online by Cambridge University Press

Let us consider the Gauss variation

$$G(\nu) = \iint \log \frac{1}{PQ}\, d\nu(Q)\, d\nu(P) - 2\int U^{\mu}(P)\, d\nu(P)$$

for any positive measure ν supported by F. Put

$$G^* = \inf_{\nu} G(\nu)$$

for the positive measures ν supported by F with total mass 1. There exists a sequence of positive measures ν_n supported by F with total mass 1 such that $G(\nu_n) \downarrow G^*$. We may suppose that $\{\nu_n\}$ is a vaguely convergent sequence by selecting a partial sequence in advance if necessary. The limiting measure μ' is a positive measure supported by F with total mass 1. As $U^{\mu}(P)$ is a finite and continuous function on F, we have

$$G^* \leqq G(\mu') \leqq \underline{\lim}_{n\to+\infty} G(\nu_n) = G^*.$$

So, we have $G^* = G(\mu')$. As is well-known ([1], § 37), in putting

$$\gamma = \int_F (U^{\mu'} - U^{\mu}) d\mu',$$

we have

(1) $U^{\mu'}(P) \geqq U^{\mu}(P) + \gamma$ on F with a possible exception of a set of logarithmic capacity zero, and

(2) $U^{\mu'}(P) \leqq U^{\mu}(P) + \gamma$ on the support of μ'.

Let us show that the latter inequality holds everywhere. In fact, the function

$$f(P) = U^{\mu'}(P) - U^{\mu}(P) - \gamma$$

is subharmonic in each component of the complement CF' of the support F' of μ', and we have

$$\overline{\lim}_{P\to M} U^{\mu'}(P) \leqq \overline{\lim}_{Q\to M} U^{\mu'}(Q)$$

at each boundary point M of F', P being points of CF' and Q being points of F'. This is owing to the fact that the logarithmic kernel satisfies the maximum principle: the inequality $U^{\lambda}(P) \leqq K$ (a constant) on the support of a positive measure λ induces the same inequality everywhere. $U^{\mu}(P)$ being finite and continuous in a neighbourhood of F', we have

$$\overline{\lim}_{P\to M} f(P) \leqq \overline{\lim}_{Q\to M} f(Q) \leqq 0$$

at each boundary point M of F'. Furthermore, let us notice that $\gamma \geqq 0$. It is because we have

$$\gamma \geqq \int (U^{\mu'} - U^{\mu}) d\lambda = \int U^{\lambda} d\mu' - \int U^{\lambda} d\mu \geqq 0$$

for the equilibrium measure λ with total mass 1 on F'. So, we have

$$\overline{\lim_{P\to\infty}} f(P) = -\gamma \leqq 0$$

which is due to $\int d\mu' = \int d\mu$. Therefore, we have $f(P) \leqq 0$ in each component of CF'. Hence, we have

(1) $U^{\mu'}(P) = U^{\mu}(P) + \gamma$ on F with a possible exception of a set of logarithmic capacity zero, and

(2) $U^{\mu'}(P) \leqq U^{\mu}(P) + \gamma$ everywhere.

It is sufficient to put $\gamma_{\mu} = \gamma$. Let us remark that this balayaged measure μ' is of finite logarithmic energy.

[II] The case where F is compact and $\mu(F) = 0$.

μ is supported by the complement CF of F. Let D_0 be a large disk containing F, and $\{D_n\}$ and $\{D_{-n}\}$ be two sequences of bounded open sets such that

$$D_0 \supset D_{-1} \supset D_{-2} \supset \cdots \supset D_{-n} \supset \cdots \to F$$

and

$$D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots \to \text{the whole plane.}$$

Let μ_n be the restricted measure of μ to

$$E_n = D_n - D_{n-1} \qquad (n = \pm 1, \ \pm 2, \ \pm 3, \ldots).$$

The support of μ_n is a compact set which has no intersection with F. Let a_n be the total mass of μ_n and μ'_n be a balayaged measure, with total mass a_n, of μ_n onto F. We have with a non-negative constant γ_{μ_n}

(1) $U^{\mu'_n}(P) = U^{\mu_n}(P) + \gamma_{\mu_n}$ on F with a possible exception of a set of logarithmic capacity zero, and

(2) $U^{\mu'_n}(P) \leqq U^{\mu_n}(P) + \gamma_{\mu_n}$ everywhere.

As we have $\mu = \sum \mu_n$ and the measure $\mu' = \sum \mu'_n$ is a positive measure supported by F with total mass 1, the series

$$\sum_{n=-\infty}^{+\infty} \gamma_{\mu_n}$$

is convergent. Denoting by γ_μ ($\geqq 0$) the sum of that series, we have

(1) $U^{\mu'}(P) = U^{\mu}(P) + \gamma_\mu$ on F with a possible exception of a set of logarithmic capacity zero, and

(2) $U^{\mu'}(P) \leqq U^{\mu}(P) + \gamma_\mu$ everywhere.

Let us remark that this balayaged measure μ' is the sum of positive measures of finite logarithmic energy.

[III] The case where F is compact and $\mu(CF) = 0$.

The support of μ is a compact subset of F. Taking a larger number R than the diameter of F, put

$$U_R^{\mu}(P) = \int \left(\log \frac{1}{PQ} - \log \frac{1}{R}\right) d\mu(Q).$$

We have

$$\log \frac{1}{PQ} - \log \frac{1}{R} > 0 \text{ and } U_R^{\mu}(P) > 0$$

for any points P and Q of F. Let

$$G_n = \{P\ ;\ U_R^{\mu}(P) > n\} \text{ and } F_n = F - G_n,$$

and μ_{1n} and μ_{2n} be the restricted measures of μ to F_n and G_n respectively. As we have

$$U_R^{\mu_{1n}}(P) \leqq n \text{ and } U_R^{\mu_{2n}}(P) \leqq n$$

on F_n, we have

$$\int U^{\mu_{1n}} d\mu_{1n} - \log \frac{1}{R}\left(\int d\mu_{1n}\right)^2 \leqq n \cdot \int d\mu_{1n}$$

and

$$U^{\mu_{2n}}(P) - \log \frac{1}{R}\left(\int d\mu_{2n}\right) \leqq n \text{ on } F_n.$$

So, μ_{1n} is of finite logarithmic energy and the logarithmic potential of μ_{2n} is bounded on F_n. Let a_n be the total mass of μ_{2n} and μ'_{2n} be a balayaged measure, with total mass a_n, of μ_{2n} onto F_n. We have with a non-negative constant γ_{μ_n}

(1) $U^{\mu'_{2n}}(P) = U^{\mu_{2n}}(P) + \gamma_{\mu_n}$ on F_n with a possible exception of a set of loga-

https://doi.org/10.1017/S002776300002434X Published online by Cambridge University Press

rithmic capacity zero, and

(2) $U^{\mu'_n}(P) = U^{\mu_{2n}}(P) + \gamma_{\mu_n}$ everywhere.

The measure

$$\mu'_n = \mu_{1n} + \mu'_{2n}$$

is a positive measure supported by F_n with total mass 1 and is of finite logarithmic energy. We have

(1) $U^{\mu'_n}(P) = U^{\mu}(P) + \gamma_{\mu_n}$ on F_n with a possible exception of a set of logarithmic capacity zero, and

(2) $U^{\mu'_n}(P) \leqq U^{\mu}(P) + \gamma_{\mu_n}$ everywhere.

Let us prove that $U^{\mu'_n}(P) - \gamma_{\mu_n}$ increases with n everywhere. Let P be any point of CF_n, ε'_n be a balayaged measure of the Dirac measure ε at P onto F_n and $\gamma_{\varepsilon n}$ be an associated non-negative constant. ε'_n and μ'_n being of finite logarithmic energy, we have

$$\begin{aligned} U^{\mu'_n}(P) - \gamma_{\mu_n} &= \int U^{\varepsilon} d\mu'_n - \gamma_{\mu n} \\ &= \int (U^{\varepsilon'_n} - \gamma_{\varepsilon n}) d\mu'_n - \gamma_{\mu n} \\ &= \int (U^{\mu'_n} - \gamma_{\mu n}) d\varepsilon'_n - \gamma_{\varepsilon n} \\ &= \int (U^{\mu'_{n+1}} - \gamma_{\mu(n+1)}) d\varepsilon'_n - \gamma_{\varepsilon n} \\ &= \int (U^{\varepsilon'_n} - \gamma_{\varepsilon n}) d\mu'_{n+1} - \gamma_{\mu(n+1)} \\ &= \int U^{\varepsilon} d\mu'_{n+1} - \gamma_{\mu(n+1)} = U^{\mu'_{n+1}}(P) - \gamma_{\mu(n+1)}. \end{aligned}$$

The required inequality holds on F_n with a possible exception of a set of logarithmic capacity zero. It holds everywhere on account of the superharmonicity of logarithmic potentials. We may suppose that $\{\mu'_n\}$ is a vaguely convergent sequence by selecting its partial sequence in advance if necessary. The limiting measure μ' is a positive measure supported by F with total mass 1, and we have

$$U^{\mu'}(P) \leqq \varliminf_{n\to+\infty} U^{\mu'_n}(P)$$

everywhere, the equality holding with a possible exception of a set of logarithmic capacity zero. So, the sequence $\{\gamma_{\mu n}\}$ is convergent. Its limit γ_{μ} is a non-negative constant. The logarithmic capacity of $G_n = F - F_n$ decreasing to zero, we have

(1) $U^{\mu'}(P) = U^{\mu}(P) + \gamma_\mu$ on F with a possible exception of a set of logarithmic capacity zero, and

(2) $U^{\mu'}(P) \leqq U^{\mu}(P) + \gamma_\mu$ everywhere.

Let us remark that this balayaged measure μ' is the vague limit of a sequence of positive measures μ'_n with total mass 1, which are supported by F and of finite logarithmic energy, and which satisfy

$$U^{\mu'_n}(P) - \gamma_{\mu n} \uparrow U^{\mu'}(P) - \gamma_\mu$$

everywhere with a convergent sequence $\{\gamma_{\mu n}\}$ of non-negative numbers and its limit γ_μ.

[IV] The case where F is compact and μ is any positive measure.

Let μ_1 and μ_2 be the restricted measures of μ to F and to CF respectively, a_1 and a_2 be their total masses respectively and μ'_1 and μ'_2 be balayaged measures, with total masses a_1 and a_2, of μ_1 and μ_2 onto F respectively. The measure $\mu' = \mu'_1 + \mu'_2$ is evidently a balayaged measure of μ onto F.

[V] The reciprocal relation in case F is compact.

We are going to prove that the reciprocal relation holds for balayaged measures obtained above. Let μ be any positive measure with total mass 1 and ν be any positive measure of finite logarithmic energy with total mass 1. As stated above, there are three cases for a balayaged measure μ' of μ onto F:

(1) It is a positive measure with total mass 1 supported by F and of finite logarithmic energy,

(2) It is the sum of positive measures μ'_n supported by F and of finite logarithmic energy,

(3) It is the vague limit of a sequence of positive measures μ'_n with total mass 1 which are supported by F and of finite logarithmic energy and which satisfy

$$U^{\mu'_n}(P) - \gamma_{\mu n} \uparrow U^{\mu'}(P) - \gamma_\mu$$

everywhere with a convergent sequence $\{\gamma_{\mu n}\}$ of non-negative numbers and its limit γ_μ.

Since ν' is of finite logarithmic energy, we have

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$$\int (U^{\nu'} - \gamma_\nu) d\mu = \int U^\mu d\nu' - \gamma_\nu = \int (U^{\mu'} - \gamma_\mu) d\nu' - \gamma_\nu = \int U^{\mu'} d\nu' - \gamma_\mu - \gamma_\nu.$$

On the other hand, it is easy to prove that

$$\int (U^{\mu'} - \gamma_\mu) d\nu = \int U^{\mu'} d\nu' - \gamma_\mu - \gamma_\nu.$$

For example, in cases (3) we have

$$\int (U^{\mu'} - \gamma_\mu) d\nu = \lim_{n\to+\infty} \int (U^{\mu'_n} - \gamma_{\mu n}) d\nu$$
$$= \lim_{n\to+\infty} \int U^\nu d\mu'_n - \gamma_\mu = \lim_{n\to+\infty} \int (U^{\nu'} - \gamma_\nu) d\mu'_n - \gamma_\mu$$
$$= \lim_{n\to+\infty} \int U^{\mu'_n} d\nu' - \gamma_\mu - \gamma_\mu$$
$$= \lim_{n\to+\infty} \int (U^{\mu'_n} - \gamma_{\mu n} + \gamma_{\mu n}) d\nu' - \gamma_\mu - \gamma_\nu$$
$$= \int (U^{\mu'} - \gamma_\mu) d\nu' - \gamma_\nu = \int U^{\mu'} d\nu' - \gamma_\mu - \gamma_\nu.$$

It is proved similarly in cases (1) and (2).

[VI] The case where F is a non-compact closed set and μ is any positive measure.

Let S_n be a closed disk of radius n centered at the origin, μ'_n be a balayaged measure of μ onto $F_n = F \cdot S_n$ and $\gamma_{\mu n}$ be the associated non-negative constant. First, let us prove

$$U^{\mu'_1}(P) - \gamma_{\mu 1} \leqq U^{\mu'_2}(P) - \gamma_{\mu 2} \leqq U^{\mu'_3}(P) - \gamma_{\mu 3} \leqq \cdots \to U^\mu(P)$$

everywhere. Let P be any point of CF_n, λ be the circular measure with total mass 1 on a small circle, outside F_n, with the center at P, λ'_n be a balayaged measure of λ onto F_n and $\gamma_{\lambda n}$ be an associated constant. Since both λ and λ'_n are of finite logarithmic energy, we have

$$U^{\mu'_n}(P) - \gamma_{\mu n} = \int (U^{\mu'_n} - \gamma_{\mu n}) d\lambda = \int (U^{\lambda'_n} - \gamma_{\lambda n}) d\mu$$
$$= \int U^\mu d\lambda'_n - \gamma_{\lambda n} = \int (U^{\mu'_n} - \gamma_{\mu n}) d\lambda'_n - \gamma_{\lambda n}$$
$$= \int (U^{\mu'_{n+1}} - \gamma_{\mu(n+1)}) d\lambda'_n - \gamma_{\lambda n}$$
$$= \int U^{\lambda'_n} d\mu'_{n+1} - \gamma_{\lambda n} - \gamma_{\mu(n+1)}$$

$$\leqq \int (U^{\lambda} + \gamma_{\lambda n}) d\mu'_{n+1} - \gamma_{\lambda n} - \gamma_{\mu(n+1)}$$

$$= \int (U^{\mu'_{n+1}} - \gamma_{\mu(n+1)}) d\lambda = U^{\mu'_{n+1}}(P) - \gamma_{\mu(n+1)}.$$

The required inequality holds on F_n with a possible exception of a set of logarithmic capacity zero. It holds everywhere on account of the superharmonicity of logarithmic potentials. By the integration with respect to the circular measure λ with total mass 1 on a large circle of radius R and with center at the origin, we have

$$\log \frac{1}{R} - \gamma_{\mu n} \leqq \log \frac{1}{R} - \gamma_{\mu(n+1)}.$$

So, the sequence $\{\gamma_{\mu n}\}$ decreases to a non-negative number δ_{μ} with $1/n$ and $\lim U^{\mu'_n}(P) > -\infty$ exists everywhere. Next, we choose a vaguely convergent subsequence of $\{\mu'_n\}$. It will be denoted again by $\{\mu'_n\}$. As $\{U^{\mu'_n}(P) - \gamma_{\mu n}\}$ is a sequence of superharmonic functions monotone increasing with n and the limiting function is not identically equal to $+\infty$, it converges to a superharmonic function. Consequently $\lim U^{\mu'_n}(P)$ is superharmonic. Take an increasing sequence $\{R_k\}$ of numbers such that each closed disk S_k of radius R_k centered at the origin has no positive mass for μ' on its boundary. We have

$$\lim_{n\to+\infty} \int_{S_k} \log \frac{1}{PQ} d\mu'_n(Q) = \int_{S_k} \log \frac{1}{PQ} d\mu'(Q)$$

in the plane with a possible exception of a set of logarithmic capacity zero for each k. Let M be a point inside S_1 at which the limit exists for all k. Since

$$\lim_{n\to+\infty} U^{\mu'_n}(M)$$

exists,

$$\lim_{n\to+\infty} \int_{CS_k} \log \frac{1}{MQ} d\mu'_n(Q)$$

exists for each k. This increases to a non-positive finite value as $k \to +\infty$. We shall denote it by α. Take any compact set K which contains a point M. We have

$$\left| \int_{CS_k} \log \frac{1}{PQ} d\mu'_n(Q) - \int_{CS_k} \log \frac{1}{MQ} d\mu'_n(Q) \right| \leqq \int_{CS_k} \left| \log \frac{MQ}{PQ} \right| d\mu'_n(Q)$$

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for any point P of K if $K\subset S_k$. If R_k is large, $|\log MQ/PQ|$ is arbitrarily small for all Q in CS_k. Hence, given $\varepsilon>0$, there are n_0 and k_0 such that

$$\left|\int_{CS_k}\log\frac{1}{PQ}\,d\mu_n'(Q)-\alpha\right|<\varepsilon \text{ for } k\geqq k_0 \text{ and } n\geqq n_0.$$

As we have

$$\lim_{n\to+\infty}\int_{S_k}\log\frac{1}{PQ}\,d\mu_n'(Q)=\int_{S_k}\log\frac{1}{PQ}\,d\mu'(Q)$$

in the plane with a possible exception of a set of logarithmic capacity zero, we have

$$\left|\lim_{n\to+\infty}\left(U^{\mu_n'}(P)-\int_{S_k}\log\frac{1}{PQ}\,d\mu'(Q)-\alpha\right)\right|=\left|\lim_{n\to+\infty}\int_{CS_k}\log\frac{1}{PQ}\,d\mu_n'(Q)-\alpha\right|<\varepsilon$$

if k is sufficiently large, where $\varepsilon>0$ is given. This shows that $U^{\mu'}(P)=\int\log 1/PQ\,d\mu'(Q)$ exists and equals $\lim U^{\mu_n'}(P)-\alpha$ on K and hence in the whole plane with a possible exception of a set of logarithmic capacity zero. Since $\lim U^{\mu_n'}(P)$ is superharmonic in the plane, the equality holds without exception. We recall that $U^{\mu_n'}(P)-\gamma_{\mu n}\leqq U^{\mu}(P)$ in the plane with the equality holding on F possibly except for a set of logarithmic capacity zero. Now we have

(1) $U^{\mu'}(P)-\gamma_\mu=U^\mu(P)$ on F with a possible exception of a set of logarithmic capacity zero, where $\gamma_\mu=\delta_\mu-\alpha\geqq 0$, and

(2) $U^{\mu'}(P)-\gamma_\mu\leqq U^\mu(P)$ everywhere.

We remark that the total mass of μ' is one. To prove it we use the fact that

$$\alpha=\lim_{k\to+\infty}\lim_{n\to+\infty}\int_{CS_k}\log\frac{1}{MQ}\,d\mu_n'(Q)$$

is a finite value. Since $MQ\geqq R_k/2$ on CS_k if k is large,

$$\alpha\leqq\lim_{k\to+\infty}\lim_{n\to+\infty}\log\,(2/R_k)\,\mu_n'(CS_k).$$

This shows that $\lim_{k\to+\infty}\lim_{n\to+\infty}\mu_n'(CS_k)=0$, whence the total mass of μ' is one.

[VII] The reciprocal relation in case F is a non-compact closed set and the uniqueness of balayaged measures.

We are going to prove that the reciprocal relation holds for balayaged measures obtained above. Let μ be any positive measure with total mass 1 and ν be a positive measure of finite logarithmic energy with total mass 1. Let $\{\mu_n'\}$ and $\{\nu_n'\}$ be the sequences of balayaged measures of μ and ν onto F_n

respectively and $\{\gamma_{\mu n}\}$ and $\{\gamma_{\nu n}\}$ be the sequences of their associated non-negative constants. We have as stated in [V]

$$\int (U^{\mu'_n} - \gamma_{\mu n})\, d\nu = \int (U^{\nu'_n} - \gamma_{\nu n})\, d\mu.$$

As we have

$$U^{\mu'_n}(P) - \gamma_{\mu n} \uparrow U^{\mu'_n}(P) - \gamma_\mu$$

and

$$U^{\nu'_n}(P) - \gamma_{\nu n} \uparrow U^{\nu'_n}(P) - \gamma_\nu$$

everywhere, we have

$$\int (U^{\mu'} - \gamma_\mu)\, d\nu = \int (U^{\nu'} - \gamma_\nu)\, d\mu.$$

Finally, let us consider the uniqueness of balayaged measures. Let μ' and μ'' be balayaged measures of μ onto F. Suppose that

(1) $U^{\mu'}(P) = U^{\mu}(P) + \gamma'_\mu$ and $U^{\mu''}(P) = U^{\mu}(P) + \gamma''_\mu$ on F with a possible exception of a set of logarithmic capacity zero, and

(2) $U^{\mu'}(P) \leqq U^{\mu}(P) + \gamma'_\mu$ and $U^{\mu''}(P) \leqq U^{\mu}(P) + \gamma''_\mu$ everywhere, γ'_μ and γ''_μ being non-negative constants.

For the circular measure λ with total mass 1 on any closed circle centered at any point P, we have

$$\int (U^{\lambda'} - \gamma_\lambda)\, d\mu = \int (U^{\mu'} - \gamma'_\mu)\, d\lambda = \int (U^{\mu''} - \gamma''_\mu)\, d\lambda.$$

So, we have

$$\int (U^{\mu'} - U^{\mu''})\, d\lambda = \gamma'_\mu - \gamma''_\mu,$$

which induces

$$\int (U^{\mu'} - U^{\mu''})(d\lambda_1 - d\lambda_2) = 0$$

for the circular measures λ_1 and λ_2 with total mass 1 on two concentric circles centered at P. Hence, we have

$$\int U^{\lambda_1 - \lambda_2}\, d\mu' = \int U^{\lambda_1 - \lambda_2}\, d\mu'',$$

which induces $\mu'(S) = \mu''(S)$ for any disk S. In conclusion, we have $\mu' = \mu''$ and $\gamma'_\mu = \gamma''_\mu$.

DEFINITION. Let F be any closed set. A point P is called a regular point of F if the balayaged measure ε' of the Dirac measure ε at P onto F coincides with ε and the associated non-negative constant γ_ε reduces to zero.

With this terminology we have the following theorem.

THEOREM 2. *Two following expressions are equivalent.*

[A] *A point P is a regular point of F.*

[B] *Let μ be any positive measure with total mass 1, μ' be the balayaged measure of μ onto F and γ_μ be the associated non-negative constant. Then, it holds that*

$$U^{\mu'}(P) = U^{\mu}(P) + \gamma_\mu.$$

Proof. First, we prove that [A] implies [B]. Let λ_n be the circular measure with total mass 1 on the closed circle of radius $1/n$ centered at P, λ'_n be the balayaged measure of λ_n onto F and $\gamma_{\lambda n}$ be the associated non-negative constant. Let us remark that $U^{\lambda'_n} - \gamma_{\lambda n}$ increases to U^ε with n everywhere. It is because we have

$$\int (U^{\lambda'_n} - \gamma_{\lambda n})\, d\lambda = \int (U^{\lambda'} - \gamma_\lambda)\, d\lambda_n = \int U^{\lambda_n} d\lambda' - \gamma_\lambda$$

for the circular measure λ with total mass 1 on any closed circle, the balayaged measure λ' of λ onto F and the associated non-negative constant γ_λ, and the quantity increases with n to

$$\int U^\varepsilon d\lambda' - \gamma_\lambda = \int (U^{\lambda'} - \gamma_\lambda) d\varepsilon = \int (U^{\varepsilon'} - \gamma_\varepsilon)\, d\lambda = \int U^\varepsilon d\lambda.$$

It follows that

$$U^{\mu'}(P) - \gamma_\mu = \lim_{n\to+\infty} \int (U^{\mu'} - \gamma_\mu) d\lambda_n = \lim_{n\to+\infty} \int (U^{\lambda'_n} - \gamma_{\lambda n}) d\mu = \int U^\varepsilon d\mu = U^\mu(P).$$

Next, we prove that [B] implies [A]. Let ε' be the balayaged measure of the Dirac measure ε at P onto F and γ_ε be the associated non-negative constant. We have

$$\int U^\mu d\varepsilon = \int (U^{\mu'} - \gamma_\mu) d\varepsilon = \int (U^{\varepsilon'} - \gamma_\varepsilon)\, d\mu$$
$$= \int U^\mu d\varepsilon' - \gamma_\varepsilon$$

for any positive measure μ of finite logarithmic energy with total mass 1.

Therefore, we have

$$\int U^{\lambda_1-\lambda_2}d\varepsilon=\int U^{\lambda_1-\lambda_2}d\varepsilon'$$

for any circular measure λ_1 and λ_2 with total mass 1 on two concentric closed circles, which implies $\varepsilon(S)=\varepsilon'(S)$ for any disk S. So, we have $\varepsilon=\varepsilon'$ and $\gamma_\varepsilon=0$.

Question. In Theorem 1, the associated non-negative constant γ_μ in the balayage of any positive measure μ onto any closed set F does not always reduce to zero. But, if the complement of F is bounded, the constant γ_μ reduces to zero. What conditions are necessary and sufficient for a closed set F in order that the associated non-negative constant γ_μ in the balayage of any positive measure μ onto F always reduces to zero?

References

[1] O. Frostman.: Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions, Meddelanden Lunds Univ. Mat. Sem., Band 3, 1935.
[2] C. de la Vallée Poussin: Le potentiel logarithmique, Gauthier-Villars, Paris, 1949.

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