BULL. AUSTRAL. MATH. SOC. 30D05, 58F23
VOL. 59 (1999) [99–104]

WANDERING DOMAINS IN THE DYNAMICS OF CERTAIN MEROMORPHIC FUNCTIONS

YUEFEI WANG

It is shown that meromorphic solutions of certain first-order nonlinear differential equations do not have wandering domains.

1. INTRODUCTION

Let f be a non-linear meromorphic function. The *Fatou set* $\mathcal{F}(f)$ of f is the largest set of the Riemann sphere where the iterates f^n of f are defined and form a normal family. The complement of $\mathcal{F}(f)$ is called the *Julia set* and is denoted by $\mathcal{J}(f)$. The Fatou set is open and completely invariant. The Julia set is closed and also completely invariant. It is also known to be the closure of the set of repelling periodic points. If U is a component of $\mathcal{F}(f)$, then $f^n(U)$ lies in some component U_n of $\mathcal{F}(f)$. If $U_n \neq U_m$ for all $n \neq m$, then U is called a *wandering domain* of f. Otherwise U is called *preperiodic* and if $U_n = U$ for some $n \in \mathbb{N}$, then U is called *periodic*. For an introduction to iteration theory, we mention Beardon [**8**], Carleson and Gamelin [**15**], and Steinmetz [**27**] as well as Milnor's lecture notes for rational functions and the survey articles of Baker [**4**] and Eremenko and Lyubich [**17**] for rational and entire functions and Bergweiler [**10**] for transcendental meromorphic functions. The classical references are Fatou [**19**] and Julia [**23**] for rational and Fatou [**20**] for transcendental entire functions.

Sullivan [**28**] proved that rational functions do not have wandering domains. Transcendental meromorphic functions, however, may have wandering domains, see [**1, 2, 3, 16, 28**], while various classes of meromorphic functions without wandering domains are known [**1, 5, 6, 9, 12, 13, 14, 18, 22, 26**]. In [**9**] and [**13**] the nonexistence of wandering domains is proved for the meromorphic function f of finite order, satisfying $f'(z) = q(z)e^{p(z)}(f(z) - z)$, where $q(z)$ is rational and $p(z)$ is a polynomial, and for meromorphic solutions of one of the following differential equations

$$f'(z) = r(z)(f(z) - z)^2,$$
$$f'(z) = r(z)(f(z) - z)(f(z) - \tau),$$
$$f'(z)^2 = r(z)(f(z) - z)^2(f(z) - \tau),$$
$$f'(z)^2 = r(z)(f(z) - z)^2(f(z) - \tau)(f(z) - \delta)$$

Received 19th June, 1998
The author would like to thank Professor Walter Bergweiler for useful suggestions and comments.

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9729/99 $A2.00+0.00.

https://doi.org/10.1017/S0004972700032639 Published online by Cambridge University Press

where $r(z)$ is a rational function and $\tau, \delta \in \mathbb{C}$.

In this note, we shall show that meromorphic solutions of the following first-order nonlinear differential equations do not have wandering domains.

THEOREM 1. *Let $q(z)$ be a rational function, $p(z)$ a polynomial and $m, n \in \mathbb{N}$, $t \in \mathbb{N} \cup \{0\}$, $a \in \mathbb{C}\backslash\{0\}$. Suppose that f is a meromorphic solution of the differential equation*

$$(f'(z))^n = q(z)p(f(z))(f'(z) - a)^t(f(z) - z)^m. \tag{1.1}$$

Then f does not have wandering domains.

THEOREM 2. *Let $q(z)$ be a rational function, $p(z)$, $p_1(z)$ two polynomials and $m, n \in \mathbb{N}$. Suppose that f is a meromorphic function of finite order, satisfying the differential equation*

$$(f'(z))^n = q(z)p(f(z))e^{p_1(z)}(f(z) - z)^m. \tag{1.2}$$

Then f does not have wandering domains.

REMARKS. 1. The solutions of the differential equations (1.1) and (1.2) may have infinitely many critical values and are, usually, not of critically finite type.

2. Rational functions always satisfy differential equations of the type (1.1). In the proofs, however, we shall only discuss the case that f is transcendental

2. LEMMAS

To prove the theorems we need the following lemmas.

LEMMA 1. (Gol'dberg [**21**] or Bank and Kaufman [**7**]) *Meromorphic solutions of any first-order algebraic differential equations are of finite order.*

Therefore the order of any meromorphic solution of (1.1) is finite.

LEMMA 2. (Bergweiler and Eremenko [**11**]) *Let f be a meromorphic function of finite order. Suppose that a is an asymptotic value of f. Then a is a limit of critical values $a_k \neq a$ of f or all singularities of f^{-1} over a are direct.*

Let U be a component of $\mathcal{F}(f)$. Then we denote by U_k the component of $\mathcal{F}(f)$ that contains $f^k(U)$, $k = 1, 2, \cdots$, as in Section 1.

LEMMA 3. *Let f be a meromorphic solution of the equation (1.1) or (1.2). Suppose that f has a wandering domain U. Then there exists m such that U_k does not contain critical or asymptotic values of f for $k \geqslant m$.*

PROOF: We first discuss the case (1.1). It follows from (1.1) that all but finitely many fixpoints of f are superattracting, and that all but finitely many critical values are superattracting fixpoints.

https://doi.org/10.1017/S0004972700032639 Published online by Cambridge University Press

According to the classification of periodic components of the Fatou set, (see, for example, [**8**] or [**15**]), each superattracting fixpoint is contained in a superattractive basin, a periodic component of period 1, of $\mathcal{F}(f)$. Therefore all but finitely many critical values are not contained in $\bigcup_{k\geqslant 1} U_k$, since U is wandering.

By Lemma 1 we may assume that f has finite order ρ. Suppose that a is an asymptotic value of f. Then, by Lemma 2, a is a limit of critical values $a_k \neq a$ of f or all singularities of f^{-1} over a are direct. If a is a limit of critical values a_k , then $a \notin \bigcup_{k\geqslant 1} U_k$ since all but finitely many critical values of f are not contained in $\bigcup_{k\geqslant 1} U_k$. On the other hand, the celebrated Denjoy-Carleman-Ahlfors Theorem states that the inverse function f^{-1} has at most $\max\{2\rho, 1\}$ direct singularities [**25**]. Now we conclude that all but finitely many asymptotic values of f are not contained in $\bigcup_{k\geqslant 1} U_k$. This proves the lemma for the case (1.1).

Similarly we can prove the lemma for the case (1.2). □

A fix point z_0 of a meromorphic function f is called *weakly repelling* if $|f'(z_0)| > 1$ or $f'(z_0) = 1$. We need the following result of Bergweiler and Terglane [**13**].

LEMMA 4. *Let f be a meromorphic function with a wandering domain U. Suppose that there exists m such that U_k does not contain critical or asymptotic values of f for $k \geqslant m$. If for some $n \geqslant m$, U_n is multiply-connected, then f has infinitely many weakly repelling fixpoints.*

3. PROOF OF THE THEOREMS

We first prove Theorem 1.

PROOF OF THEOREM 1: Suppose that f has a wandering domain U. Then it follows from Lemma 3 that there exists m such that U_k does not contain critical or asymptotic values of f for $k \geqslant m$. If for some $n \geqslant m$, U_n is multiply-connected, then f must have infinitely many weakly repelling fixpoints by Lemma 4. This contradicts the fact that f has at most finitely many weakly repelling fixpoints since all but finitely many fixpoints of f are superattracting fixpoints.

Therefore for each $k \geqslant m$, U_k is simply-connected. Now we apply the quasiconformal methods of Sullivan [**28**]. For a given number $K > 1$, we consider K-quasiconformal self-maps ϕ of the sphere which fix $0, 1, \infty$, such that $f_\phi = \phi \circ f \circ \phi^{-1}$ is meromorphic. Since U is an eventually singular-value free, simply-connected wandering domain, the *Beltrami coefficients* of the $\{\phi\}$ depend on infinitely many parameters, or equivalently the deformation family $\{f_\phi\}$ depends on infinitely many parameters, see [**5, 8**] or [**28**].

It is clear that a pole z_0 of f corresponds to the pole $\phi(z_0)$ of f_ϕ, with the same

https://doi.org/10.1017/S0004972700032639 Published online by Cambridge University Press

multiplicity. On the other hand, it follows from Lemma 1 and (1.1) that f has finite order and $f' \neq a$. Hence if we express $p(f)$ in the form

$$C_0 \prod_{0 \leqslant j \leqslant l} (f - b_j)$$

where k is a non-negative integer, and C_0, b_j are complex numbers, $(0 \leqslant j \leqslant l)$, then we have from (1.1) that the critical points of f correspond to the fixpoints, b_j-points, $(0 \leqslant j \leqslant l)$, of f, with finitely many exceptions. It is also easy to see that the fixpoints, b_j-points $(0 \leqslant j \leqslant l)$ of f correspond to the fixpoints, $\phi(b_j)$-points of f_ϕ, with finitely many exceptions. Thus we find that $\left(f'_\phi(z)\right)^n / \left\{ \left[\prod_{0 \leqslant j \leqslant l} \left(f_\phi(z) - \phi(b_j)\right)\right] \left(f_\phi(z) - z\right)^m \right\}$ has only finitely many zeros and poles, as many as q has. The Hölder continuity of ϕ at ∞ gives that $\left|\phi(z)\right| = O\left(|z|^K\right)$ and $\left|\phi^{-1}(z)\right| = O\left(|z|^K\right)$ as $z \to \infty$. This implies that $\rho(f_\phi) \leqslant K\rho(f)$, where $\rho(f)$ denotes the order of f. Therefore the order of $\left(f'_\phi(z)\right)^n / \left\{ \left[\prod_{0 \leqslant j \leqslant l} \left(f_\phi(z) - \phi(b_j)\right)\right] \left(f_\phi(z) - z\right)^m \right\}$ is also at most $K\rho(f)$. We conclude that

$$\frac{\left(f'_\phi(z)\right)^n}{\left[\prod_{0 \leqslant j \leqslant l} \left(f_\phi(z) - \phi(b_j)\right)\right] \left(f_\phi(z) - z\right)^m} = q_\phi(z) e^{r_\phi(z)} \tag{3.1}$$

for some rational function q_ϕ of the same degree as q and some polynomial r_ϕ of degree at most $K\rho$. (3.1) gives that the deformation family $\{f_\phi\}$ depend only on finitely many parameters, which is a contradiction. The proof of the Theorem 1 is complete. ∎

Similarly we can prove Theorem 2.

References

[1] I.N. Baker, 'An entire function which has wandering domains', *J. Austral. Math. Soc. Ser. A* **22** (1976), 173–176.

[2] I.N. Baker, 'Wandering domains in the iteration of entire functions', *Proc. London Math. Soc. (3)* **49** (1984), 563–576.

[3] I.N. Baker, 'Some entire functions with multiply-connected wandering domains', *Ergodic Theory Dynamical Systems* **5** (1985), 163–169.

[4] I.N. Baker, *Iteration of entire functions: an introductory survey*, Lectures on complex analysis (World Scientific, Singapore, London, 1987), pp. 1–17.

[5] I.N. Baker, J. Kotus and Y. Lü, 'Iterates of meromorphic functions IV: Critically finite functions', *Results Math.* **22** (1992), 651–656.

[6] I.N. Baker and A. Singh, 'Wandering domains in the iteration of compositions of entire functions', *Ann. Acad. Sci. Fenn. Ser. A I Math.* **20** (1995), 149–153.

https://doi.org/10.1017/S0004972700032639 Published online by Cambridge University Press

[7] S. Bank and R. Kaufman, 'On meromorphic solutions of first-order differential equations', *Comment. Math. Helv.* **51** (1976), 289–299.

[8] A.F. Beardon, *Iteration of rational functions* (Springer-Verlag, Berlin, Heidelberg, New York, 1991).

[9] W. Bergweiler, 'Newton's method and a class of meromorphic functions without wandering domains', *Ergodic Theory and Dynamical Systems* **13** (1993), 231–247.

[10] W. Bergweiler, 'Iteration of meromorphic functions', *Bull. Amer. Math. Soc.* **29** (1993), 151–188.

[11] W. Bergweiler and A. Eremenko, 'On the singularities of the inverse to a meromorphic function of finite order', *Revista Mat. Iberoamericana* **11** (1995), 355–373.

[12] W. Bergweiler, M. Haruta, H. Kriete, H.G. Meier and N. Terglane, 'On the limit functions of iterates in wandering domains', *Ann. Acad. Sci. Fenn. Ser. A I Math* **18** (1993), 369–375.

[13] W. Bergweiler and N. Terglane, 'Weakly repelling fixpoints and the connectivity of wandering domains', *Trans. Amer. Math. Soc.* **348** (1996), 1–12.

[14] W. Bergweiler and Y. Wang, 'On the dynamics of composte entire functions', *Ark. Mat.* **36** (1998), 31–39.

[15] L. Carleson and T. Gamelin, *Complex dynamics* (Springer-Verlag, Berlin, Heidelberg, New York, 1993).

[16] A. Eremenko and M. Lyubich, 'Examples of entire functions with pathological dynamics', *J. London Math. Soc.* **36** (1987), 454–468.

[17] A. Eremenko and M. Lyubich, 'The dynamics of analytic transforms', *Leningrad Math. J.* **36** (1990), 563–634.

[18] A. Eremenko and M. Lyubich, 'Dynamical properties of some classes of entire functions', *Ann. Inst. Fourier* **42** (1992), 989–1020.

[19] P. Fatou, 'Sur les équations fonctionelles', *Bull. Soc. Math. France* **47** (1919), 161–271; **48** (1920), 33–94, 208–314.

[20] P. Fatou, 'Sur l'itération des fonctions transcendantes entières', *Acta Math.* **47** (1926), 337–360.

[21] A. Gol'dberg, 'On sing-valued solutions of first-order differential equations', (in Russian), *Ukrain Mat. Zh.* **8** (1956), 254–261.

[22] L. Goldberg and L. Keen, 'A finiteness theorem for a dynamical class of entire functions', *Ergodic Theory and Dynamical Systems* **6** (1986), 183–192.

[23] G. Julia, 'Sur l'itération des fonctions rationelles', *J. Math. Pures Appl.* **4** (1918), 47–245.

[24] J. Milnor, *Dynamics in one complex variable: introductory lectures* (IMS Stony Brook, New York, 1990).

[25] R. Nevanlinna, *Eindeutige analytische Funktionen* (Springer-Verlag, Berlin, Heidelberg, New York, 1953).

[26] G.M. Stallard, 'A class of meromorphic functions with no wandering domains', *Ann. Acad. Sci. Fenn. Ser. AI. Math.* **16** (1991), 211–226.

[27] N. Steinmetz, *Rational iteration* (Walter de Gruyter, Berlin, 1993).

https://doi.org/10.1017/S0004972700032639 Published online by Cambridge University Press

[28] D. Sullivan, 'Quasiconformal homeomorphisms and dynamics I: Solution of the Fatou-Julia problem on wandering domains', *Ann. Math.* **122** (1985), 401–418.

Institute of Mathematics
Chinese Academy of Sciences
Beijing 100080
China
e-mail: wangyf@math03.math.ac.cn

https://doi.org/10.1017/S0004972700032639 Published online by Cambridge University Press